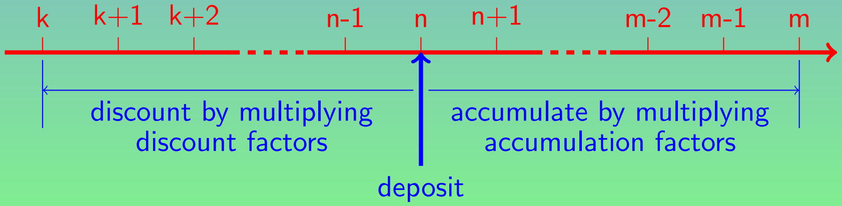
# Equation of Values

**Definition**

The comparison date is the date to let accumulated or discounted values equal for both direction of payments (e.g. payments to the bank and money received from the bank).

A time diagram is often helpful in solving such a problem.



**Example (Exercise 2.1)**

In return for a promise to receive $\$2,000$ at the end of four years and $5000 at the end of ten years, an investor agrees to pay $3000 immediately, and to make an additional payment at the end of three years. Find the amount of the additional payment if $i^{(4)} = 0.06$.

### Example (Exercise 2.2)

You have an inactive credit card with a \$1000 outstanding unpaid balance. This particular credit charges interest at the rate of 18% compound monthly. You are able to make a payment of \$200 one month from today and \$300 two months from today. Find the amount that you will have to pay three months from today to completely pay off this credit card debt.

## Example (Exercise 2.3)

At a certain interest rate the present values of the following two payment patterns are equal:

(i) \$200 at the end of 5 years plus \$500 at the end of 10 yeas.

(ii) \$400.94 at the end of 5 years.

At the same interest \$100 invest now plus \$120 at the end of 5 years will accumulate to $P$ at the end of 10 years. Calculate $P$.

# Unknown Time

**Example (Exercise 2.6)**

Find how long \$1000 should be left to accumulate at 6% effective in order that it will amount to twice the accumulated value of another \$1000 deposited at the same time at 4% effective.

**Example (Exercise 2.7)**

You invest \$3000 today and plan to invest another \$2000 two years from today. You plan to withdraw \$5000 in $n$ years and another \$5000 in n + 5 years, exactly liquidating your investment account at that time. If the effective rate of discount is equal to 6%. find n.

**Example (Exercise 2.8)**

The present value of two payments of \$100 each to be made at the end of $n$ years and $2n$ years is \$100. If $i = 0.08$, find $n$.

**Question:**
Given payments $s_1, \ldots, s_n$ paid at $t_1, \ldots, t_n$, respectively, find the time $t$ such that the single payment $s_1 + \cdots + s_n$ at $t$ is equivalent to the payments $s_1, \ldots, s_n$ made separately.
**Exact Solution:** We discount all the payment to $t = 0$:

$$s_1 v^{t_1} + \cdots + s_n v^{t_n} = (s_1 + \cdots s_n) v^t.$$

So

$$t = \frac{\ln(s_1 v^{t_1} + \cdots + s_n v^{t_n}) - \ln(s_1 + \cdots s_n)}{\ln(v)}.$$

**Approximated Solution (method of equated time):** Replace each $v^{t_i}$ by the simple discount $1 - dt_i$ function:

$$s_1(1 - dt_1) + \cdots + s_n(1 - dt_n) = (s_1 + \cdots s_n)(1 - d\bar{t}).$$

$$(s_1 + \cdots + s_n) - d(s_1 t_1 + \cdots + s_n t_n) = (s_1 + \cdots s_n) - d\bar{t}(s_1 + \cdots s_n).$$

$$s_1 t_1 + \cdots + s_n t_n = \bar{t}(s_1 + \cdots s_n).$$

$$\bar{t} = \frac{s_1 t_1 + \cdots + s_n t_n}{s_1 + \cdots s_n}.$$

**Rule of 72:**

Solve $(1+i)^t = 2$ for $t$: The exact solution is

$$t = \frac{\ln(2)}{\ln(1+i)} = \left(\frac{\ln(2)}{i}\right)\left(\frac{i}{\ln(1+i)}\right)$$

If we approximate $\frac{i}{\ln(1+i)}$ at the median of the interest rates: 0.08, then we have

$$t \approx \frac{\frac{0.08\ln(2)}{\ln(1.08)}}{i} \approx \frac{.7205174674}{i} \approx \frac{72}{100i}$$

i.e. it takes approximately $\frac{72}{\text{percentage interest rate}}$ years for an investment to double.

# Unknown Rate of Interest

**Example (Exercise 2.13)**

Find the nominal rate of interest convertible semiannually at which the accumulated value of \$1000 at the end of 15 years is \$3000.

**Example (Exercise 2.14)**

Find the effective rate of interest at which payments of \$300 at the present, \$200 at the end of one year, and \$100 at the end of two years will accumulate to \$700 at the end of two years.

**Example (Exercise 2.15)**

You can receive one of the following two payment streams:

(i) \$100 at time 0, \$200 at at time $n$, and \$300 at time $2n$.

(ii) \$600 at time 10.

At an annual effective interest rate of $i$. the present values of the two streams are equal. Given $v^n = 0.75941$, determine $i$.

**Example (Exercise 2.16)**

It is known that an investment of \$1000 will accumulate to \$1825 at the end of 10 years. If it is assumed that the investment earns simple interest at rate $i$ during the 1st year, $2i$ during the 2nd year,..., $10i$ during the 10th year, find $i$.

**Example (Exercise 2.17)**

It is known that an amount of money will double itself in 10 years at a varying force of interest $\delta_t = kt$. Find an expression for $k$.

### Example (Exercise 2.18)

The sum of the accumulated value of 1 at the end of three years at a certain effective rate of interest $i$, and the present value of 1 to be paid at the end of three years at an effective rate of discount numerically equal to $i$ is 2.0096. Find the rate $i$.

# Determining Time Periods I

There are three different ways to count number of days:

**Definition**

"**actual/actual**":
Count exact number of days (count also the leap day) and use 365 days per year. Simple interest using "actual/actual" counting method is called **exact simple interest**.

For example, the Canada Treasury Bill's quoted rate is the simple interest rate using the actual/actual rule

$$\text{Face value} = \text{Price}\left(1 + \frac{\text{time}}{365} r\right)$$

https://www.canada.ca/en/department-finance/programs/financial-sector-policy/securities /securities-technical-guide/determining-bond-treasury-bill-prices-yields.html

# Determining Time Periods II

**Definition**

"**30/360**":
Count always 30 days per month and 360 days per year, and use the formula

$$\# \text{ of days} = 360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1).$$

For example, for February, 25, 2018, $Y$ is 2018, $M$ is 2, and $D$ is 25. Simple interest using "30/360" counting method is called **ordinary simple interest**.

# Determining Time Periods III

**Definition**

"**actual/360**":
Count exact number of days and use 360 days per year. Simple interest using "actual/360" counting method is called **banker's rule**.

For example, the U.S. Treasury Bill's quoted rate is the simple discount rate using the banker's rule

$$\text{Price} = \text{Face value}\left(1 - \frac{\text{time}}{360} r\right)$$

https://www.treasurydirect.gov/marketable-securities/understanding-pricing/#id-bills-822455

### Example (Exercise 2.20)

A sum of $\$10,000$ is invested for the months of July and August at 6% simple interest. Find the amount of interest earned:

a) Assuming exact simple interest.
b) Assuming ordinary simple interest.
c) Assuming the Banker's Rule.

# Practical Examples

**Example (Exercise 2.22)**

A bill for \$100 is purchased for \$96 three months before it is due. Find:

a) The nominal rate of discount convertible quarterly earned by the purchaser.

b) The annual effective rate of interest earned by the purchaser.

## Example (Exercise 2.23)

A two-year certificate of deposit pays an annual effective rate of 9%. The purchaser is offered two options for prepayment penalties in the event of early withdrawal:

**A:** a reduction in the rate of interest to 7%.

**B:** loss of three months interest.

In order to assist the purchaser in deciding which option to select, compute the ratio of the proceeds under Option A to those under Option B if the certificate of deposit is surrendered:

a) At the end of 6 months.

b) At the end of 18 months.

## Example (Exercise 2.24)

The ABC Bank has an early withdrawal policy for certificates of deposit (CDs) which states that interest still be credited for the entire length the money actually stays with the bank, but that the CD nominal interest rate will be reduced by 1.8% for the same number of months as the CD is redeemed early. An incoming college freshman invests $5000 in a two-year CD with a nominal rate of interest equal to 5.4% compounded monthly on September 1 at the beginning of the freshman year. The student intended to leave the money on deposit for the full two-year term to help finance the junior and senior years. but finds the need to withdraw it on May 1 of the sophomore year. Find the amount that the student will receive for the CD on that date.

## Example (Exercise 2.26)

A savings and loan association pays 7% effective on deposits at the end of each year. At the end of every three years a 2% bonus is paid on the balance at that time. Find the effective rate of interest earned by an investor if the money is left on deposit:

a) Two years.

b) Three years.

c) Four years.

## Example (Exercise 2.27)

A bank offers the following certificates of deposit (CDs):

| Term years | Nominal Annual interest rate (convertible semiannually) |
|---|---|
| 1 | 5% |
| 2 | 6% |
| 3 | 7% |
| 4 | 8% |

The bank does not permit early withdrawal, and all CDs mature at the end of the term. During the next six years the bank will continue to offer these CDs. An investor deposits $1000 in the bank. Calculate the maximum amount that can be withdrawn at the end of six years.