Is $e^{\pi} > \pi^{e}$? Pre-Service Teachers Making Sense of Numbers

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Abstract

This paper is about making conclusions from the attempts of two calculus II pre-service teachers who were presented with the following problem: How can it be shown that e^π>π^e without the use of a calculator? Jane employed knowledge of calculus to solve the problem but she evinced gaps in her mathematical background when using operations of logarithmic and exponential functions. On the other hand, John used two approaches to solve the problem: one using calculus I techniques and the other, infinite series ideas. Interestingly his second approach enabled him to amend his first approach which resulted in a correct solution. Jane demonstrated gaps in mathematical understanding that were not witnessed in John's work. The problem makes an excellent exercise in a first-semester course in calculus.

Key wards: Problem solving, calculus, curve sketching, concavity, critical points, composition of functions, Series, inequality, exponential function and logarithmic functions.

Introduction

Learning mathematics involves solving problems that increase in sophistication as students advance from one grade level to the next, a gradual process which should "support students' development of number sense and understanding of operation" (NCTM, 2000 p. 117). Number sense, which is a perception of a number's ordinal and cardinal meaning (National Council of Teachers of Mathematics, 1989 and NCTM, 1989), needs to be learned and taught. This is so because "number sense is a holistic concept of quantities, numbers, operations, and their relationships, which should be efficiently and flexibly applied to daily life situations" (Yang & Wu, 2010).

Learning number sense starts very early as students are being introduced to number operations. It is important for students to understand number concepts because such an understanding lays a "foundation for problem solving, strategy development, and generalization abilities" (Yang & Wu, 2010, p. 55). Students' number sense continues to develop from one grade level to the next and throughout their educational career. As students continue to encounter more sophisticated and advanced mathematics problems, number sense and problem-solving strategies play a significant role in the learning of mathematics.

This paper is about a problem that the researcher posed to a class of Calculus II students. The question was, "How can you show that $e^{\pi} > \pi^{e}$ without a calculator?" Two students, Jane and John (pseudonyms), who were both preservice teachers for the class, were selected for a task-based interview because of the nature of their approach and the strategies they used to tackle the problem. As John's first approach will demonstrate, one solution to showing without a calculator that the inequality $e^{\pi} > \pi^{e}$ is true produces an interesting application of the calculus techniques of graph sketching. In fact, this solution of John's would make an excellent exercise in a first-semester course in calculus. The two students are discussed as follows.

Jane

When Jane was presented with this problem, she thought about it for a moment. Since she was not supposed to use a calculator or any technological tools, she struggled to find a solution. Since it was a calculus class, she thought she might use her knowledge of calculus to solve the problem but was not sure how. She remembered the problem they did at the beginning of the semester (f(x) = xlnx-x), for which they were supposed to sketch out the problem and discuss its characteristics. Based on this previous experience, Jane decided to use a similar strategy, as outlined below.

> Jane: Maybe I should introduce variables to come up with an equation that I can investigate.
>
> Researcher: How do you plan to set it up? (Jane thought for a moment)
>
> Jane: How about expressing $e^{\pi} > \pi^{e}$ in the form of a function as $f(x) = x^{\pi} - \pi^{x}$ and investigate the nature of the function?
>
> Researcher: Why did you decide to replace e with x rather than replace π with x?
>
> Jane: We could either set $f(x) = x^{\pi} - \pi^{x}$ or $f(x) = e^{x} – x^{e}$. We will get a similar function either way, and the analysis will be similar to make our conclusions.

Jane's thinking was to analyze the concavity of the function to try to prove that $e^{\pi} > \pi^{e}$. She deduced that if she could determine that all the values of f(x) and f ``(x) (as x tends to e) are positive and, consequently, that the function is concave up, then she could conclude somehow that $e^{\pi} > \pi^{e}$. Jane wanted to find the second derivative of f(x) to determine the concavity of her function as illustrated in her solution below in Figure 1.

On the right-hand side, Jane drew four Cartesian planes with four scenarios. She noted that only the first two could be useful in analyzing concavity and answering the question as to whether or not $e^{\pi} > \pi^{e}$. The first scenario was useful for responding to the question. The second scenario was useful as well, but if it is concave down, in that case, it could offer proof by contradiction, since that can only happen if $e^{\pi} < \pi^{e}$.
Jane abandoned this approach because the expression $(x^{\pi-1} = \pi^{x-1} \ln \pi)$was too complicated for her to solve, but it did provide an opportunity to think about the best way forward. The goal for Jane at this point was to express e^{π} and π^{e} so that they might be expressed either with the same natural logarithm or the same base e. Jane tried the option of expressing her function in

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terms of the natural logarithm, but this turned out to be rather complicated, and she didn't make much progress.

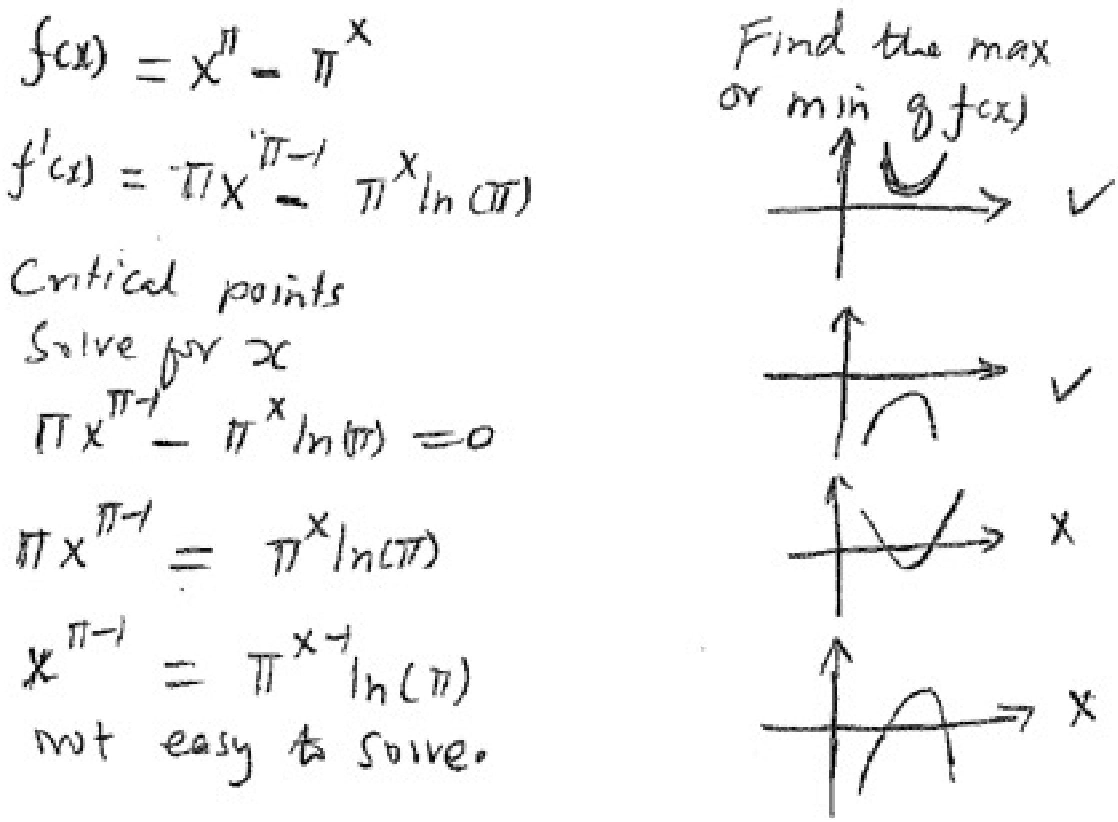

Figure 1: Jane's first step

Researcher: I see you are trying to use the natural logarithm to solve the problem?

Jane: Yes but I realized that it would be complicated to solve.

Researcher: So?

Jane: I think I will like to try expressing π^e to base e and hope it will also not be as complicated!

Expressing π^e in terms of base e was also a challenge for Jane that the researcher wanted to investigate further.

Researcher: Why do you want to express π^e to base e?

Jane: So that I can compare exponents and maybe get an idea about the relationships between π^e and e^π.

Jane was good at changing the base of a logarithmic expression to another logarithmic expression, but it was a challenge to convert π^e to a form of e to the power of something. The researcher then suggested an idea for Jane.

Researcher: Can you try $\pi^e = e^{e\ln\pi}$ that you learned in college algebra?

Jane: Is $\pi^e = e^{e\ln x}$?

Researcher: What do you think?

Jane: Not sure.

Researcher: Yes, note that $e^{e\ln\pi} = e^{\ln\pi^e} = \pi^e$.

Jane: Not sure why $e^{\ln\pi^e} = \pi^e$.

To understand what challenges Jane was going through, I posed the following question:

Researcher: Can you simplify the expression $\ln e^{\pi^e}$?

Jane: $\ln e^{\pi^e} = \pi^e$.

Researcher: Why?

Jane: Because each is the inverse of the other which allows me to write the equation $\log_e e^{\pi^e} = \pi^e$. This follows from the definition of a logarithm to the base e.

Jane's concept of inverse function was that it allows the e of the base and the e of the number to cancel each other implied that the concept of logarithmic function and composition of function was not well unpacked in Jane's thinking.

Researcher: Can the same be applied to $e^{\ln\pi^e}$?

Jane: No, because $e^{\ln\pi^e}$ is different from $\ln e^{\pi^e}$ and that rule does not apply here because there are no two es to cancel in the expression.

Jane also demonstrated that she preferred to work with the notation $\log_e e$ rather than the equivalent form $\ln e$. It seemed to her to be more intuitive to cancel the two e's in $\log_e e$. This step is not so easily "seen" when using the form $\ln e$.

The problem with Jane was that she had knowledge-gaps in the concept of the composition of functions. The researcher reminded Jane of the identity $(fog)(x) = (gof)(x) = x$ which holds if the functions f and g are inverses of each other.

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Researcher: Note that $e^{\ln \pi^e} = \pi^e$ and likewise $\ln e^{\pi^e} = \pi^e$, using the above identity.

Jane: Got it! I think I can solve it now.

Jane paused for a moment, thought a little bit, and noted:

Jane: Since we want to show that $e^{\pi} > \pi^{e}$, and that $\pi^{e} = e^{e\ln\pi}$, then $e^{\pi} > e^{e\ln\pi}$ Since the exponential function is increasing, we need to show that $\pi > e \ln \pi$ right?

Researcher: yes

Jane: Okay.

Jane went on to do the problem as follows:

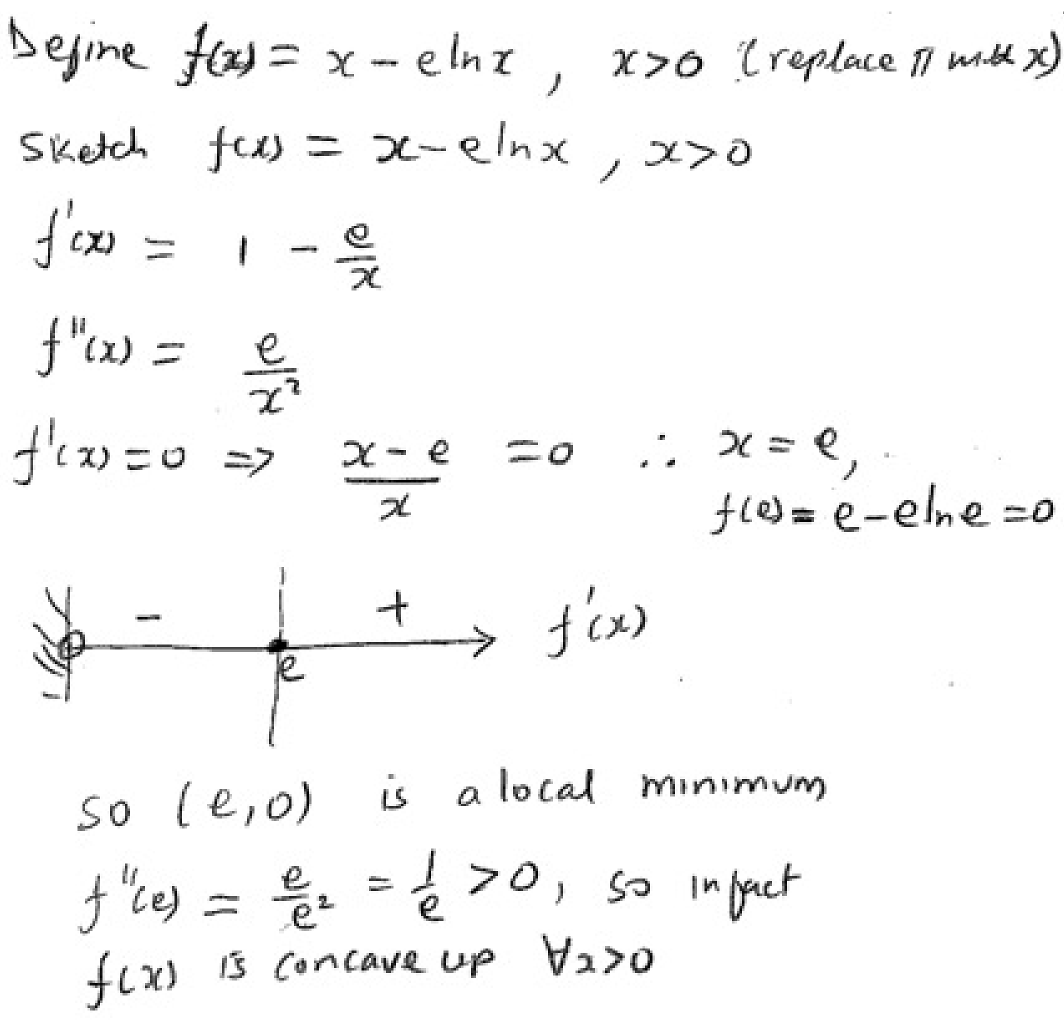

Figure 2: Jane's second step

Researcher: So, what does that mean?

Jane: Let us sketch the graph.

Researcher: How are you going to sketch it?

Jane thought for a moment and wrote the following:

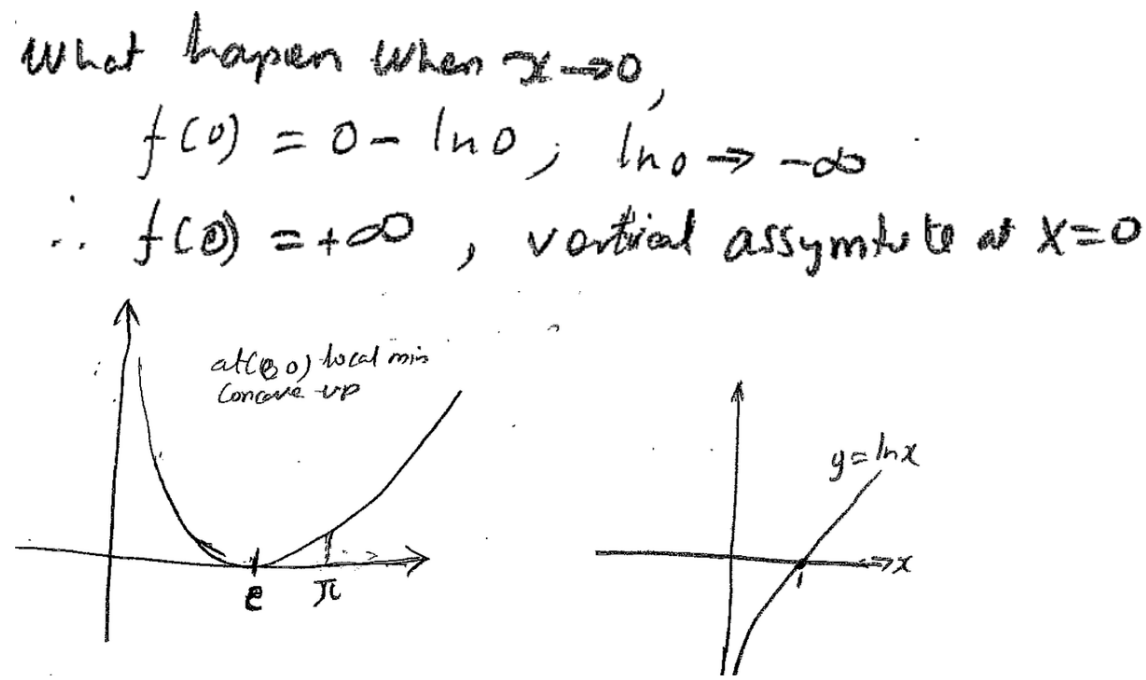

Figure 3

When Jane was asked what she could say about the graph in figure 3 (f(x) = x-elnx), she was initially unsure what else she might say about x = e.

Researcher: What do you think is happening with x = e regarding x and elnx?

Jane: I know that when x>e, f(x) = x-elnx is increasing, and on the other hand, when x<e, the function is decreasing and is zero at x = e.

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Jane then recalled that at the beginning of the derivation she had set f(x) = x-elnx, x>o (she replaced π with x). The critical question is now, what happens when x = π? Jane looked at the figure above and observed:

Jane: Notice that f(π)>0, however f(π) = π-elnπ>0, therefore π>elnπ meaning that $e^{\pi}>\pi^{e}$

Jane managed to solve this problem, but she did have challenges using operations of the natural logarithm function and was confused when representing the given expressions graphically to justify the original inequality.

John

John, on the other hand, carried out an interesting investigation and came up with two approaches to solving the problem. The first approach was similar in some respects to Jane's, whereas the second approach was a little bit different, but was inspired and informed by the first approach. As opposed to Jane, John had completed the second part of the calculus sequence. John acknowledged that it was not readily apparent that $e^{\pi} > \pi^{e}$, "since the base e of e^{π} is smaller than the base π of π^{e} ".

John's first approach used natural logs, and the second approach was to use his knowledge of series and sequences. As will be illustrated below, John started with approach 2 first, hit a wall and went back to approach 1, and then made connections that led to the completion of approach 2.

Approach 1

John started out by working backward to try to solve the problem. Unlike Jane, John decided at the outset to make use of natural logarithms of e^{π} and π^{e} as shown below.

First, we will work backward from $e^{\pi} > \pi^{e}$.

Taking the natural log of both sides, we get: $ln(e^{\pi}) > ln(\pi^{e})$.

This gives us: $\pi * ln(e) > e * ln(\pi)$

Rewriting this, we get: $\pi - e * ln(\pi) > 0$. If this can be shown to be true, we will be finished.

Using this, we define our function as follows, replacing π with x.

Define $f : \mathbb{R}^{+} \to \mathbb{R}$ by $f(x) = x - e * ln(x)$. Then $f'(x) = 1 - \frac{e}{x}$.

Setting $f'(x) = 0$, we get $1 - \frac{e}{x} = 0$. Thus, $x = e$ is the only critical point.

Suppose that $x > e$. Then

$f'(x) = 1 - \frac{e}{x} > 1 - \frac{e}{e} = 1 - 1 = 0.$

Thus, when $x > e$, f is increasing.

Since $\pi > e$, then

$f(\pi) = \pi - e * ln(\pi) > f(e) = e - e * ln(e) = e - e = 0,$

so $\pi - e * ln(\pi) > 0$, (which is what we wanted to show from above.)

Thus, $\pi > e * ln(\pi) = ln(\pi^{e})$.

Thus, $e^{\pi} > e^{ln \pi^{e}} = \pi^{e}$

Therefore, $e^{\pi} > \pi^{e}$.

As opposed to Jane, John had a more advanced grasp of the operations of the logarithmic and exponential functions.

Approach 2

John also comes up with the following solution, which was followed by an interview with the researcher.

We have that $e^x = \sum_0^\infty \frac{x^k}{k!}$.

Pick $a = \frac{\pi}{e} - 1$.

Then $e^a = \sum_0^\infty \frac{a^k}{k!} = 1 + a + \sum_0^\infty \frac{a^k}{k!} > 1 + a$

Thus, $e^a > 1 + a$

Substituting $a = \frac{\pi}{e} - 1$ into the inequality, we get

$e^{\frac{\pi}{e}-1} > 1 + \frac{\pi}{e} - 1 = \frac{\pi}{e}$

Thus, $e^{\frac{\pi}{e}-1} > \frac{\pi}{e}$

Multiplying by e on both sides of the inequality, we get

$e^{\frac{\pi}{e}} > \pi$

Raising both sides of the inequality to the power of e, we get

$e^{\frac{\pi}{e}*e} > \pi^e$

Therefore, $e^\pi > \pi^e$

Researcher: What was your thinking process about this problem?

John: After thinking about the problem for a while, I thought about using the series:
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Researcher: So why did you choose this series? Remembering that our problem considered the inequality $\pi^e > e^\pi$, how did you think this was going to help us to solve our problem?

John: I did not know of a real number expansion for π to some power, but I thought we could use this series for e^x to write e^π as an infinite series

Researcher: What were your main challenges working through this?

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John: The main challenge was what to pick for a ($a = \frac{\pi}{e} - 1$). Initially, I was plugging in π for x. In the series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, I was trying to get something to work with $e^\pi = \sum_{k=0}^{\infty} \frac{\pi^k}{k!} = 1 + \pi + \pi^2 + \ldots$ I was playing around with the idea that you could write $\pi^e = \pi^{2+(e-2)} = \pi^2.\pi^{e-2}$. I was playing with this because π^2 shows up in this expression as $\pi^2.\pi^{e-2}$, and somewhere I was getting $\frac{\pi^2}{6}$ to show up, which you can represent with another series as well. So, I was trying to see if I could compare the two series to make something happen, but it never worked out.

Researcher: So, where did this lead you?

John: This work stopped for a while, which is why I started with the first proof (approach 1) and then came back to this one. Because taking this number a = $\frac{\pi}{e} - 1$ made the proof work. I arrived at that by looking at the first proof. There is no clear connection, but if you notice $\frac{\pi}{e} - 1$, you see that it looks similar to $1 - \frac{e}{x}$, for x to be π (see approach 1). For this proof, picking a ($\frac{\pi}{e} - 1$) was the hard part, and at the time I might not have seen that connection without first having had my experience with approach 1. It came by accident, because I was looking at $f'(x) = 1 - \frac{e}{x}$, which is similar to the choice of a = $\frac{\pi}{e} - 1$, which is used for this approach. That is basically how to approach 1 informed approach 2. So, as I stated before, I started with approach 2, then hit a wall, went to approach 1, and then learned something that I applied to approach 2.

Conclusion

Knowledge gained through problem solving is meaningful when applied to solving other problems that elicit deeper thinking and connections. In certain instances, problem-solving in such a situation may help discover gaps in our mathematical knowledge: in this case, operations of logarithmic and exponential functions.

Graphical representation was an important element in Jane's ability to solve the problem. But it should be noted that producing a graph using a calculator does not necessarily provide mathematical insight.

It was obvious to Jane that $\ln e^\pi = \pi$ but it was more challenging for her to notice that $e^{e \ln \pi} = e^{\ln \pi^e} = \pi^e$. The way functions, inverse functions, and compositions of functions are taught needs to be looked at more carefully: in the case of Jane, the concept of compositions of functions was not well unpacked. Problems of this kind elicit prior knowledge of the student. In the case of John, with his application of a formula for an infinite series to help solve the original inequality, is an important issue to discuss with students regarding applying classroom learning to solving real-world problems.

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