Finalexam-Due 12/3/2024 Tuesday.

November 26, 2024

1. Consider three independent Poisson arrival processes $N_i(t)$, $t \geq 0$ with rates λ_i for $i = 1, 2$, all starting at $N_i(0) = 0$. Let T be an independent exponential distribution with parameter 1. Let $X_i = N_i(T)$.

(1) Find the joint distribution of X_1 and X_2.

(2) Find $\mathbb{E}[X_1|X_2]$.

2. Let X_n be a Markov chain with state space $\{0, 1, \cdots, 2N\}$ with transition matrix P satisfying

$$p_{i,i-1} = p_{i,i+1} = 1/2, \text{ for } 1 \leq i \leq 2N-1$$
$$p_{0,N} = p_{2N,N} = 1.$$

Find the stationary distribution π.

3. Let U_i be a sequence of i.i.d random variables with exponential distribution ($\lambda = 1$). Define

$$N = \min\{n \geq 0 : U_1 + \cdots + U_n \geq 1\}.$$

Find $\mathbb{E}N$.

4. Find the extinction probabilities for the following branching processes.

(a) An individual has 0 offspring with probability 3/4 and 4 offspring with probability 1/4.

(b) A cell dies with probability 1/5 and divides with probability 4/5.

(c) The offspring distribution has pgf $G(s) = e^{(s-1)/3}$.

5. Let N be a Poisson process with intensity λ. Find the covariance of $N(t)$ and $N(s)$.

6. Let $W_1, W_2, \cdots$ be the event times in a Poisson process of rate λ, and let $N(t)$ be the number of points in the interval $(0, t]$. Evaluate

$$\mathbb{E} \sum_{k=1}^{N(t)} (W_k)^2.$$

7. Let $Y_1, Y_2, \ldots$ be i.i.d normal random variables with mean 2 and variance 1. Let $N(t)$ be a Poisson process with intensity λ, independent of the Y_i , and let $M(t) = Y_1 + \cdots + Y_{N(t)}$ (with $M(t) = 0$ if $N(t) = 0$). Prove that

$$\mathbb{E}\Big[\frac{M(t)}{N(t)}\Big|N(t) > 0\Big] = \frac{\mathbb{E}M(t)}{\mathbb{E}N(t)}$$

8. Prove or disprove the following statement: for every Markov chain $X_0, X_1, \cdots$ and every real valued function f, the sequence $f(X_0), f(X_1), \cdots$ also forms a Markov chain.