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Unpacking the Algorithm: Rethinking Elementary Pre-Service Teachers' Strategies for Solving Multi-digit Addition Problems

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Abstract

This paper details an exploratory study of 14 elementary prospective teachers (PTs) solving purposefully crafted, two-digit addition problems. The numbers in each problem were chosen to elicit diverse solution strategies. We coded 95 responses based on the ways the PTs completed the calculation (for example, by referring to the number as individual digits and/or by using strategies that attended to the value of the number). The findings suggest that although an overwhelming number of strategies explicitly referred to the numbers as distinct digits while adding, there was some evidence that PTs utilized more diverse strategies or sub-strategies (e.g., counting up or down strategies, derived facts of doubles or base ten knowledge). Our study seeks to unpack for mathematics teacher educators the typical narrative of PTs' reliance on algorithmic strategies while highlighting their use of more nuanced strategies as well.

Keywords: elementary school, mathematical reasoning and problem solving, number sense, preservice teacher education

Background

A major goal of mathematics teacher educators (MTEs) is to provide new teachers with numerous experiences that challenge, revise, or reframe what they know about teaching and learning mathematics in ways that will benefit all children in their practice (Ball, 1988). Those learning to teach must negotiate what they already know about mathematics with what is known about how children learn mathematics, as well as the most effective strategies for fostering that learning. One concept on which prospective teachers (PTs) often believe they have a firm grasp is whole-number computation using the basic operations and procedures for adding, subtracting, multiplying and dividing numbers. Most PTs in elementary education programs have had years of experience with algorithms for the operations. But because one purpose of algorithms in mathematics is to simplify, generalize, and condense complex conceptual material into an easy-to-follow recipe, PTs still need opportunities to reflect on the conceptual underpinnings of these algorithms. When PTs learn to unpack the algorithms they know and use, they can also learn to how to help their students employ a diverse set of solution strategies for arithmetic computation, both algorithmic and not, which is crucial work to developing students' number sense (Welder & Simonsen, 2011). The existing research on pedagogical strategies such as number talks supports the notion that teachers need to learn how to help children discuss and connect multiple strategies for basic computations (Humphreys & Parker, 2015). Well-developed number sense has been shown to be a major indicator in students' future mathematical success, especially with more advanced concepts like algebra (Boaler, 2016, Hiebert et al., 1997). In order to help their students construct a diverse set of strategies for computations, teachers must attend to the numbers they choose in arithmetic problems that they pose to their students (Land, Drake, Sweeney, Franke, & Johnson, 2015).

Posing multi-digit computation problems is a process that requires a great deal of thoughtful attention to the appropriateness of the numbers selected, the ways in which students might solve the problem, and how the problem might leverage students' conceptual knowledge (Humphreys & Parker, 2015; Land et al., 2015; Kilpatrick et al, 2001; Parrish, 2010). Consider the following problem posed in Land et al., (2015, p. 65): 100 + ____ = 99 + 44. Although the problem can be solved algebraically by adding the numbers on the right side of the equation and then subtracting 100 from that sum, the numbers were purposefully chosen so that students could notice that "100 is one more than 99, so the answer must be one less than 44" without ever having to explicitly add 99 and 44 and then subtract 100 (Land et al., 2015, p. 65). Carefully crafted arithmetic problems are a crucial step in preparing children to mentally compute and manipulate numbers (Kilpatrick et al., 2001). Problems that involve purposefully chosen numbers are more likely to elicit children's number sense and strategic competence, which leads to "further insights into the properties of numbers and operations" (p. 214).

In addition, Parrish (2010) argues that the role of mental computation helps "students to focus on number relationships and use these relationships to develop efficient, flexible strategies with accuracy" (p. 13). For example, Parrish (2010) suggests that children might leverage their existing place value knowledge and experience with solving multiplication problems as "groups of" when mentally computing 12 x 49. In this case, a teacher could expect her students to potentially leverage their knowledge of 12 groups of 50 when solving 12 x 49 and these kinds of computations can help children build a flexible number sense while also affording them opportunities to justify their thinking in multiple ways based on the numbers posed in the problem (Hiebert et al., 1997; Kilpatrick et al., 2001; Parrish, 2010, 2011).

The seminal research on Cognitively Guided Instruction (CGI) (Carpenter, Fennema, & Franke, 1996) argues that young children begin to develop their number sense by modeling actions in a story problem by representing all of the quantities in the problem. Later, children learn and leverage more sophisticated strategies as they build fact fluency and flexibility in how they solve more challenging problems. More specifically, teachers can support young children's number sense by attending to how purposeful number choice can support and/or extend children's mathematical thinking; children can learn more sophisticated strategies that leverage their number sense as they solve increasingly difficult problems. For example, students can count up from one number to another in order to find a missing value (Fuson & Willis, 1988) (e.g., To solve 43 + x = 52, a child can count up from 43 by saying, "43...44, 45, 46, 47, 48, 49, 50, 51, 52. The answer is 9"). Alternately, students can use derived facts or their knowledge of tens and/or doubles to strategically solve problems (Steinberg, 1985). For example, if a student is asked to solve 6+7, they may utilize a derived fact of doubles to say “I know that 6+7 is 13 because 6+6 is 12 so one more is 13.”

Procedural fluency in arithmetic depends on leveraging a diverse set of strategies for solving arithmetic computation problems. Therefore, it is incumbent on teachers to understand how to elicit these diverse strategies from students. By the same token, MTEs must also understand how PTs approach such problems in order to better structure their instruction on the importance of number choice in the elementary grades.

PTs may be more susceptible to default to using algorithmic strategies when performing basic operations than young children are, due largely to the years of experience they have with those algorithms. PTs’ knowledge and use of algorithmic strategies prompted this study. In one classroom episode in an elementary mathematics methods class, the first author asked her PTs to find three or more different ways to mentally solve the problem 70-26, written horizontally. The PTs were asked to record their thinking on paper after they had devised a solution. After ten minutes, many of the PTs came to the correct conclusion of 44, but in vastly different ways. Some PTs claimed that they simply transposed the numbers so that the problem was written vertically and applied the standard U.S. algorithm for subtraction that is commonly taught in the United States (see Figure 1 below).

$$\begin{array}{r} {}^{6}\cancel{7}\,{}^{1}0 \\ -\ 26 \\ \hline 44 \end{array}$$

Figure 1
Example of a Standard U.S. Algorithm.

Others left the problem as it was written and relied less on algorithmic strategies to solve the problem. For example, one student said that 70-26 was the same as 74-30, which would result in the same difference. Other PTs compensated the subtrahend (26) to become 30, subtracted this from 70, and then added the 4 back from the compensation.

Later in the class, the first author asked her PTs a second, similar question: to mentally solve 72-35, this time written vertically. Nearly all the PTs solved the problem using the standard U.S. algorithm, and few PTs leveraged the same strategies as they had when the problem was written horizontally.

Noting the differences in how PTs responded to these the two problems, we were curious as to why this specific problem elicited different solution strategies: was it the numbers selected? Was it the way the problem was written, horizontal versus vertical? Was it the fact that the PTs were asked to mentally solve this and could not refer to a calculator for assistance? This prompted the following guiding questions for this study:

1. What are the strategies that PTs leverage when asked to solve multi-digit addition problems and was there a difference in their strategies when the problems were written vertically versus horizontally (if at all)?
2. What was the role of number choice in how the PTs solved the problems, if any at all?

Our intention with this study is to help inform mathematics teacher educator practices for eliciting PTs mathematical content knowledge about multidigit addition, which could serve as a springboard for discussions about how teachers need to carefully craft mathematical tasks that attend to number choice and children's invented strategies for solving similar problems.

Framing the Study

The recent focus on the importance of number choice has been based in part on the following theoretical assumptions:

1. Writing arithmetic computation problems horizontally instead of vertically prompts students to rely less on standard algorithms (Humphreys & Parker, 2015).
2. Certain combinations of numbers in arithmetic computation problems affect the types of strategies that students use to solve them (Land, et al, 2015).

For this study, we investigated those assumptions with PTs. We chose to focus only on the experiences of PTs in an elementary mathematics content course, still knowing that there are greater implications to what happens to the prospective teachers after they leave their content courses. Moreover, the focus on PTs allowed us to investigate how PTs interact with arithmetic problems with purposefully chosen numbers, which in turn allows us (as MTEs) to prompt discussions about number choice in both content and methods classes and emphasize its importance in the elementary mathematics curriculum. Prospective elementary teachers are college students with ample experience in carrying out arithmetic computations. They have practiced their strategies for the basic operations and reified their thinking into conceptually-dense procedures that they implement every time they are required to do them. Thus, the process of unpacking PTs' procedural strategies for solving arithmetic problems is an important step in helping MTEs structure instruction for PTs.

In addition, this study falls into the tradition of (but is not a direct example of) the framework for teacher learning by Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson (2009). This framework identifies three major aspects of learning to teach: representing practice, decomposing practice, and approximating practice. We consider this study to be a decomposition of practice, or a "naming of parts," as Grossman et al., describe it (p. 2068); however, we have not decomposed practice explicitly for the participating PTs, but instead attempt to name the parts of PTs strategies for solving two-digit addition problems. It is our intention that this decomposition can help MTEs in the future better understand how to help PTs decompose, represent, and approximate the practice of teaching addition.

Thus, we investigated how PTs mentally solve two-digit addition problems with purposefully chosen addends. Without a deep understanding of the diversity of strategies that can be used to solve multi-digit addition problems, PTs may face challenges when facilitating conversations in their classrooms that foster the development of robust number sense in children. And, without a clear understanding of the “lay of the land” of PTs already formed understandings of multi-digit addition, it is difficult for MTEs to structure lessons on the importance of number choice in elementary mathematics.

Methodology

According to Humphreys and Parker (2015), writing arithmetic problems horizontally can “discourage the use of rote procedures” (p. 11). The methodology of this quantitative, exploratory study was structured to investigate this claim, but the data that was collected revealed much more about PTs’ thinking than we anticipated.

Setting and Data Collection. Participants in this study were 14 elementary PTs who were enrolled in a mathematics content course for elementary education majors at a large university in the southern United States during the spring 2015 semester. Students take this course within the first or second year of their coursework and prior to being admitted to the teacher preparation program. The study was carried out within the course, just before students began their exploration of topics related to multi-digit addition and subtraction. To stimulate dialogue and facilitate the collection of video recording data, PTs were paired with each other. Each participant was given a warm-up problem, three horizontally-written problems, and three comparable vertically-written problems on notecards. PTs were then asked to solve each of the problems mentally and to verbally communicate their solution to their partners; PTs videotaped each other’s strategies using iPads. For many problems, which required regrouping, this data collection format seemed to encourage PTs to think aloud. Thus, we captured a great deal of *exploratory talk*, which Cazden (2001) considers speaking “without the answers fully intact,” as opposed to condensed and edited (i.e. intact) versions of their solutions.

The addends of each problem were purposefully chosen so that each addition problem contained specific numerical attributes. The problem types listed in Table 1 are numbered by the order in which each problem type was given to the participants.

Each numbered type of problem shares the same numerical attributes as all the other problems of the same type number. The lettered designations in the type name refer to whether the problem was presented horizontally (H) or vertically (V). We created two problems for each problem type and orientation so that each partner had the opportunity to solve one of each type and orientation of problem. PT pairs followed an alternating scheme for viewing problems and reporting solutions so that each participant had the opportunity to report first for at least half of the problem types, which we hoped would minimize any one participant’s ability to “reflect” her partner’s strategy. In fact, we saw little evidence of the PTs reflecting their partner’s responses, but that may be explained by the fact that participants who were responsible for their partners’ recording solutions were more focused on the demands of operating the technology than listening to their partners' responses.

Table 1

Attributes of problems given to PTs

Problem type & orientation	**Problem pair**	**Order of solution reporting**	**Numerical attributes of problem**
Type 1H	A. 21+39 B. 12+28	Partner 1 Partner 2	● Ones digits sum to ten; problem could elicit the use of a regrouping by multiples of ten strategy. No regrouping necessary in the tens place
Type 1V	A. 18+22 B. 29+31	Partner 2 Partner 1	
Type 2H	A. 47+38 B. 28+19	Partner 1 Partner 2	● The ones place value digits in both numbers is close to ten; problem could elicit the use of a rounding-up strategy. Only the ones place would need regrouping into a higher place value
Type 2V	A. 29+38 B. 48+37	Partner 2 Partner 1	
Type 3H	A. 68+73 B. 57+64	Partner 1 Partner 2	Both the ones place and tens place need to be regrouped into a higher place value; problem could elicit the use of compensation, estimation, etc.
Type 3V	A. 54+67 B. 63+78	Partner 2 Partner 1	

The PTs also completed a written debrief of their strategies after problem types 1H-3H and then again after problem types 1V-3V (Table 2). PTs were asked to describe their "mental pictures" of the problems because in an early test-run of the data collection methodology, we noticed students gesturing in the air as if rewriting a horizontally-written problem vertically. The PTs were asked to record their mental strategy in a written debrief as a means of confirming their strategy.

Table 2

Debriefing questions.

Timing of debrief	**Question**
After problem types 1H-3H (horizontal)	Review the problems you solved. Describe how you mentally solved the problems. What was your mental picture of the problems as you solved them?
After problem types 1V-3V (vertical)	Review the problems you solved. Describe how you mentally solved the problems. What was your mental picture of the problems as you solved them? Did you use a different strategy when solving the second round of problems or was it the same?

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The primary data source that we analyzed were the transcripts of the videos PTs took of each other solving multi-digit addition problems. Data from the ensuing class discussion (which was also video-recorded) and participants' written debriefs were considered secondary, contextualizing sources of data.

Data Coding and Analysis

Types of Strategies. The data we collected yielded a total of 95 participant responses, that were then transcribed and analyzed. A "response" is defined as the complete set of one participant's utterances following her initiation of the problem, which was the moment she picked up an index card. To address the first research question, we began by reviewing the transcripts and categorizing the PTs' responses into three broad categories, as informed by the extensive research about strategies for solving similar problems (Carpenter, et al., 1999; Land et al., 2015): Purposeful redistribution, Adding by place value, and concatenated digits. In the description below, we describe each of the codes and offer an example from the PTs' in our data:

1. **Purposeful Redistribution**: regrouping a number by any place value so that it is easier to add.
   Example response to 21+39: "Well, I got 21 plus 39. What I would do is carry the 1 to this number [points at 39] so it would be 40 so 40 plus 20 is 60."
2. **Adding by Place Value**: adding a number by its place value and regrouping when necessary.
   Example response to 12+28: "So I have 12 plus 28. And that would be 40...8 plus 2 is 10. 20 plus 10 is 30. So 30 plus 10 is 40. And that last 10 is what I got from 8 plus 2. So the answer is 40."
3. **Concatenated Digits**: "Treating the numbers as columns of place-value neutral digits" (Humphreys & Parker, 2015, p. 7).
   Example response to 28+19: "I have 28 plus 19. And I know that 8 plus 9 is 17. Carry the 1. And 2 plus 1 is 3 plus 1 is 47."

These codes described the overarching strategy contained in each response. However, these broad categories did not capture all the thinking evidenced in PTs' responses. Many responses also displayed the use of *sub-strategies* (similar to the ancillary strategies identified by Hope and Sherrill (1987)) within the overarching strategy in order to handle one-digit addition facts. For example, participants may have used an overarching strategy that was coded as "concatenated digits" while they made use of a "doubles" fact for adding the ones digits. We defined a sub-strategy to be a strategy used within an overarching strategy that cannot be described as the *same as* the overarching strategy. Sub-strategies were coded based on previous research on student's strategies for one-digit addition problems (Carpenter et al., 1999; Fuson & Willis, 1988; Land et al., 2015; Steinberg, 1985):

1. **Landmark numbers**: breaking numbers into parts to reach a landmark number (usually ten) and then adding remaining part to the landmark number.

Example response to 57+64: "Fifty-seven plus 64. I know that 7 plus 3 is 10. 7 plus 4 is 11. So... I would carry the one. 6 plus 5 is 11. 11 plus 1 is 12. so the answer would be 121."

2. **Doubling**: adding two numbers that are close together by doubling the smaller one and compensating.

Example response to 29+38: "So, just by looking at these 8 plus 8 which is 16 plus a 1 is 19 - oh no - 17. 8 plus 8 is 16. So plus 1 which makes it a 9 makes ... 17. so 8 plus 9 is 17 and place the 7 underneath the 8 and carry the 1. And say 1 plus 3 is 3, 1 plus 2 is 3 plus 3 is 6. So it would be sixty...67. The total is 67."

3. **Counting up**: Counting up from one addend by the other addend in order to reach the sum.

Example response to 57 + 64: "So I got 57 plus 64. And I took the smaller number on bottom. So 7 plus 4 is 11. And then 6 plus 5 is 11 plus 1 is 12 so we have....whispers to herself: 7, 8, 9, 10,11. 121."

When devising our coding scheme, we also looked for responses that may have included more than one sub-strategy. Two example of this are below:

- *Example response to 57+64*: "57 plus 64. 4 plus 4 is 8. And you have three left over from 7. Hold on. I am confused. 7 plus 4 is... 4 plus... 7 plus 4. 4 plus 4 is 8. you have three left over from the 7 which makes it 9, 10, 11 (counting up to 11). 6 plus 5, 6 plus 6 is 12, you subtract the 1, which makes it 11. And you carry the 1 which makes it somethin'.... 7, 8,9, 10, 11. 111."
- *Example response to 48+37*: "48 plus 37. So 7 plus 7 is 14 plus 1 is 15. So the five and carry the 1. 1 plus 4 is 5; 6,7,8. 85."

Based on the exemplars for our codes, the research team individually coded the entire data set, which included 95 responses in all from 14 PTs. We then reconvened and reviewed how each person coded the data and resolved disagreements. Because the responses were brief in nature, we determined that it was manageable to code and resolve disagreements as a group about each of the coded responses. The following section discusses our findings and their implications.

Findings

Our findings suggested that although an overwhelming majority of the responses (87.6%) used a concatenated digits strategy, PTs also explicitly leveraged a diverse set of more basic sub-strategies such as landmark numbers (e.g., adding numbers so that they more easily sum to 10), doubling (e.g., adding 6+7 using the knowledge of 6+6), and counting up or down (e.g., counting up from one of the addends in order to add two numbers) in order to add one-digit numbers within their chosen overarching strategy.

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Table 3
Distribution of Strategies used by PTs

Overarching Strategy	Number of responses	Sub-strategies	Number of responses
Concatenated Digits	85	Landmark numbers	4
		Doubling	7
		Counting up	12
		Doubling and Counting up	4
		None observed	58
Purposeful Redistribution	6	Landmark numbers	3
		Doubling	2
		None observed	1
Adding by Place Value	5	None observed	5

Table 3 shows how many PTs used a specific overarching strategy/sub-strategy combination. Table 4 shows which problem types elicited the greatest variety of solution strategies.

Our analysis concluded that the majority of PTs used a concatenated digits strategy to solve their addition problems (87.6%). However, 31.8% of those responses involved the use of a distinct sub-strategy. For example, one student gave the following response:

> Forty-seven plus 38. I have to count with my fingers. Seven plus 8 I think is 15. [Silently counts up from 8 to 15 with her fingers] Seven plus 8 is 15. So I rearrange the number this 38 underneath, is 47 plus 38 would be 15, carry the 1. 4 plus 3 is 7 plus 1 is 85. So the answer would I hope 85.

We can see from her language that she treated the numbers 47 and 38 as concatenated digits; not only did she fail to mention the place values of the digits in the numbers, she also explicitly stated that she mentally rearranged the numbers from a the given horizontal arrangement to a vertical one, as one would do in order to carry out the standard U.S. algorithm for multi-digit addition.

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plus 3 is 7 plus 1 is 85. So the answer would I hope 85.

Table 4
Solution Strategies Reorganized by Problem.

Problem type & orientation	Problem pair	Overarching strategies		Sub-strategies	
Type 1H	A. 21+39 B. 12+28	Purposeful Redistribution	2	None observed	
		Adding by Place Value	3		
		Concatenated Digits	9		
Type 2H	A. 47+38 B. 28+19	Purposeful Redistribution	1	Landmark numbers	1
		Concatenated Digits	11	Counting up	2
				Doubles	1
				Landmark numbers	1
Type 3H	A. 68+73 B. 57+64	Purposeful Redistribution	1	Doubles	1
		Concatenated Digits	12	Counting up	4
				Doubles	1
				Landmark numbers	1
Type 1V	A. 18+22 B. 29+31	Concatenated Digits	14	Counting up	2
Type 2V	A. 29+38 B. 48+37	Purposeful Redistribution	1	None observed	
		Concatenated Digits	14	Counting up	2
				Doubles	6
				Landmark numbers	1
Type 3V	A. 54+67 B. 63+78	Concatenated Digits	14	Counting up	5
				Doubles	3
				Landmark numbers	1

Note that Student 1's response displays two telling markers of the standard US algorithm: 1) she uses the phrase "carry the one," and 2) at no point does she refer to the actual values of the numbers that 6 and 7 seven represent, she treats them as concatenated digits. However, it is important to also note that within the framework of the standard U.S. algorithm, she also used a doubling strategy, strategically grouping the regrouped 10 (referred to as a "carried" 1) with the 6 from 63 so that she could just double the 7 from 78. In fact, her answer to the doubling process is the one part of the solution that she does not doubt when her partner indicates that she has reported an incorrect answer, implying that she may be most confident in her computational skills when using one of these basic one-digit addition strategies.

Horizontal Versus Vertical Orientation Findings

We found that horizontally-written problems did seem to elicit a greater variety of overarching strategies from these students, supporting one of the theoretical assumptions about number choice: 20.0% of the responses to horizontally-written problems involved the use of an overarching strategy other than digit language as compared to 2.3% of the responses to vertically-written problems. In contrast, only 27.5% of all responses to horizontally-written problems used a distinct sub-strategy versus 46.5% of the responses to vertically-written problems. Our findings do not refute the assumption by Humphreys and Parker (2015) that horizontally-written problems may elicit more diverse strategies for solving arithmetic problems. Instead, our findings seem to imply that vertically written problems opened opportunities for PTs to be more explicit in their use of basic strategies for adding one-digit numbers; that is, some of the PTs in our small study explicitly did not use automated facts to solve two-digit problems by leveraging their knowledge of single-digit addition. This finding is interesting because it might be easily assumed that the PTs might *only* apply concatenated digits strategies to vertically-written problems, which was not the case in this study with all PTs and across all solutions.

We posit that problems written in the vertical form seemed to elicit more diverse sub-strategies than those written horizontally; this difference may be because when PTs were presented with problems written horizontally, they mentally rearranged the problem to be vertical and applied a more traditional algorithm to their solution. Eight of the 14 PTs explicitly claimed in their written debriefs that they visualized horizontally-written problems as written vertically and then applied the form of the standard U.S. algorithm that treated the numbers as concatenated digits in the solution strategy prior to solving the problem. For example, one student reflected after solving the horizontally-written problems:

> I was taught at a very young age to add double digit numbers by rearranging them so that one number is on top of another. Thus, that is what I picture in my head when I try to solve the problems mentally.

Then after solving the vertically-written problems, the same student reflected:

> I didn't have to mentally solve the problem (other than some mental math)

> because the problems were already arranged in the way I would [en]vision it in my head. I used some strategy because it's still faster for me to solve when the numbers are arranged like that [vertically stacked].

Yet when the PTs were presented with vertically arranged problems, PTs who leveraged the standard U.S. algorithm did not necessarily need to rearrange the problem and might have had more opportunities to focus on other strategies that they could use to solve the problem (i.e., using doubles, base ten knowledge, etc). For example, one student noted in the horizontal problems that they rearranged the problems to be vertical and used a concatenated digits strategy. Yet in the vertically stacked problems, the PT stated, "It was the same strategy two digits on top of each other. Right now [Start by adding the] first [right column], then carry, then add the left side. I used adding doubles and using my fingers."

Purposeful Number Choice Findings

We also analyzed the overarching and sub-strategies based on the purposeful numbers that we selected for our problems. First, we noticed that in Type 1 problems (refer to Table 1) that asked the PTs to consider when the ones digits added to 10, there was the greatest diversity of overarching strategies and that there was a difference in the number of solution strategies that leveraged a concatenated digits conception of two-digit numbers based on whether the problem was horizontally or vertically written. Secondly, we noticed that all problems of Type 2 and 3 from Table 1 presented more sub-strategies across problems written both horizontally and vertically.

Furthermore, the purposeful selection of numbers in the problems afforded PTs more opportunities to leverage conceptually-based solution strategies. For example, when we crafted the problems 29+38 and 48+37, which both had ones digits that were only one number apart, we anticipated that PTs might leverage doubles or some kind of purposeful redistribution based on their knowledge of making tens in order to solve the problem, which was the case in our findings. Although we were not necessarily expecting students to use a counting-up strategy with problems like 29+38, our findings speak to the research that designing problems with purposeful number choice in mind can in fact elicit certain strategies from students (Humphries & Parker, 2015; Land et al., 2015; Parrish, 2010).

Although we did not approach this exploratory study assuming PTs would leverage a monolithic set of procedures to faithfully follow, we were surprised by the nuanced set of sub-strategies that the PTs used. Within the strategies that leveraged the standard U.S. algorithm and a concatenated digits strategy, some PTs also used other strategies that drew from their knowledge of base ten, doubles, and/or landmark numbers, which can help them when they design and implement mathematics tasks in the future (Carpenter et al., 1999; Philipp, 2008).

Discussion and Implications

As we designed the study, we made several assumptions about PTs' knowledge of adding multi-digit numbers that later helped us to question and reframe our conceptions. As some of the PTs solved the problems, they consistently relied on certain strategies that aided them in solving basic one-digit addition problems within the framework of the two-digit addition problem (e.g., using doubles to solve each problem even though we did not intend for PTs to use the same strategy for all problems). Some of the PTs in our study also engaged in meta-cognition, or self-awareness, about the strategies they used to solve the problems. The findings from this study have led us as MTEs to resist being disappointed with the overabundance of PTs' use of developmental strategies for doing single-digit addition (that is, showing that they are not as automatic in their basic facts as we might expect). Instead, what we have learned from this study is to remind ourselves as MTEs the potential for our students to utilize more deliberate strategies like derived facts or using landmark numbers of 10 and to use these in-class experiences as a springboard for PTs learning more about the role of number choice and solution strategies in their own practice as teachers. Moreover, PTs' awareness of their use of these developmental strategies bodes well for developing their awareness of elementary students' trajectories for learning basic addition facts and leveraging those facts for solving multi-digit addition problems.

Secondly, PTs should have more experience learning to attend to the role of number choice when solving and designing mathematics problems. As MTEs, we again assumed that our PTs might use more efficient and diverse strategies for solving the problems, but this was not the case. Some PTs correctly answered the problem by using the strategy of doubling even though it was not *the most efficient* method of solution. For example, consider the following student response:

> 57 plus 64. 4 plus 4 is 8. And you have three left over from 7. Hold on. I am confused. 7 plus 4 is... 4 plus... 7 plus 4. 4 plus 4 is 8. you have three left over from the 7 which makes it 9, 10, 11 (counting up to 11). 6 plus 5, 6 plus 6 is 12, you subtract the 1, which makes it 11. And you carry the 1 which makes it somethin'.... 7, 8,9, 10, 11. 111.

This student employed a doubling strategy to add 7 and 4, when perhaps "making 10" would have served her better in her solution. Also, she used doubling to add 6+5+1 even though she had to both compensate for the "missing" one when she doubled 6, and then include an extra one that was actually a regrouped ten.

Nonetheless, MTEs can elicit a variety of solution strategies from PTs that spur conversations about the efficiency of each valid strategy presented, which is a similar practice for teachers and their students in mathematics classrooms (Humphreys & Parker, 2015). Better understanding the mathematical thinking of PTs will allow MTEs to investigate and emphasize the importance of number choice with their prospective elementary teacher students. However, we also feel that as this study falls into the decomposing practice aspect of the framework by Grossman, et al (2009), simply exploring the way PTs solving two-digit addition problems contributes to the practice of mathematics teacher educators.

Finally, our study also highlighted the need for more collaboration across content and methods courses in teacher preparation program. Although the genesis for the study arose from a moment in a mathematics methods course, the study was appropriately designed to be implemented in an elementary content course with a focus on PTs' conceptual and procedural knowledge and the role of number choice when posing computational tasks.

Because we want our PTs to integrate the experiences across their teacher preparation program, MTEs should collaborate more to engage in research that explores PTs' mathematical thinking in various settings. When MTEs collaborate to research their own practices and their PTs' mathematical thinking, they can also revise and improve their courses in ways that further support PTs' thinking about teaching mathematics.

This study provides one example of an investigation into prospective teachers' strategies for solving addition problems with purposefully chosen addends. The importance of number choice in influencing the diversity and type of strategies that PTs use is still unfinished. Future iterations of this study might examine addition using different number choices or investigate PTs' knowledge of the other three whole number operations or rational number operations. PTs might also examine authentic examples of student solution strategies and compare them to their own methods. When PTs are prepared to unpack the strategies and algorithms that they might see from their students, they will be more prepared to help children see the nuance and complexity of their own thinking.

References

Ball, D. L. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics*, *8*(1), 40-48.

Boaler, J. (2016). *Mathematical mindsets: unleashing students' potential through creative math, inspiring messages and innovative teaching*. San Francisco, CA: Jossey-Bass.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively Guided Instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 403-434.

Carpenter, T. P., Fennema, E., Franke, M., Levi, L., & Empson, S. (1999). *Children's mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Cazden, C. (2001). *Classroom discourse: The language of teaching and learning*. Portsmouth, NH: Heinemann.

Fuson, K., & Willis, G. (1988). Subtracting by counting up: More evidence. *Journal for Research in Mathematics Education, 19*(5), 402-420.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. Teachers College Record, 111(9), 2055- 2100.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Hope, J., & Sherrill, J. (1987). Characteristics of unskilled and skilled mental calculators. *Journal for Research in Mathematics Education*, *18*(2), 98-111.

Humphreys, C., & Parker, R. (2015). *Making number talks matter: Developing mathematical practices and deepening understanding, Grades 4-10*. Portland, MN: Stenhouse Publishers.

Kilpatrick, J., Swafford, J., Findell, B., National Research Council (U.S.). Mathematics Learning Study Committee, & National Academy of Sciences - National Research Council, Washington, DC. Center for Education. (2001). Adding it up: Helping children learn mathematics. Washington, DC: National Academy Press.

Land, T. J., Drake, C., Sweeney, M., Franke, N., & Johnson, J. (2015). Transforming the task with number choice: kindergarten through grade 3. Reston, VA: The National Council of Teachers of Mathematics.

Parrish, S. (2010). *Number talks: Helping children build mental math and computation strategies, grades K-5*. Sausalito, CA: Math Solutions.

Parrish, S. D. (2011). Number talks build numerical reasoning. *Teaching Children's Mathematics*, *18*(3), 198-206.

Philipp, R. A. (2008). Motivating prospective elementary school teachers to learn mathematics by focusing upon children's mathematical thinking. Issues in Teacher Education, 17(2), 7-26.

Steinberg, R. (1985). Instruction on derived facts strategies in addition and subtraction. *Journal for Research in Mathematics Education*, *16*(5), 337-355. doi:10.2307/749356

Welder, R., & Simonsen, L. (2011). Elementary teachers' mathematical knowledge for teaching prerequisite algebra concepts. *IUMPST (Issues in the Undergraduate Mathematics Preparation of School Teachers): The Journal. Vol 1 (Content Knowledge), 1*, 1-16. Retrieved March 13, 2018, from http://www.k-12prep.math.ttu.edu/journal/1.contentknowledge/welder01/article.pdf