Take-Home Final, MATH5382

Due: 12/10/2025 Wednesday.

1. Using the definition of conditional expectation,
(1) Prove that $\mathbb{E}(\xi|\mathcal{G}) = \xi$ if ξ is $\mathcal{G}$-measurable.
(2) Prove that $\mathbb{E}(\xi|\mathcal{G}) = \mathbb{E}\xi$ if ξ and $\mathcal{G}$ are independent.
(3) Find $\mathbb{E}[\xi|I_A]$.

2. Given the joint density of (X, Y) by f, find the $\mathbb{E}[X|Y]$.

3. Let X_n be a sequence of random variables with $\sup_n \mathbb{E}|X_n| < \infty$. Is such a sequence uniform integrable? Is such a sequence tight?

4. Present an example that $X_n \Longrightarrow X$ but $F_n(x) \to F(x)$ fails for some x.

5. Prove that $X_n \to a$ in probability if and only if $X_n \Longrightarrow a$.

6. Let X_n be an i.i.d sequence random variables with mean m. Let T be a $\mathbb{N}$-valued random variable independent of $\{X_n\}$. Find the $\mathbb{E}\sum_{n=1}^{T} X_n$.

7. Prove that $\mathbb{E}|X| < \infty$ iff $\sum_{n=1}^{\infty} \mathbb{P}(|X| > n) < \infty$.