PROSPECTIVE TEACHERS' UNDERSTANDINGS: FUNCTION AND COMPOSITE FUNCTION

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Abstract

The current education reform efforts have placed greater emphasis on conceptual understanding and have focused attention on teacher preparation especially on the adequacy of teachers' mathematical knowledge of the material they will be engaged in teaching. This paper discussed the responses of 29 prospective elementary and special education mathematics specialists to questions focused on conceptualization of the function concept as well as facility with composite functions. The results point to the conclusion that many of the prospective teachers held historical definitions tied to formulaic rules and this negatively affected their ability to solve composite function problems.

Over the past decade, many researchers have turned their attention from studying students' understanding of elementary school content knowledge to examining the content and pedagogical understandings held by inservice and prospective teachers. In particular, this shift resulted from the growing disfavor over quantifying the mathematical content knowledge of teachers by the number of courses taken or scores attained on standardized tests (Ball, 1991). Researchers felt that such information does not reflect the teachers' understandings of the material that they will be teaching nor how they will transmit those understandings.

This study investigated the content knowledge of prospective teachers by examining their responses to a cognitively-guided instrument designed to answer the following questions: (1) What are the definitions of functions that these prospective teachers would be willing to accept and what definition would they consider the best? and (2) How do prospective teachers interpret the composition of functions and how does this reflect on the understanding of the function concept? These two questions arose from analyzing the previous research both on prospective teachers' understandings of the function concept (Even, 1993; Ebert, 1993; Wilson, 1993) and typical student misconceptions associated with the function concept. The next section discusses the various conceptions of the function definition held by mathematics students and teachers.

Theoretical Framework

The function concept is one of the central concepts underlying mathematics (Ferrini-Mundy & Graham, 1991). Even though it is fundamental to mathematics, many students hold primitive understandings and firmly rooted misconceptions (Davis, 1984, Tall & Vinner, 1981). For example, students, typically, understand a function to be a formula (Breidenbach, Dubinsky, Hawks & Nichols, 1992) connecting function with actions of substitution. This formulaic view of the function concept has ties to historical perspectives. The conceptualization of the function concept moved from a curve described by a motion during the 17th century to being conceived as a formulaic rule composed of variables in the 18th century. In particular, this conception involved analytical expressions composed of variables and constants that represent a relation between the variables restricted by the distinction that the graph must not contain any "sharp" corners. This definition focused on dynamic-dependencies between the variables and appears to contain linkages to the function remaining similar over its domain and the connection of a function to a singularity of rule. Even (1993) provided the following present-day definition of function (consistent with the formal Dirichlet-Bourbaki definition): f is a function from one set to another, say **A** to **B**, if f is defined as a subset of the Cartesian product of **A** (the domain) and **B** (the range or codomain), such that for every a **A** there is exactly one b **B** such that (a,b) f. This definition does not rely on the linkages associated with the 18th century definition rather it appeals to static, set-theoretic notions of arbitrariness and univalence. The arbitrariness aspect of functions, as conceived in the modern definition, refers to the relationship between the two sets as well as the nature of the sets themselves. The modern definition of function expanded the definition to include many relationships not previously considered functions i.e., functions defined on split domains, discontinuous functions, functions with a finite number of exceptional points, functions defined by a graph, functions composed of arbitrary correspondences, functions defined by more than one rule, and functions not unilaterally accepted by mathematicians (Leinhardt et al., 1990; Malik, 1980; Vinner & Dreyfus, 1989). These exemplify the notion that the relation between the two sets does not have to exhibit any regularity to be considered a function. The other arbitrariness characteristic dealt with the arbitrary nature of the sets themselves. The sets no longer needed be composed of numbers. For example, the sets could be composed of a group of symbols, mathematical operations, or functions.

The main requirement of the modern definition of the function concept is univalence. The univalence requirement requires that for each element in the set **A**, called the domain of the function, there is associated only one element of **B**, called the range of the function. This requirement was found to be necessary during the development of differential calculus. Without the univalence requirement, higher-order differentials were difficult, if not impossible, to distinguish based upon a distinction of the independent and dependent variable involved. Therefore, the requirement of univalence allowed mathematicians to overcome the difficulties of multi-valued symbols and kept analysis manageable. Even though univalence is useful in the management of analysis, it does not appear to be a well understood or situated concept for both students and prospective teachers (Even, 1993).

Researchers have pointed to the position that many students and some prospective teachers do not hold a modern conception of functions (Dreyfus & Eisenberg, 1983, 1987; Even, 1993; Ferrini-Mundy & Graham, 1991; Markovits, Eylon, & Bruckheimer, 1983, 1986; Marnyanskii, 1975; Vinner, 1983; Vinner & Dreyfus, 1989). In a study conducted by Vinner and Dreyfus (1989), students were asked to define the term "function" and the responses generated were categorized into six classifications:

1. The Formal Dirichlet-Bourbaki definition;
2. A dependence relation between two variables (y depends on x);
3. A rule which requires a certain amount of regularity;
4. An operation or process;
5. A formula, algebraic expression, or equation; or
6. A representation typically in a meaningless graphical or symbolic form.

Students generally appear to hold to the requirement that the function be reasonable and describable by a formula (Graham & Ferrini-Mundy, 1990). One possible reason proffered for this has been that the general experience students and prospective teachers have with functions are almost exclusively built around functions whose rule of correspondence is given by a formula (Even, 1993; Vinner & Dreyfus, 1989). The result, however, has been that students tend to exclude many of the functions that are acceptable under the modern definition but were not acceptable under historical definitions such as:

> Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case the second variable is said to be a function of the first;
>
> or
>
> Any mathematical expression containing a variable x, that has a definite value when a number is substituted for x, is a function of x (Hight, 1968).

Leinhardt et al. (1990) and Vinner & Dreyfus (1989) pointed out that many of the errors produced during student classification of relations as functions or not functions do not occur as a result of the lack of acceptance of the Dirichlet-Bourbaki definition. They found students accept the Dirichlet-Bourbaki definition of the function but when involved in classification tasks, students rely upon personal experience that is closely connected to historical definitions of function.

Therefore, these errors may not be a consequence of a misconception but rather a "missed" concept. As a result, it is the purpose of this study to examine if prospective teachers hold a "historical" definition of function an what implications such a working definition would have on their understanding and facility with composite functions.

Methodology

The participants of this study were 29 prospective teachers completing their mathematical content experiences for either an Elementary Education degree (K-8 certification) or a Child and Family Development degree (Pre-kindergarten certification preparing students to work with public or private preschool, day care, or Head Start programs). The course in which these participants were enrolled during the Spring of 1997 was an Advanced Mathematics for Elementary Teachers taught at a regional state university. All of these students were required to take this capstone course after completing classes in number systems, geometry, precalculus, and statistics although a few were concurrently taking the fifth prerequisite course of calculus. The purpose of the specialization was to provide prospective teachers with additional training in a particular area beyond the typical two mathematical content courses thereby preparing these students to take be mathematical specialists in an elementary schools.

During the Spring of 1997, an assessment (selected items are shown in appendix A) was administered during the middle of the semester. This pretest anticipated discussions of mathematical patterning and its linkage with functional understandings thereby providing the instructor with a broad-scoping view of the students' understandings and beliefs prior to instruction. The instrument was both piloted and externally reviewed by an expert panel of mathematicians, mathematics educators, and psychometricians who evaluated the test with respect to content, wording, and reasonableness of the questions. The complete instrument consisted of 32 items of which 7 were open-ended requiring the participant to explain his or her understanding and 25 were either computational, short-answer, or multiple-choice items. Students were informed to attempt each item but leave it blank if they were unable to answer a question. Such an instruction "ensured" that the results, especially regarding the objective items, were not tainted by mere guessing.

Analysis of the participants' responses focused upon qualitative aspects of the replies consistent with those set forth by Silver and Cai (1993). In particular, analysis centered on identifying the various response types provided to each of the items and quantifying the number of respondents displaying similar response characteristics in reference to the answer, the explanation type, the usage of graphical representations, and other salient characteristics peculiar to an item. In order to accomplish the quantification, the data collected were double-coded by raters and examined for consistency between codings. For any responses evidencing a discrepancy between the two codings, the response was reviewed and a consensus was reached concerning the final coding of the response.

Results and Discussion

This study sought to answer the following questions: (1) What are the definitions of functions that these prospective teachers would be willing to accept and what definition of function would they consider the best? and (2) How do prospective teachers interpret the composition of functions and how does this reflect on the understanding of the function concept?

Definitions of the function concept

In order to answer the first question, three particular items (**T1**, **M2**, and **O3**) focused on the prospective teachers' acceptance of various definitions identified by Vinner and Dreyfus (1989) in their research study. Task **T1** presented six definitions of the function: (A) A function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set; (B) A function is a dependence relation between two variables (y depends on x); (C) A function is a rule which connects the value of x with the value of y; (D) A function is a computational process which produces some value of one variable (y) from any given value of another variable (x); (E) A function is a formula, algebraic expression, or equation which expresses a certain relation between factors; and (F) A function is a collection of numbers in a certain order which can be expressed in a graph. The participants were then asked to mark the item as a true or false definition of function.

Each statement was accepted by at least 75% of the respondents with most of the respondents (26 of the 29 respondents or about 89%) accepting definition (B). Examination at the definition level revealed that of the 29 respondents, 23 accepted definition (A), 21 accepted definition (C), 21 accepted definition (D), 22 accepted definition (E), and 21 accepted definition (F). In addition to these results, a variety of patterns emerged from the data (see Table 1) and one of particular interest was that 7 respondents accepted all six definitions, 10 respondents selected five of the six, 7 respondents chose four of the six; 4 respondents accepted three of the six, and 1 respondent selected only two of the definitions.

	Participant Number																												
Def	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
A		**B**	X	**B**	**B**		**B**		**B**	X	X	**B**		**B**	X		**B**	X	**B**	X	**B**	**B**	**B**	**B**		**B**	X	X	**B**
B		X	X		X	X	X	**B**	X	X	X	X	X	X	**B**	X	X	X	X	X	X	X	X	X	X	X	X		X
C		X		X	X	X	X	X	X		**B**	X	X		X		X		X	X		X	X	X	X	X	X		X
D	**B**		**B**	X	X	X	X	X	X	X	X	X	**B**	X		X	X	X	X	X				X	X				X
E	X		X		X	**B**		X	X	**B**	X	X	X	X	X	**B**		**B**	X	**B**		X		X	**B**	X		**B**	X
F	X	X			X		X	X	X	X	X	X	X	X	X	X	X	X	X	X		X		X	X		**B**	X	

X -- denotes the definition was accepted as true

B — denotes the definition was accepted as true and selected as the best definition

Table 1: Data of responses to T1 and M2

In item **M2**, the prospective teachers were asked to select the definition that best describes the mathematical concept of function (denoted with a **B** in table 1). Given the number of participants accepting definition (B), it might seem that it would be the definition of choice; however, analysis revealed a divergence between the level of acceptance and the choice of a "best" definition. As seen in table 1, 15 of the 29 respondents selected definition (A), only 2 selected definition (B), 1 selected definition (C), 3 selected definition (D), 7 selected definition (E), and 1 selected definition (F). Interestingly, of the 23 participants accepting definition (A), only 15 respondents considered it the best definition. However, this selection of the modern definition of function, definition (A), by just over 50% of the prospective teachers is consistent with the results detected by Even (1993) in her study examining prospective secondary school teachers.

This pattern did not hold true when the prospective teachers were presented with item **O3** which asked for participants to "Define the mathematical concept: function". Even though some of these prospective teachers accepted the modern definition of the function concept and quite possibly chose it to be the "best" definition, many of them did not or chose not to provide it for this question. In particular, only 21 of the 29 participants supplied a response to this item and of those, only one respondent provided a definition consistent with definition (A). Three other participants provided statements that combined aspects of definition (A) and definition (E). In fact, definition (E) or some related definition was provided most often. Besides the three participants combining definitions (A) and (E), an additional six provided responses consistent with definition (E) or statements comprised of definition (E) and added constraints such as graphability or dependency. Of the remaining 11 respondents, 5 made either direct or indirect illusions to functions being "one-to-one", 2 participants provided statements consistent with definition (B) and the other 4 responses were unclassifiable. Interestingly, when examining the responses which indicated that a function was "one-to-one" corroborating evidence pointed to the conclusion that many of these respondents considered the phraseology "one-to-one" to mean that "*one* x is mapped *to one* y". In other words, the correspondence between x and its image y is immutable in the sense that the rule connecting these two entities does not change from one application to the next.

There are at least three possible reasons for the identified inconsistency associated with the prospective teachers' responses. First, as Even (1993) pointed out, prospective teachers may consider the modern definition as a viable definition, however they chose to explain the concept of function in alternative terms since conceptualization of the modern definition is difficult. Another possible explanation is that these prospective teachers felt that additional constraints needed to be attached to the definition which in turn obscured the essential components. A third possible interpretation could be that these prospective teachers, although recognizing the validity of the modern definition, were either unable to articulate the modern definition, chose to not respond, or responded with a definition which was an attempt to classify their understandings in a phrase or sentence composed of borrowed or personal terminology when asked to provide their own definition. In any case, this study revealed that most of these prospective teachers, although choosing mathematics as their area of specialization, either did not provide the modern definition of the function concept or were generally inconsistent in their selection of the definition.

Since many of the participants linked the function concept with definition (E), the results associated with item **T4** which stated "Every function can be expressed by a certain computational formula (e.g., y=2x+1 or y=3sin (π+ x))" should not be surprising. Of the 28 respondents to this item, 16 indicated agreement. Breidenbach et al. (1992) attribute this phenomenon to students and pre-service teachers dependence on an action conceptualization of function which focuses on the act of substituting numbers for variables and calculating to obtain a result. This attachment to formulas coupled with the already identified desired immutability of the correspondence brings to question the nature of the formula. According to Leinhardt et al. (1990), students feel that closed rules comprised of arbitrary associations, such as a random number function, are not functions. Two particular items, **T5** and **M6**, focused on this issue and the participants' responses indicated that many of these prospective teachers were unaccepting of arbitrarily associated function. In response to item **T5**, which asked the participants to respond to the statement "Every function expresses a certain regularity (the values of x and y can not be matched in a completely arbitrary manner", 16 of the 27 respondents felt the statement was correct. In a more dramatic fashion, the desire to not have the function defined entirely arbitrarily was seen in the prospective teachers' responses to item **M6** which presented three propositions describing functions. One of these, proposition III, stated that "For every value of x we choose the corresponding value of y in an arbitrary way (e.g., by throwing dice)". Of the 28 respondents to this item, 24 selected choices which did not include proposition III thereby making it evident that most of these prospective teachers were not accepting of a function being defined by a random correspondence scheme. Thus, about half of the prospective teachers (14 of the 29 participants) held formulaic conceptions of function while being unaccepting of randomly-defined functions

The question that now arises is one of how does the inconsistent or lack of selection of modern definition of the function concept coupled with a formulaic view impact the prospective teachers' facility with composite functions. This next section delineates both how the prospective teachers organized the function concept when dealing with composite functions and how they handled the underlying symbolism used to represent the composite functions.

Composite Functions

One major topic which has the concept of function as an integral component is that of composite functions. The implications of how these prospective teachers situate the function concept should be evident when they work with composite functions. Items **O7**, **O8**, **O9**, and **M10** looked at a variety of aspects associated with composition, i.e., finding $(F \circ G)(x)$ where $F(x)$ and $G(x)$ include variables, finding $F(x)$ given $(F \circ G)(x)$ and $G(x)$, finding $(F \circ G)(x)$ where $F(x)$ and $G(x)$ do not include variables, and examining the zeros of a composition respectively.

The responses to item **O7** which presented two functions, i.e., $F(x) = 1 + \frac{4}{x-1}$ and $G(x) = 2 + \frac{5}{x-1}$, and asked participants to describe the composition of the two functions revealed that many of the prospective teachers had difficulty establishing the composition. In particular, eleven of the prospective teachers did not respond to the item leaving a total of 18 respondents to this item. Of those, only 7 were able to correctly establish the composition although some of the responses contained errors as part of the simplification process from

$$(F\circ G)(x) = 1 + \frac{4}{2 + \frac{5}{x-1} - 1} \text{ to } (F\circ G)(x) = \frac{-20}{x+4} + 5.$$

One prospective teacher established $(G\circ F)(x)$ rather than $(F\circ G)(x)$, a couple others concluded that "the composition will always be positive" without providing a formulated composition, and several found $(F\circ G)(2)$. The rest of the prospective teachers responding to this item (5 of the 18) generally considered the composition symbol to connote multiplication and as a result computed $F(2)$ and $G(2)$ to find their product.

When the prospective teachers were presented with item **O8** which indicated that $H(x) = (F\circ G)(x) = \cos^2 x$, $G(x) = \sin x$, and expected participants to deduce the nature of the function $F(x)$, only 2 of the 22 responding prospective teachers were able to identify that $F(x) = x^2 - 1$. A couple of prospective teachers came close to establishing the correct nature of $F(x)$; however they concluded that "$F(x) = \sin x + 1$ since $\sin^2 x + 1 = \cos^2 x$" or "$\cos^2 x = F(\sin x)$ so $F(x) = x^2 + 1$". Most of the other respondents considered the composition symbol, $\circ$, to stand for multiplication. In particular, 14 respondents attempted to make sense of the composition in a manner similar to the one shown in figure 1.

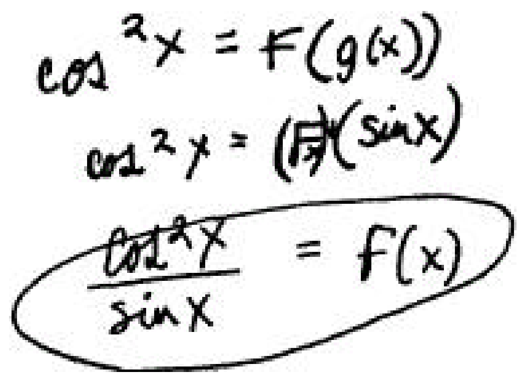

Figure 1: Example of improper usage of functional notation

This response was indicative of how some of the prospective teachers would consider the amalgam of symbols $F(G(x))$ to be $F \bullet (G(x))$ rather than the composite of two functions $F(x)$ and G(x).

These two items (**O7** and **O8**) both had the characteristic that the underlying functions had specific variables into which students could mechanically manipulate to establish the composition; however, item **O9** did not have this property. Prior to deliberating the case of composition, a discussion of several items which focused on constant functions in general will provide needed illumination. In particular, it was evident from two items on the assessment, **T11** and **T12**, that many of the prospective teachers did not generally consider constant functions as a function. When participants were presented with the statement "There exists a function all of whose values are equal to each other" in item **T11**, 15 of the 27 respondents disagreed with the statement. In a similar fashion, when participants responded to the statement "As the value of the independent variable of a function changes, the value of the dependent variable also changes" in item **T12**, 20 of the 28 respondents agreed with the assertion. As a result, these response indicated that many of the prospective teachers believed that variation in the independent variable must correspond with variation in the dependent variable. Since a constant function can be described as a many-to-one correspondence, information from item **M13**, "A function can be: (I) one-to-one, (II) one-to-many, or (III) many-to-one" provided additional information. It was interesting to find that in response to this item only 5 of the 28 respondents selected a statement which indicated acceptance of a function having a felt that a many-to-one correspondence.

This lack of acceptance of a constant function as a function by students has been identified in prior research studies (Barnes, 1988; Markovits et al., 1986; Marnyanskii, 1975). Several possible origins for this error have been proposed: (a) the stress of covariation in the 18th century definition, (b) the use of contrasting meanings of constant and variable and associating function with variable, or (c) the resistance to accept many-to-one correspondences (Markovits et al., 1986; Marnyanskii, 1975; Vinner, 1983; Vinner & Dreyfus, 1989). An additional characteristic imposed on the function concept by some students is the causality factor. From experience, students imply the connection of the dependency characteristic with a causal connection (Marnyanskii, 1975).

Since many of these prospective teachers misclassified constant functions, then it should be no surprise to find that many of them had difficulty with item **O9** that contained only constant functions. This item presented the two functions to be composed, i.e., $f(x)=4$ and $g(x)=2$, and then prompted the participants to determine the value of $(f \circ g)(x)$ at $x=7$. Only seven of the 27 respondents were able to correctly establish that $(f \circ g)(7) = 4$. Four other respondents made comments similar to the following "*f*(*g*(7)), I can't do it because there is no formula for me to plug 7 in". The rest of the respondents (16 of the 27) interpreted the composition symbol to be multiplication and made computations similar to that shown in figure 2.

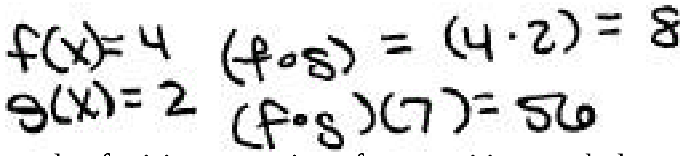

Figure 2: Example of misinterpretation of composition symbol as multiplication

The product of 56 was obtained most often although some respondents arrived at unique answers such as 2744, 14, 18, 28, and 392. Each of these came with its own set of computational techniques, for instance 2744 was obtained in this manner: $(f \circ g)(7) = f(7)\ \ g(7) = (4\ \ 7)\ \ (2\ \ 7) = 2744$.

A possible explanation for the majority of prospective teachers being unable to solve many of these composition problems may reside with the prospective teachers not completely recognizing or understanding the composition notation. Eisenberg (1991) chronicled some of the notational difficulties encountered in reference to functional symbolism. He noted that the "*f*(*x*) notation is confusing because *f*(*x*) is usually read to mean *x* under the function *f* goes to . . ." (Eisenberg, 1991, p. 149). In addition, the symbolism *f*(*x*) is a complex of symbols which students are not used to thinking of as a variable. The complex, *f*(*x*), is not considered by many students to be a variable because a variable to them is a single symbol, such as the y in $y = 3x + 4$, and *f*(*x*) is composed of four symbols (Dunham & Osborne, 1991). In particular, when presented with items **T14** and **T15**, 26 of the 28 respondents indicated that more than single symbols could be variables but of those 26, 6 indicated in item **T15** that *f*(*x*) was not a variable. As a result, the prospective teachers may have been confused by the larger set of symbols such as $(F \circ G)(x)$ or $F(G(x))$ and considered the composition symbol, $\circ$, to correspond, as was seen in many of the responses, to a symbol for multiplication. As a result, the complexes of $(F \circ G)(x)$ or $F(G(x))$ might have been considered to be a "weird" mathematical symbolism where the components correspond to variables. The responses indicated a belief that the symbols *F*, *G*, and *x* were autonomous variables and that by following order of operations a solution could be achieved.

An additional item, **M10**, was included to determine if the prospective teachers could establish the zeros of a composite function. In this item, participants were presented with the information that the function, *f*(*x*), had two zeros when $x=1$ and $x=4$ and from this they needed to determine what values of z would make $f(4z)=0$. In order to arrive at a correct response, the prospective teachers' must to identify and solve two equations, $4z=1$ and $4z=4$, since the values of 1 and 4 made $f(x)=0$. Only 9 of the 28 respondents correctly established that $z=1/4$ and $z=1$ would make $f(4z)=0$. Since the solution of these equations are quite accessible, it seems evident that many of the prospective teachers could not analyze the impact of the composition. In particular, 8 respondents answered with "unable to determine" and 7 others selected "z=0 only". Two additional prospective teachers selected the response "z=4 and z=16 only" and another two chose "z=1 and z=4 only". Evidently, many of the prospective teachers were unable to examine a composite function to establish the zeros of the composite function given the zeros of the "outer function". Further studies will need to be conducted to examine if the difficulties evidenced with composition reside with misunderstanding of the function concept, with the notation surrounding the composition description, or with some combination of these.

Conclusion

The results of this study point to the conclusion that many of these prospective teachers were inconsistent when dealing with the modern definition of function. In particular, this study identified that many of the prospective teachers held a rule-based interpretation of the function concept and corresponding with that interpretation was a strong tendency of these prospective teachers to negate constant functions as being functions. This errant connection revealed an inherent weakness to this conceptualization especially when faced with composite functions containing constant functions.

In order to gain a true perspective on these results, one should consider the training of these prospective teachers and the other factors influencing their development. The National Council of teachers of Mathematics seeks that all teachers "have not only a thorough understanding of the mathematics they are teaching but also a vision of where that mathematics is leading" (NCTM, 1991, p. 135). Additionally, an excerpt from the "Common Experiences in the Mathematical Education of Teachers" section of the NCTM's Professional Standards for Teaching Mathematics, shown below, reveals the foundational knowledge of functions expected for K-4 teachers:

> Teachers need to experience the development of mathematical language and symbolism and how these have influenced the way we communicate mathematical ideas. Also, experience in representing and solving problems requiring the use of variables is important. To build bridges for their students to the mathematics that comes later in the school curriculum, teachers must have *a basic understanding of the concepts of functions and their use* (italics mine) in the growth of mathematical ideas. Understanding different representations of functions (tabular, graphical, symbolic, verbal), how to move among these representations, and the strengths and limitations of each is fundamental. The distinction between continuous and discrete approaches in the solution of mathematical problems should also be a part of the experiences provided for these teachers and should be introduced initially at an intuitive and informal level (NCTM, 1991, pp. 136-7).

The selection calls for initial treatments to be informal and intuitive but also contains a noticeable lack of appeal to the Dirichlet-Bourbaki definition of function. At the 5-8 level, the only mention of functions arose in the section entitled "Concepts of calculus" where the following statement is made: "Functions, graphs, and the notion of limits should be explored, starting with concrete problems such as maximizing the volume of a box that can be folded from a rectangular sheet of grid paper. The rate of change of the volume of the box as a function of the height of the box can be investigated in a way that introduces the concepts of differentiation and integration in an appropriate manner for teachers of grade 5 and above" (NCTM, 1991, p. 138). In neither description was a mention of a formal treatment of the nature of the function mentioned. Treatments were to begin informally and intuitively and then proceed to exploring concrete situations. Tying such explorations to concrete situations further enforces the view of functions as a rule based entity.

The informal and intuitive presentation giving way to concrete explorations is mirrored in the NCTM's Curriculum and Evaluation Standards for School Mathematics which describes the content of the K-8 curriculum. Activities such as "guess my rule" were suggested to introduce the function concept and the presentation was to be "informal and relatively unburdened by symbolism" (NCTM, 1989, p. 98). In doing this, students should come to an "understanding that functions are composed of variables that have a dynamic relationship: Changes in one variable result in change in another" (NCTM, 1989, p. 98). In particular, the primary arena for the study of functions arises in the study of patterns in both informal settings such as in architecture, plowed fields, and physical patterns, and more formal mathematical settings and this study "should focus on the analysis, representation, and generalization of functional relationships" (NCTM, 1989, p. 98). These types of presentations further corroborate the perception that the function concept is rule-based, non-arbitrary, and needs to contain both independent and dependent variables.

As a result, historical definitions of functions appear to be the primary characterization required from and presented to prospective K-8 teachers. However, the noticeable lack of treatment of the formal mathematical definition of the function concept in the K-8 mathematical preparation of teachers brings to rise two questions. First, would the presentation of the Dirichlet-Bourbaki definition at the K-8 level be a hindrance or a benefit to K-8 student understanding of the function concept? In other words, is the definition so complex that students may not be sufficiently prepared to explore the subtle nuances of the definition or would early introduction, although not completely understood, set the stage for understanding rather than permitting or reinforcing a working definition which is considered incomplete by mathematicians at this time? Secondly, is it sufficient to merely push the K-8 teacher's understanding of functions to a level just above that which is expected of the most advanced K-8 student? Finding answers to these questions may not be feasible from this study; however, if prospective teachers are allowed to enter classrooms with such incomplete notions then those could be passed on to students.

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Appendix A

TASK **T1**

Please mark the following six definitions of function as being true or false

A. T F A function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set.

B. T F A function is a dependence relation between two variables (y depends on x).

C. T F A function is a rule which connects the value of x with the value of y.

D. T F A function is a computational process which produces some value of one variable (y) from any given value of another variable (x).

E. T F A function is a formula, algebraic expression, or equation which expresses a certain relation between factors.

F. T F A function is a collection of numbers in a certain order which can be expressed in a graph.

TASK **M2**
Which of the above definitions best describes the mathematical concept of function as you understand it?

A B C D E F

TASK **O3**
Define the mathematical concept: function.

TASK **T4**

T F Every function can be expressed by a certain computational formula (e.g. y = 2x+1 or y=3sin(π+x)).

TASK **T5**

T F Every function expresses a certain regularity (the values of x and y can not be matched in a completely arbitrary manner).

TASK **M6**
Which of the following propositions describe functions? (x and y are natural numbers)

I. If x is an even number then y = 2x+5; otherwise y = 1-3x.

II. If x = 0 then y = 3;
If x > 0 then to find the corresponding value of y we add 2 to the value of y corresponding to x-1.

III. For every value of x we chose the corresponding value of y in an arbitrary way (e.g. by throwing dice).

I	II	III	I & II	I & III	II & III	All	None
A	B	C	D	E	F	G	H

TASK **O7**
If $F(x) = 1 + \frac{4}{x-1}$ and $G(x) = 2 + \frac{5}{x-1}$ where the domain of $F(x)$ and $G(x)$ is $x \geq 2$, then describe the composition $F \circ G$.

TASK **O8**
If $H(x) = (F \circ G)(x)$ where $H(x) = \cos^2 x$ and $G(x) = \sin x$, then find $F(x)$.

Show your work or explain your answer.
F(x) = ___________________________________

TASK **O9**
Let f(x) = 4 and g(x) = 2. What is the value of $(f \circ g)(x)$ when x = 7?

$(f \circ g)(7)$ = ________________

Explain how you found your answer.

TASK **M10**

If $f(x) = 0$ only when $x = 1$ and $x = 4$, then for what values of z does $f(4z) = 0$?

- A $z = 4$ and $z = 16$ only
- B $z = 1$ and $z = 4$ only
- C $z = {}^1/_4$ and $z = 1$ only
- D $z = 0$ only
- E Unable to determine

TASK **T11**

T F There exists a function all of whose values are equal to each other.

TASK **T12**

T F As the value of the independent variable of a function changes, the value of the dependent variable also changes.

TASK **M13**

A function can be:

I	one-to-one
II	one-to-many
III	many-to-one

I	II	III	I & II	I & III	II & III	All	None
A	B	C	D	E	F	G	H

TASK **T14**

T F Only single symbols, such as y, can be a variable.

TASK **T15**

T F The set of symbols, $f(x)$, is a variable.