Methods of Repaying A Loan

Amortization method: Pay back the loan by periodic payments.

Sinking fund method: Deposit money into a sinking fund, and pay a lump-sum payment at the end of term. Either pay interest of the loan periodically (Common practice), or pay all the accumulated value at the end.

Finding the Outstanding Loan Balance I

Let

$$pmt = \text{the periodic payment,}$$

$$bl_k = \text{the loan balance immediately after the kth payment,}$$

$$pr_k = \text{the principal repaid in the kth payment,}$$

$$in_k = \text{the interest potion in the kth payment,}$$

and

$$P = \text{loan amount.}$$

Finding the Outstanding Loan Balance II

There are two ways to find the outstanding loan balance bl_k after the kth payment:

Prospective Method:

$$\text{loan balance} = \text{present value of future payments.}$$

After the kth payment, the current value of all future payments is given by

$$pmt(v + v^2 + \cdots + v^{n-k}) = pmt\frac{v(1 - v^{n-k})}{1 - v} = pmt\frac{1 - v^{n-k}}{i}.$$

So

Finding the Outstanding Loan Balance III

The outstanding loan balance after the kth payment is

$$bl_k = pmt\frac{1 - v^{n-k}}{i} = (pmt)a_{\overline{n-k}|}.$$

TVM solver:

N	I	PV	PMT	FV	P/Y	C/Y	E/B?
n-k	I		-pmt	0			
		$\Downarrow$ bl_k					END

Finding the Outstanding Loan Balance IV

Retrospective Method:

$$\begin{aligned} \text{loan balance} &= \text{accumulated value of loan} \\ &\quad -\text{accumulated value of past payments} \end{aligned}$$

Finding the Outstanding Loan Balance V

The outstanding loan balance after the kth payment is

$$bl_k = Pu^k - pmt\frac{u^k - 1}{i} = Pu^k - (pmt)s_{\overline{k}|}.$$

TVM solver:

N	I	PV	PMT	FV	P/Y	C/Y	E/B?
k	I	-P	pmt				
				$\Downarrow$ bl_k			END

Example (Exercise 5.1)

A loan of $\$1,000$ is being repaid with quarterly payments at the end of each quarter for five years at 6% convertible quarterly. Find the outstanding loan balance at the end of the second year.

Example (Exercise 5.2)

A loan of $\$10,000$ is being repaid by installments of $\$2000$ at the end of each year, and a smaller final payment made one year after the last regular payment. Interest is at the effective rate of 12%. Find the amount of outstanding loan balance remaining when the borrower has made payments equal to the amount of the loan.

Example (Exercise 5.3)

A loan is being repaid by quarterly installments of \$1500 at the end of each quarter at 10% convertible quarterly. If the loan balance at the end of the first year is $\$12,000$, find the original loan amount.

Example (Exercise 5.4)

A $\$20,000$ loan is to be repaid with annual payments at the end of each year for 12 years. If $(1+i)^4 = 2$, find the outstanding balance immediately after the fourth payment.

Example (Exercise 5.5)

A \$20,000 mortgage is being repaid with 20 annual installments at the end of each year with an effective annual interest rate $i = 0.04$. The borrower makes five payments and then is temporarily unable to make payments for the next two years. Find the revised payment to start at the end of the 8th year if the loan is still to be repaid at the end of the original 20 years.

Example (Exercise 5.7)

A husband and wife buy a new home and take out a $\$150,000$ mortgage loan with level annual payments at the end of each year for 15 years on which the effective rate of interest is equal to 6.5%. At the end of 5 years they decide to make a major addition to the house and want to borrow an additional $\$80,000$ to finance the new construction. They also wish to lengthen the overall length of the loan by 7 years (i.e. until 22 years after the date of the original loan). In the negotiations the lender agrees to these modifications. but only if the effective interest rate for the remainder of the loan after the first 5 years is raised to 7.5%. Find the revised annual payment which would result for the remainder of the loan.

Amortization Schedules I

Amortization Schedule: A list of interests and principals paid in each payments, and the outstanding loan balances.

Clearly the principal repaid in the kth payment equals the loan balance at the $(k-1)$th payment minus the loan balance at the kth payment, i.e.

$$\begin{aligned} pr_k &= bl_{k-1} - bl_k = (pmt)a_{\overline{n-k+1}|} - (pmt)a_{\overline{n-k}|} \\ &= pmt((v + \cdots + v^{n-k} + v^{n-k+1}) - (v + \cdots + v^{n-k})) \\ &= (pmt)v^{n-k+1}. \end{aligned}$$

So

$$\begin{aligned} P &= (pmt)v^n + (pmt)v^{n-1} + \cdots (pmt)v \\ &= pr_1 + pr_2 + \cdots + pr_n. \end{aligned}$$

Amortization Schedules II

We have

$$pr_k = pr_m u^{k-m}.$$

$$pr_k \longrightarrow pr_{k+1} = pr_k u$$

Amortization Schedules III

In the kth payment:

$$\begin{aligned} pr_k &= (pmt)v^{n-k+1} \\ &= pr_j u^{k-j} \\ &= pmt - in_k. \end{aligned}$$

$$\begin{aligned} in_k &= pmt(1 - v^{n-k+1}) \\ &= bl_{k-1} i \\ &= pmt - pr_k. \end{aligned}$$

Example (Exercise 5.8)

A loan is being repaid with quarterly installments of $1,000 at the end of each quarter for five years at 12% convertible quarterly. Find the amount of principal in the sixth instalment.

Example (Exercise 5.10)

A loan is being repaid with a series of payments at the end of each quarter for five years. If the amount of principal in the third payment is \$100, find the amount of principal in the last five payments. Interest is at the rate of 10% convertible quarterly.

Example (Exercise 5.13)

A loan of L is being amortized with payments at the end of each year for 10 years. If $v^5 = 2/3$, find the following:

a) The amount of principal repaid in the first 5 payments.

b) The amount due at the end of 10 years, if the final 5 payments are not made as scheduled.

Example (Exercise 5.14)

A 35-year loan is to be repaid with equal installments at the end of each yeas. The amount of interest paid in the 8th installment is \$135. The amount of interest paid in the $22nd$ installment is \$108. Calculate the amount of interest paid in the 29th installment.

Example (Exercise 5.15)

A 10-year loan of L is repaid by the amortization method with payments of \$1000 at the end of each year. The annual effective interest rate is i. The total amount of interest repaid during the life of the loan is also equal to L. Calculate the amount of interest paid during the first year of the loan.

Example (Exercise 5.16)

A bank customer borrows X at an annual effective rate of 12.5% and makes level payments at the end of each year for n years.

(i) The interest portion of the final payment is \$153.86.

(ii) The total principal repaid as of time $n-1$ is \$6009.12.

(iii) The principal repaid in the first payment is Y.

Calculate Y.

Example (Exercise 5.19)

On a loan of $\$10,000$ interest at 9% effective must be paid at the end of each year. The borrower also deposits $\$X$ at the beginning of each year into a sinking fund earning 7% effective. At the end of 10 years the sinking fund is exactly sufficient to pay off the loan. Calculate X.

Example (Exercise 5.20)

A borrower is repaying a loan with 10 annual payments of $1000. Half of the loan is repaid by the amortization method at 5% effective. The other half of the loan is repaid by the sinking fund method in which the tender receives 5% effective on the investment and the sinking fund accumulates at 4% effective. Find the amount of the loan.

Example (Exercise 5.21)

A borrows $\$12,000$ for 10 years and agrees to make semiannual payments of $\$1000$. The lender receives 12% convertible semiannually on the investment each year for the first 5 years and 10% convertible semiannually for the second 5 years. The balance of each payment is invested in a sinking fund earning 8% convertible semiannually. Find the amount by which the sinking fund is short of repaying the loan at the end of the 10 years.

Example (Exercise 5.22)

A borrower takes out a loan of $3000 for 10 years at 8% convertible semiannually. The borrower replaces one third of the principal in a sinking fund earning 5% convertible semiannually and the other two thirds in a sinking fund earning 7% convertible semiannually. Find the total semiannual payment.

Example (Exercise 5.23)

A payment of $\$36,000$ is made at the end of each year for 31 years to repay a loan of $\$400,000$. If the borrower replaces the capital by means of a sinking fund earning 3% effective, find the effective rate paid to the lender on the loan.

Example (Exercise 5.24)

A borrows \$1000 for 10 years at an annual effective interest rate of 10%. A can repay this loan using the amortization method with payments of P at the end of each year. Instead. A repays the loan using a sinking fund that pays an annual effective rate of 14%. The deposits to the sinking fund are equal to P minus the interest on the loan and are made at the end of each year for 10 years. Determine the balance in the sinking fund immediately after repayment of the loan.

Differing Payment Periods And Interest Conversion Periods

Example (Exercise 5.25)

An investor buys an annuity with payments of principal and interest of \$500 per quarter for 10 years. Interest is at the effective rate of 8% per annum. How much interest does the investor receive in total over the 10-year period?

Example (Exercise 5.26)

A borrows \$10000 for five years at 12% convertible semiannually. A replaces the principal by means of deposits at the end of every year for five years into a sinking fund which earns 8% effective. Find the total dollar amount which A must pay over the five- year period to completely repay the loan.

Example (Exercise 5.27)

A borrower is repaying a loan with payments of $3000 at the end of every year over an unknown period of time, If the amount of interest in the third installment is $2000, find the amount of principal in the sixth installment. Assume that interest is 10% convertible quarterly.

Example (Exercise 5.28)

A borrows \$5000 for 10 years at 10% convertible quarterly. A does not pay interest currently and will pay all accrued interest at the end of 10 years together with the principal. Find the annual sinking fund deposit necessary to liquidate the loan at the end of 10 years if the sinking fund earns 7% convertible semiannually.

Varying Series Of Payments

Example (Exercise 5.29)

A loan is repaid with payments which start at \$200 the first year and increase by \$50 per year until a payment of \$1000 is made, at which time payments cease. If interest is 4% effective, find the amount of principal in the fourth payment.

Example (Exercise 5.30)

A borrower is repaying a \$1000 loan with 10 equal semi-annual payments of principal. Interest at 6% convertible semiannually is paid on the outstanding balance each half-year. Find the price to yield an investor 10% convertible semiannually.

Example (Exercise 5.31)

A borrows \$2000 at an effective rate of interest of 10% per annum and agrees to repay the loan with payments at the end of each year. The first payment is to be \$400 and each payment thereafter is to be 4% greater than the preceding payment, with a smaller final payment made one year after the last regular payment.

a) Find the outstanding loan balance at the end of three years.

b) Find the principal repaid in the third payment.

Example (Exercise 5.32)

A has money invested at effective rate i. At the end of the first year A withdraws 162.5% of the interest earned, at the end of the second year A withdraws 325% of the interest earned, and so forth with the withdrawal factor increasing in arithmetic progression. At the end of 16 years the fund is exactly exhausted. Find i.

Example (Exercise 5.33)

A 10-year loan of $2000 is to be repaid with payments at the end of each year. It can be repaid under the following two options:

(i) Equal annual payments at an annual effective rate of 8.07%.

(ii) Installments of $200 each year plus interest on the unpaid balance at an annual effective rate of i.

The sum of the payments under Option (1) equals the sum of the payments under Option (ii). Determine i.

Example (Exercise 5.34)

A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is $1000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment. Calculate the outstanding loan balance immediately after the 40 payment is made.