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The Mathematical Preparation of Prospective Elementary Teachers: Reflections from Solving an *Interesting Problem*

Corresponding Author:
Mark W. Ellis
California State University, Fullerton
23 Sycamore Lane
Buena Park, CA 90621
mellis@fullerton.edu

José Contreras
The University of Southern Mississippi
20 Cornerstone Rd
Hattiesburg MS 39402
jose.contreras@usm.edu

Armando M. Martínez-Cruz
California State University, Fullerton
154 McMarthy Hall
Fullerton, CA 92834
Amartinez-cruz@fullerton.edu

Abstract

Problem solving tasks offer valuable opportunities to strengthen prospective elementary teachers' knowledge of and disposition toward mathematics, providing them with new experiences doing mathematics. Mathematics educators can influence future instruction by modeling effective pedagogical strategies that engage students in making sense of processes of mathematical reasoning. What follows is a description of a well-designed task and the role played by one mathematics educator in engaging prospective teachers in processes of mathematical reasoning. It is hoped that this description can shed light on some general ideas about the design and implementation of problem solving tasks in courses for future elementary teachers.

Introduction

The mathematics courses for future elementary teachers are among the most important courses taught in any mathematics department. This assertion is based in the fact that the activities these teachers will conduct in their classrooms will influence students' achievement in and attitudes toward mathematics (Hill, Rowan, & Ball, 2005; Midgley, Feldlaufer, & Eccles, 1989) as well as their future decisions about taking more or fewer (to none) mathematics courses. While it is true that recruiting students into the study of mathematics is not solely the responsibility of elementary teachers, there is little doubt that they play an important role in setting up students' trajectories in mathematics (Midgley, Feldlaufer, & Eccles, 1989).

Prior research has demonstrated that we, university mathematics educators, must do a better job preparing the teachers who will work in the elementary grades. The mathematical knowledge of many U.S. elementary teachers is algorithmic and fragile, susceptible to error when extended beyond routine calculation or challenged with identifying conceptual connections (Ball, 1990; Conference Board of the Mathematical Sciences, 2000; Ma, 1999; Simon & Blume, 1994). Many teachers believe mathematics to be an illogical collection of loosely related facts and skills (Putnam, Heaton, Prawat, & Remillard, 1992; Raymond, 1997) and often harbor less than positive feelings toward the subject (Cornell, 1999).

This matter is not simply one of attitude but of experience. Based on the findings of observational studies of mathematics classrooms in the U.S., it is safe to assume that prospective elementary teachers in the U.S. are likely to have learned mathematics as nothing more than a collection of facts to recall and algorithmic skills to mimic (Stigler & Hiebert, 1997; Weiss, Pasley, Smith, Banilower, & Heck, 2003). This historical pattern of teaching mathematics expects students to accept what they are told without troubling over whether things make sense, and with little to no need for justification. As professional organizations are calling on elementary teachers to develop mathematical thinkers and problem solvers (e.g., NCTM, 2000; National Research Council, 2001), we are drawn to the words of Cohen and Ball (1990): “How can teachers teach a mathematics that they never learned, in ways that they never experienced?” (p. 238).

With this in mind it is incumbent on the university mathematics educator to offer opportunities for prospective elementary teachers to engage in authentic processes of doing mathematics that allow them to “make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures” (NCTM, 2000, p. 3). Problem solving tasks offer valuable opportunities to strengthen future elementary teachers’ knowledge of and disposition toward mathematics, providing them with new experiences doing mathematics (e.g., Toluk-Uçar, 2009).

What follows is a narrative reflection on the use of a well-designed task and the role played by one mathematics educator in engaging prospective elementary teachers in processes of mathematical reasoning. Notes were taken by a second mathematics educator who sat in as the class discussed this problem which had been assigned as homework with particular attention paid to the processes of problem solving (described below). It is hoped that this rich description provides insights into some general ideas about the selection and implementation of problem solving tasks in mathematics courses for future elementary teachers.

Choosing and Implementing Problem Solving Tasks

Good problem solving tasks share some common features (Smith, Grover, & Henningsen, 1996). First, they are accessible to a wide range of students yet have no quick solution. Second, they require some amount of investigation or data gathering. Third, there are multiple mathematical paths to a solution or solutions. Fourth, they present opportunities for generalizations to be formed about mathematical relationships. Fifth, they require problem solvers to justify their steps and conclusions based on the givens. And, lastly, they allow for sense-making in that solutions and generalizations can be understood by reference to the original problem context.

The importance of task implementation cannot be underestimated. In their research on middle school teachers’ use of problem solving tasks, Smith, Grover, and Henningsen (1996) observed that more often than not teachers disrupted students’ opportunities for doing mathematics by transforming the tasks into something more algorithmic and disregarding processes of reasoning, focusing instead on the verity of students’ answers. University mathematics educators can influence future teachers’ instruction by modeling effective pedagogical strategies that engage and sustain their students in processes mathematical reasoning including: a) recognizing patterns; b) expressing patterns symbolically; and c) justifying approaches through proof (NCTM, 2000).

Each of these processes was modeled while solving a problem in the beginning unit of a mathematics course for prospective elementary teachers. The problem described herein forms the basis for 4-5 class sessions (each 75 minutes) with plenty of assignments related to current or future classroom discussions. Students are asked to work with one or more classmates in small groups in order to get used to listening to others’ thinking and expressing their own ideas in a safe environment. The description that follows offers insight into the instructor's pedagogical actions as students wrestled with the problem and gained experience with the processes of mathematical reasoning. The voices of students are included when possible to illustrate their thinking around this problem.

An Interesting Problem

The following problem (Fig. 1) appears in the first chapter of Musser, Burger, and Peterson's (2003) *Mathematics for Elementary Teachers: A Contemporary Approach.* The column labeling and the grid have been added for easier reference. This task is hereafter referred to as an *interesting problem* for it engages students with several productive mathematical discussions and concepts, such as discovering patterns, generating symbolic expressions, and developing algebraic rules.

The integer numbers greater than 1 are arranged as shown:

C1	**C2**	**C3**	**C4**	**C5**
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	

a. In which column will 100 fall?
b. In which column will 1000 fall?
c. How about 1999?
d. How about 99,997?

Fig. 1. An *interesting problem*
(Musser, Burger, and Peterson, p. 26; used with permission)

Recognizing Patterns

Before addressing the questions posed in the textbook, students are asked to complete the next five rows in the array. This allows them the chance to explore and understand implicit information in the situations they are called to solve, and it is not a difficult task to complete for the majority. Students are then asked to describe verbally the pattern that *all* the numbers follow, something that is markedly more challenging. Using a familiar analogy, students are told to describe the pattern as if they were teaching their instructor to dance. This contextualization helps tremendously; students offer suggestions such as, "move over four steps, go down, move over four to the left." More importantly, their anxiety level is noticeably decreased when they recognize that some mathematical situations can be made concrete. The dancing analogy, of course, is just a twist of "acting out" the problem.

Next, students are asked to describe the type of numbers that they see in each column and how many numbers they find in each row. This again is a very accessible task. Many say that numbers in column 2 are even while the numbers in column 5 are odd. Others notice that the numbers in some columns "go by" a certain amount. For instance, the numbers in column 5 "go by 8". Sometimes students mention that the numbers in column 1 always have a decimal part of 0.125 when divided by 8. Still others may note that if "you subtract the first number (5) in column 5 from another number in that column, and divide by 8, you get a whole number." The instructor's role during this process is to acknowledge those who have an idea and ask them to convey it to the rest.

While these prospective elementary teachers are often quick to generalize ideas from (very) few instances, in order to push their thinking further they are asked to justify their claims (Lannin, 2005). Many can explain why the numbers in column 5 "go by 8". A common line of reasoning argues that, "there are four numbers in each line and one skips a line for the next number so you have four more." Fewer can explain why the numbers in column 3 "go by 4" or why "every other number in column 2 is a multiple of 8." Typically, no one can explain right away about the 0.125 or the division-by-8 result. Preparing students to answer such questions requires the introduction of some algebraic tools.

Before moving to that, students are encouraged to solve the first two parts of the problem: "In which column will 100 fall?" And "In which column will 1,000 fall?" As a quick way to take a poll of students' answers they are asked to show with their fingers in which column (1-5) they believe the given value will fall. In a recent semester, 26 of 30 students claimed that 100 would fall in C4. Before verifying this solution, students are asked to explain why 100 cannot fall in C3. Most are eager to offer the observation that C3 is an "odd" column (made up of only odd numbers) and 100 is an even number.

When asked to share the strategies used to determine 100 falls in C4, roughly one-half admit to listing all numbers through 100 . In order to emphasize the idea that there can be multiple pathways to the answer in mathematics—and to shed light on the utility of recognizing and extending patterns—students are asked what other strategies could be used to determine that 100 would fall in C4. Some combine the information about the list in C5 with the "right-left" construction of the array and others discover patterns in the array such as the way the multiples of ten appear (e.g., 10 is in C1, 20 and 30 are in C3, 40 and 50 are in C1, and so on). Memorably, one student said she looked at 100/4 since each row has just one multiple of 4. The quotient of 25 was then identified as an odd factor. The student concluded that 100 would be in C4 since all of the "odd multiples" of 4 (the product of 4 and an odd number) were in C4.

This first foray into problem solving gets everyone involved and pushes students to think more flexibly about mathematics. While their early strategies work fine for the first two parts of the problem, students quickly recognize that they need to develop more mathematically powerful strategies in order to figure out the last two parts of the problem.

Expressing Patterns Symbolically

In scaffolding students' deeper exploration of the mathematical patterns in the array, students are introduced to the concept of sequence, as a list of numbers, and told to think of the numbers in each column as a sequence. As practice, they are asked to produce the first 20 terms of the sequence in C5 knowing that they "go by" 8. A t-table with columns for term position and term value is provided as a means to organize their work (Figure 2). After they have produced the list, students are asked to consider the idea of position: 5 is the first term, 13 is the second, 77 is the ninth, and so on. Finally, the idea of *common difference, d,* is defined as the change in value of numbers from one position to the next in a sequence. Students quickly recognize this as the mathematical term for "go by."

This idea is applied to several other arithmetic sequences for which students are asked to produce several terms when given the first term and the common difference.

Position	**Value**
1	5
2	13
3	21
4	29
5	37

Fig. 2. A T-table for the sequence in C5

With this foundation in place, students are challenged to work on developing a formula for the general case, n, of each sequence. Through group discussion, students determine that the common difference is the "multiplier" and must be the coefficient for n. Further trial and error leads many groups to figure out that $f(n) = 8n$ does not work without having an "offset" or constant. For those who do not recognize the need to subtract 3 for the formula in C5, they are told to leave it for a moment and work on the formula for C1. Since this requires addition rather than subtraction, students often more readily see that for C1 the generalized formula is $f(n) = 8n + 1$. With this discovery, they quickly realize the constant for C5 is -3, giving $f(n) = 8n + (-3)$.

Before working on other examples, students are asked to substitute other values of n to check their formulas for C5 and C1. This promotes the habit of verifying that the answer makes sense in relation to the original problem and moves students away from simply *getting an answer*. Once the formula for a sequence has been found, students use it to determine a particular term in the sequence (say, *What's the* 300^{th} *term of the sequence given by* $f(n) = 8n +1$?) and to decide whether or not a specific number is a term of the sequence. The latter of these poses some challenge in that students must reverse their thinking and "undo" the formula. For instance, *Is 989 a term in the sequence of C1?* Many times students quickly determine the 989^{th} term, having misunderstood the question. Allowing them to discuss their thinking in groups is an effective way for students to catch this error for themselves. Peers can often reword the question to help a classmate make sense of it. One such case is a student who posed the following question to her group when they were confused by the query about 989: "If the sequence 9, 17, 25,… was extended would the number 989 ever appear?"

The discussion of expressing sequences symbolically ends with some talk about the power of having a general formula: we can determine *any* term, decide if a number is an element of the sequence or not (including the position in which it is if the answer is "yes"), and more importantly, describe in a single and general way all the elements of the sequence.

Justifying Students' Approaches

Important as students work on this problem are the ways in which their approaches and discoveries are justified or proven. Many times their justifications are given immediately, but other times students are asked to wait until more tools have been acquired. Two specific instances of students' discoveries are examined below. The first claim is that the numbers in the sequence in C1 have a decimal portion of 0.125 when divided by 8. As mentioned earlier, students could not explain this claim immediately. However, once they had developed the formula for C1 to be $f(n) = 8n + 1$, students are asked to revisit their earlier claim and given a few minutes to do so independently or in small groups. When invited to share their findings, typically several students are able to explain that since the term values are given by 8n + 1, when this is divided by 8, we get $\frac{8n+1}{8} = \frac{8n}{8} + \frac{1}{8} = n + \frac{1}{8}$ which is of the form “a whole number plus .125.” Students are then challenged to use similar approaches to determine the decimal portion in each column and to prove that the difference between a number in C1 and 9 is evenly divisible by 8. Thus students learn to justify their claims mathematically as a means to “prove” their validity.

The second claim pertains to counting the terms of a sequence. One student determined the number of terms in a given sequence, 9, 17, 25, …, 1017 (which are values in the first column), using the following method. First, she computed the value that “would appear” before 9 in this sequence. This is 1 since the common difference is 8 (and 1 = 9 – 8). Next, she computed the difference between the last term given (in this case, 1017) and the value before 9. This difference is 1017 – 1 = 1016. Finally, she divided this answer by 8 and found that $\frac{1016}{8} = 127$. She then claimed that there are 127 terms. In other words, 1017 is the 127^{th} term in the sequence in C1. Although the idea is correct and other students checked that this method works, they could not justify their reasoning mathematically. At this point, with students’ interest piqued, the instructor can guide students through the steps of tracing a proof.

One approach for doing so involves first translating the student’s method into words and then using algebra to provide a proof. For instance, the student did the following computation: $\frac{1017-1}{8}$., we recognize that the student found the 0^{th} *term* when she noted that subtracting the common difference, 8, from 9 gives a value of 1. Translating this into mathematical words we get $\frac{n^{th}\ term - 0^{th}\ term}{common\ difference}$. The general algebraic formula for the n^{th} term of an arithmetic sequence is given as $f(n) = dn + a$, where d is the common difference and $a = first\ term\ given - d$. (When finding the formula for a given arithmetic sequence, a first term was given and the common difference obtained by subtracting two consecutive terms.) So, her computation can be expressed algebraically as $\frac{n^{th}\ term - 0^{th}\ term}{common\ difference} = \frac{(dn+a)-(a)}{d} = \frac{dn}{d} = n$. The formula $f(n) = dn + a$ becomes $f(0) = d*0 + a = a$. In, other words the 0^{th} term is a, and this is what we substituted.

The exploration of this *interesting problem* was brought to closure with a summary of what was learned and an opportunity to pose further “interesting problems.” The latter was quite attractive to students as they were asked to modify different aspects of the given problem and predict how the changes would alter the resulting patterns. Their ideas included putting more or fewer columns, starting the sequence in a different location, and changing the direction in which the numbers are written in each row. They were excited to be “making problems” and eager to continue the exploration beyond the boundaries of the time allotted in class. Such discourse is mathematically productive and gives students a sense of ownership of the ideas being offered.

Looking Back

Students are rarely exposed to working with one single problem over several class sessions, even as mathematics majors. Here, an *interesting problem* served as a vehicle through which to investigate patterns, as a reason to introduce more sophisticated tools (such as rules for functions), as a context to model pedagogical skills that teachers will need, and as a means to illustrate processes of mathematical reasoning and justification that teachers are increasingly being called on to demonstrate and foster in the classroom. Another benefit of the problem is the rich environment it creates within which discussions of other mathematical ideas and attitudes toward mathematics arise naturally. All the time, students had opportunities to consider and comment on a) the appropriateness of mathematical language and symbol use (for instance, the case of the equal sign), b) multiple approaches to solving a mathematics problem, c) the connection between the common difference in an arithmetic sequence and the slope of a line, and d) the idea of having fun while doing mathematics!

Reflecting on how much was covered, there is no doubt that progress was made in the mathematical preparation of these future elementary teachers. It is hoped that these experiences as learners of mathematics will serve as a point of reference for them as they go on to create their own learning environments, fostering the next generation of mathematicians.

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