Differing Payment And Interest Conversion Periods

Example (Exercise 4.1)

Find the accumulated value 18 years after the first payment is made of an annuity on which there are 8 payments of $\$2,000$ each made at two-year intervals. The nominal rate of interest convertible semiannually is 7%.

Example (Exercise 4.2)

Find the present value of a ten-year annuity which pays \$400 at the beginning of each quarter for the first 5 years. increasing to \$600 per quarter thereafter. The annual effective rate of interest is 12%.

Example (Exercise 4.3)

A sum of \$100 is placed into a fund at the beginning of every other year for eight years. If the fund balance at the end of eight years is \$520, find the rate of simple interest earned by the fund.

Example (Exercise 4.4)

An annuity-immediate that pays \$400 quarterly for the next 10 years costs \$10,000. Calculate the nominal interest rate convertible monthly earned by this investment.

Example (Exercise 4.8)

The present value of a perpetuity paying 1 at the end of every three years is $125/91$. Find i.

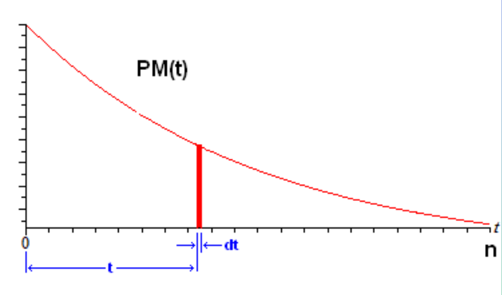
Example (Exercise 4.9)

Find the present value of an annuity on which payments are \$100 per quarter for five years, just before the first payment is made, if $\delta = 0.08$.

Continuous Annuity I

The payment for a continuous annuity is a continuous function $PM(t)$ while the total amount of payments is the area under the curve $PM(t)$.
To find the present and accumulated values of a continuous annuity, let's consider the present and accumulated values of the payments in a time interval $[t, t + dt]$.

Continuous Annuity II



So the present value at $t = 0$ of the payments in a time interval $[t, t + dt]$ is given by $PM(t)\frac{1}{a(t)}dt$, and

$$PV = \int_0^n \frac{PM(t)}{a(t)}\, dt$$

Continuous Annuity III

The accumulated value at $t = n$ is given by

$$FV = (PV)a(n) = a(n) \int_0^n \frac{PM(t)}{a(t)}\, dt.$$

The present and the accumulated values of $PM(t) \equiv 1$ (which is called "**continuous payment at the rate of** 1 **per annum**") are denoted by

$$\bar{a}_{\overline{n}|} \quad \text{and} \quad \bar{s}_{\overline{n}|}$$

respectively.

Continuous Annuity IV

Let $a(t)$ be an accumulate function. Then

$$\bar{a}_{\overline{n}|} = \int_0^n \frac{1}{a(t)}\, dt.$$

$$\bar{s}_{\overline{n}|} = \bar{a}_{\overline{n}|} a(n) = a(n) \int_0^n \frac{1}{a(t)}\, dt.$$

Continuous Annuity V

For compound interest with annual effective rate of interest i and the force of interest $\delta = \ln(1+i)$, $a(t) = (1+i)^t = e^{\delta t}$. So

$$\bar{a}_{\overline{n}|} = \int_0^n \frac{1}{(1+i)^t}\, dt = \int_0^n e^{-\delta t}\, dt = \frac{1 - e^{-\delta n}}{\delta} = \frac{1 - v^n}{\delta}.$$

$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^{n-t}\, dt = \int_0^n (1+i)^t\, dt = \int_0^n e^{\delta t}\, dt = \frac{e^{\delta n} - 1}{\delta} = \frac{u^n - 1}{\delta}.$$

Example (Exercise 4.17)

There is $\$40,000$ in a fund which is accumulating at 4% per annum convertible continuously. If money is withdrawn continuously at the rate of $\$2400$ per annum, how long will the fund last?

Example (Exam FM Sample Question 21)

Payments are made to an account at a continuous rate of $(8k + tk)$, where $0 \leq t \leq 10$. Interest is credited at a force of interest $\delta_t = \frac{1}{8+t}$. After time 10, the account is worth 20,000. Calculate k.

Payments Varying in Arithmetic Progression I

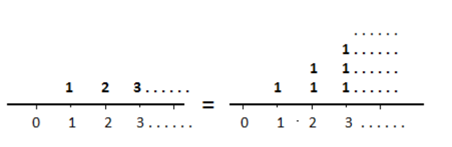
In this section we consider the annuities which is increasing or decreasing in a fixed amount.

Increasing Perpetuity-Immediate:

Consider the perpetuity-immediate which pays 1 at the end of the 1st period, 2 at the end of the 2nd period, 3 at the end of the 3rd period, ... The present value is denoted by,

$$(Ia)_{\overline{\infty|}}$$

Clearly



Payments Varying in Arithmetic Progression II

So

$$\begin{aligned}(Ia)_{\overline{\infty|}} &= a_{\overline{\infty|}} + va_{\overline{\infty|}} + v^2 a_{\overline{\infty|}} + \cdots \\ &= a_{\overline{\infty|}}(1 + v + v^2 + \cdots) \\ &= a_{\overline{\infty|}}\ddot{a}_{\overline{\infty|}} \\ &= \frac{1}{id}.\end{aligned}$$

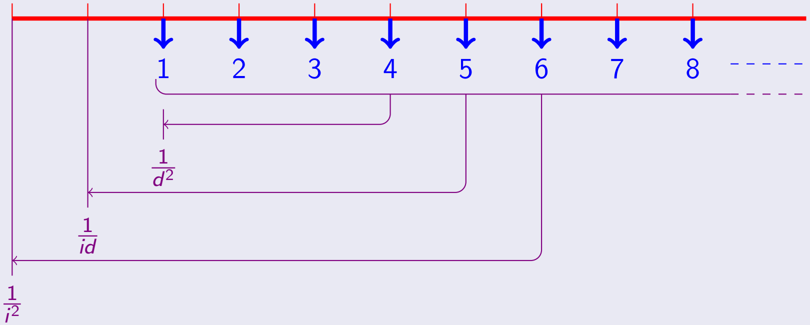
Note that from $i = \frac{d}{v}$ and $d = \frac{i}{u}$ we have

$$\frac{v}{d} = \frac{1}{i}, \qquad \frac{u}{i} = \frac{1}{d}.$$

So

$$u(Ia)_{\overline{\infty|}} = \frac{u}{id} = \frac{1}{d^2}, \quad v(Ia)_{\overline{\infty|}} = \frac{v}{id} = \frac{1}{i^2}.$$

Payments Varying in Arithmetic Progression III

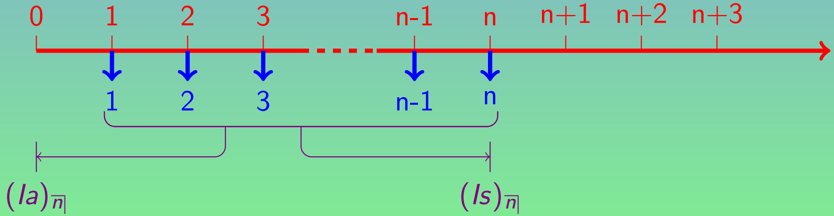


Payments Varying in Arithmetic Progression IV

Increasing and Decreasing Annuities:

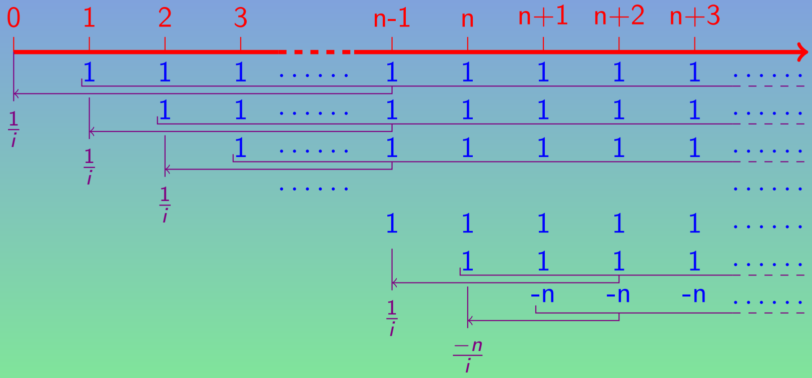
The annuity which pays $1, 2, \ldots, n$ is called increasing annuity, and its values are denoted by

$$(Ia)_{\overline{n}|},\ (Is)_{\overline{n}|}.$$



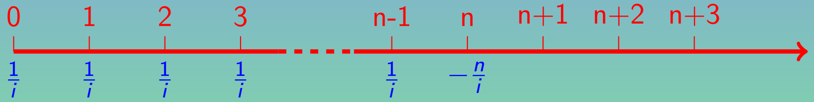
which is equivalent to the perpetuities:

Payments Varying in Arithmetic Progression V



Payments Varying in Arithmetic Progression VI

which equals the payments



Therefore

Payments Varying in Arithmetic Progression VII

$$(Ia)_{\overline{n}|} = \frac{1}{i}\ddot{a}_{\overline{n}|} - \frac{n}{i}v^n.$$

$$(Is)_{\overline{n}|} = \frac{1}{i}\ddot{s}_{\overline{n}|} - \frac{n}{i}.$$

TVM solver:

N	I	PV	PMT	FV	P/Y	C/Y	E/B?
n	I		-1/i	n/i	1	1	BEG
		$\Downarrow$ $(Ia)_{\overline{n}|}$					

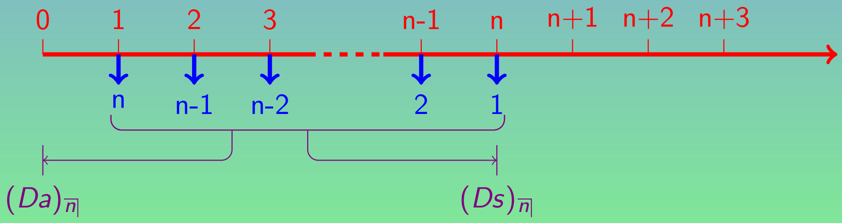
N	I	PV	PMT	FV	P/Y	C/Y	E/B?
n	I	0	-1/i		1	1	BEG
				$\Downarrow$ A			

$(Is)_{\overline{n}|} = A - \frac{n}{i}$

Payments Varying in Arithmetic Progression VIII

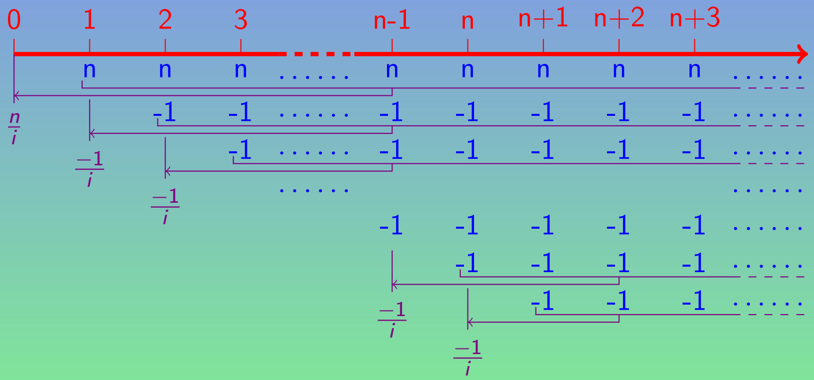
The annuity which pays $n, n-1, \ldots, 2, 1$ is called decreasing annuity, and its values are denoted by

$$(Da)_{\overline{n|}}, \ (Ds)_{\overline{n|}}.$$



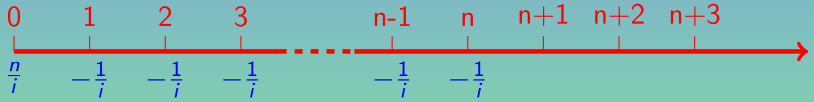
which is equivalent to the perpetuities:

Payments Varying in Arithmetic Progression IX



Payments Varying in Arithmetic Progression X

which equals the payments



So

Payments Varying in Arithmetic Progression XI

$$(Da)_{\overline{n}|} = \frac{n}{i} - \frac{1}{i} a_{\overline{n}|}.$$

$$(Ds)_{\overline{n}|} = u^n \frac{n}{i} - \frac{1}{i} s_{\overline{n}|}.$$

TVM solver:

N	I	PV	PMT	FV	P/Y	C/Y	E/B?
n	I		-1/i	0	1	1	BEG
		$\Downarrow$ A					

$(Da)_{\overline{n}|} = \frac{n}{i} - A$

N	I	PV	PMT	FV	P/Y	C/Y	E/B?
n	I	-n/i	1/i		1	1	BEG
				$\Downarrow$ $(Ds)_{\overline{n}|}$			

Example (Exercise 4.24)

Find the present value of a perpetuity under which a payment of 1 is made at the end of the first year, 2 at the end of the second year, increasing until a payment of 24 is made at the end of the 24-th year, and thereafter payments are level at 24 per year forever. The annual effective interest rate is 0.085.

Example (Exercise 4.25)

A perpetuity-immediate has annual payments of $1, 3, 5, 7, \ldots$. If the present value of the sixth and seventh payments are equal, find the present value of the perpetuity.

Example (Exercise 4.26)

If X is the present value of a perpetuity of 1 per year with the first payment at the end of the second year and $20X$ is the present value of a series of annual payments $1, 2, 3, \ldots$ with the first payment at the end of the third year, find d.

Payment Varying in Geometric Progression I

Example (Exercise 4.28)

Find the present value of a 20-year annuity with annual payments which pays \$600 immediately and each subsequent payment is 5% greater than the preceding payment. The annual effective rate of interest is 10.25%.

Payment Varying in Geometric Progression II

Example (Exercise 4.30)

Annual deposits are made into a fund at the beginning of each year for 10 years. The first 5 deposits are \$1000 each and deposits increase by 5% per year thereafter. If the fund earns 8% effective, find the accumulated value at the end of 10 years.

Payment Varying in Geometric Progression III

Example (Exercise 4.31)

A perpetuity makes payments starting five years from today. The first payment is \$1000 and each payment thereafter increases by $k\%$ per year. The present value of this perpetuity is equal to \$4096 when computed at $i = 25\%$. Find k.

Payment Varying in Geometric Progression IV

Example (Exercise 4.32)

An employee currently is aged 40, earns $\$40,000$ per year, and expects to receive 3% annual raises at the end of each year for the next 25 years. The employee decides to contribute 4% of annual salary at the beginning of each year for the next 25 years into a retirement plan. How much will be available for retirement at age 65 if the fund can earn a 5% effective rate of interest?

Payment Varying in Geometric Progression V

Example (Exercise 4.33)

A series of payments is made at the beginning of each year for 20 years with the first payment being \$100. Each subsequent payment through the tenth year increases by 5% from the previous payment. After the tenth payment, each payment decreases by 5% from the previous payment. Calculate the present value of these payments at the time the first payment is made using an annual effective rate of 7%.

Example (Exercise 4.37)

A perpetuity has payments at the end of each four-year period. The first payment at the end of four years is 1. Each subsequent payment is 5 more than the previous payment. It is known that $v^4 = 0.75$. Calculate the present value of this perpetuity.

Example (Exercise 4.38)

A perpetuity provides payments every six months starting today. The first payment is 1 and each payment is 3% greater than the immediately preceding payment. Find the present value of the perpetuity if the effective rate of interest is 8% per annum.

Continuous Varying Annuities I

The present value of the continuous annuity $PM(t) = t$ is denoted by $(\bar{I}\bar{a})_{\overline{n}|}$.

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n tv^t \, dt.$$

Example (Exercise 4.40)

Evaluate

$$(\bar{I}\bar{a})_{\overline{\infty}|}$$

if $\delta = 0.08$

Example (Exercise 4.43)

A one-year deferred continuous varying annuity is payable for 13 years. The rate of payment at time t is $t^2 - 1$ per annum, and the force of interest at time t is $\frac{1}{1+t}$. Find the present value of the annuity.