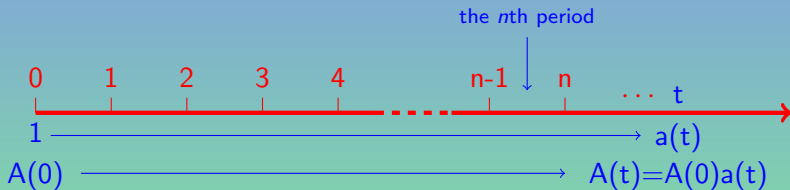


# Time Diagram

Time diagram is a very useful tool to analyze the values of investments. Almost all the problem need to be solved by drawing a time diagram first.



## Accumulation and Amount Functions

The investment amount  $a(t)$  at time  $t$  of the initial investment 1 is called the accumulation function.

The investment amount  $A(t)$  at time  $t$  of any initial investment is called the amount function.

clearly

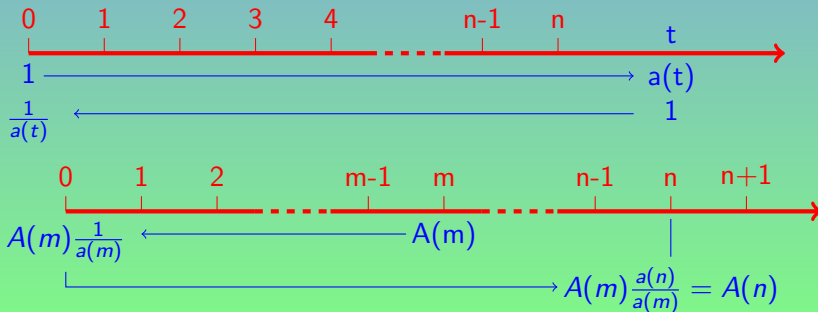
$$A(t) = A(0)a(t) \quad \text{and} \quad a(t) = \frac{A(t)}{A(0)}.$$

# Discount Function

## Discount Function

The discount function  $b(t)$  is a function which gives the accumulated value 1 at time  $t$  of an original investment of  $b(t)$

Clearly  $b(t) = \frac{1}{a(t)}$ .

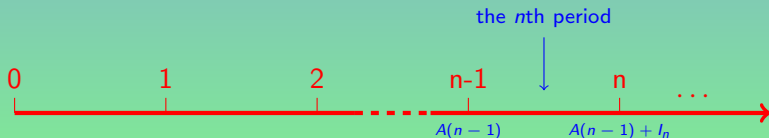


# Interest

## Interest

The amount of increasing of the investment during the  $n$ th period is called the interest eared during that period, i.e.

$$I_n = A(n) - A(n-1) \quad \text{and} \quad A(n) = A(n-1) + I_n$$



# Effective Rate of Interest

## Effective Rate of Interest

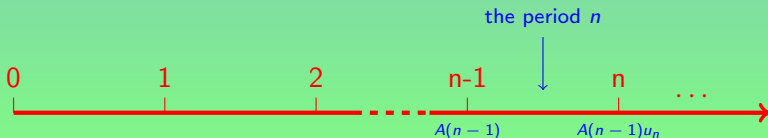
The effective rate of interest during the  $n$ th period is the ratio of interest to the beginning amount of the period, i.e.

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{\text{interest}}{\text{initial amount}}.$$

We have  $A(n) = A(n-1)(1 + i_n) = A(n-1)u_n$  where

$$u_n = 1 + i_n = \frac{A(n)}{A(n-1)} = \frac{a(n)}{a(n-1)}$$

is called the accumulation factor for the  $n$ th period  $[n-1, n]$ .



### Example (Exercise 1.1)

Consider the amount function  $A(t) = t^2 + 3t + 5$

- a) Find the corresponding accumulation function  $a(t)$ .
- b) Find  $I_n$  and  $i_n$ .

### Example (Exercise 1.4)

It is known that  $a(t)$  is of the form of  $kt^2 + b$ . If \$100 invested at time 0 accumulates to \$172 at time 3, find the accumulated value of \$100 at time 10 if it is invested at time 5.

### Example (Exercise 1.5)

Assume that  $A(t) = 350 + 3t$ .

- a) Find  $i_4$ .
- b) Find  $i_{11}$ .



### Example (Exercise 1.6)

Assume that  $A(t) = 250(1.09)^t$ .

- a) Find  $i_4$ .
- b) Find  $i_{11}$ .

### Example (Exercise 1.8)

Assume that  $A(4) = 2500$  and  $i_n = 0.05n$ . Find  $A(9)$ .

# Simple Interest

## Simple Interest

The accumulation function of the simple interest is given by

$$a(t) = 1 + it,$$

where  $i$  is the interest rate, and  $t$  is the time.

Note that the effective rate of a simple interest investment is not constant. It is decreasing:

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1 + in) - (1 + i(n-1))}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}.$$

## Effective Rate of a Simple Interest $i$

$$i_n = \frac{i}{1 + i(n-1)}$$

### Example (Exercise 1.9)

- a) At what rate of simple interest will \$500 accumulate to \$620 in 800 days?
- b) In how many years will \$500 accumulate to \$630 at 7.8% simple interest?

### Example (Exercise 1.10)

At a certain rate of simple interest \$1,000 will accumulate to \$1,110 after a certain period of time. Find the accumulated value of \$500 at a rate of simple interest  $\frac{3}{4}$  as great over twice as long a period of time.

### Example (Exercise 1.11)

Simple interest of  $i = 4\%$  is being credited to a fund. In which period is this equivalent to an effective rate of  $2.5\%$ ?

# Compound Interest I

If the interest earned is periodically reinvested, then we have the compound interest. Assume the initial investment 1, at the end of the 1st period, the amount becomes  $(1 + i)$ . Then if we reinvest the whole  $(1 + i)$  at the beginning of the 2nd period, then at the end of the 2nd period, the amount becomes  $(1 + i)(1 + i) = (1 + i)^2$ , etc. So if we reinvest the whole amount at the end of every period, then at the end of the  $n$ th period, the amount is

$$(1 + i)^n.$$

# Compound Interest II

## Compound Interest

The accumulation function of the compound interest compound once per period is given by

$$a(t) = (1 + i)^t,$$

where  $i$  is the interest rate, and  $t$  is the time.

Compound interest is used almost exclusively for financial transactions over one or multi-years. Simple interest is occasionally used for short-term transactions over a fractional period. So unless stated otherwise, we will use compound interest.

Note that for the accumulation function of compound interest  $a(t) = (1 + i)^t$ ,  $a(0) = 1$  and  $a(1) = 1 + i$ . The linear function connect the points  $(0, 1)$  and  $(1, 1 + i)$  is  $1 + it$ , which is the accumulation function of the simple interest.



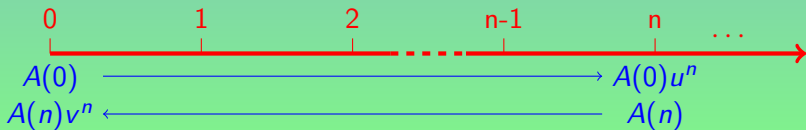
For compound interest, the future value  $A(n)$  at the time  $n$  of an initial investment  $A(0)$  is given by

$$A(n) = A(0)u^n$$

where  $u = 1 + i$  is the accumulation factor per period. Conversely, the present value  $A(0)$  of a future amount  $A(n)$  at the time  $n$  is given by

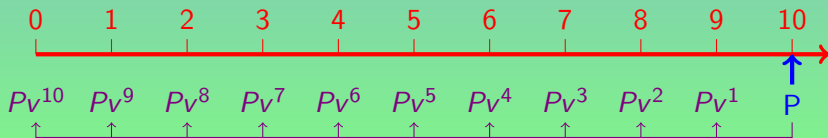
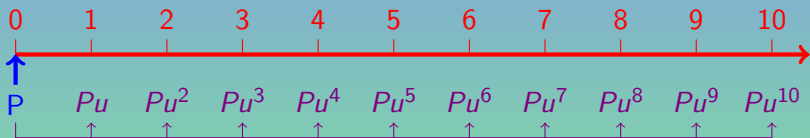
$$A(0) = A(n)v^n$$

where  $v = \frac{1}{u} = \frac{1}{1+i}$  is called the discount factor per period.



## Payment Moving Rule

For compound interests, if a payment  $P$  is moved  $n$  periods to the right (future), then its value becomes  $Pu^n$ , and if it is moved  $m$  periods to the left (past), then its value becomes  $Pv^m$



What about in the middle of a period when  $t = n + k$ ,  $0 < k < 1$ ?  
There are 2 ways to do it:

### Compound Interest Throughout

Compound interest throughout:

$$A(t) = A(0)u^t, \quad t = n + k.$$

### Compound Interest for Whole Periods Only

Compound interest with simple interest during final fractional period:

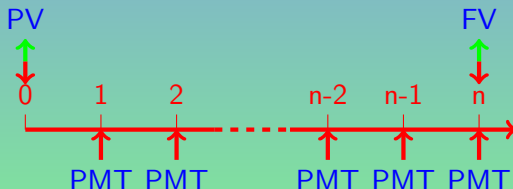
$$A(t) = A(0)u^n(1 + ki).$$

# TVM Solver

The TVM (time-value of money) solver of a financial calculator can compute any one of the following values if the other values are given: present value (PV); periodic payments (PMT); future value (FV); nominal percentage interest rate (I); the number of total periods (n).

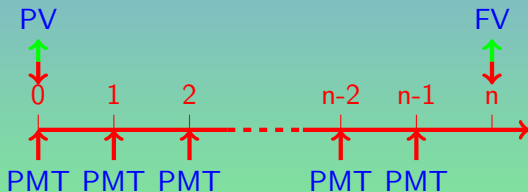
## TVM Solver: End Mode

Here is the time diagram of payments represented by the end mode of TVM solver. It is called the end mode because the  $k$ th periodic payment is at the end of the  $k$ th period  $[k - 1, k]$



## TVM Solver: Beginning Mode

Here is the time diagram of payments represented by the beginning mode of the TVM solver. It is called the beginning mode because the  $k$ th periodic payment is at the beginning of the  $k$ th period  $[k - 1, k]$ .



We can use either mode if  $PMT=0$ .

Keep in mind that **cash flow in different directions must have opposite signs**. For example, if the amount of a loan is positive, then the payments to the bank will be negative.

One easy way to remember this is: if the quantities on one side of an equation keep their signs, then the quantities on the other side of the equation need to have the opposite of their signs. For example,

$$100u^{10} = FV$$

If we want the future value  $FV$  to be positive, then we need to input the present value 100 as  $-100$ , and vice versa.

### Example (Exercise 1.13)

It is known that \$600 invested in compound interest for two years will earn \$264 interest. Find the accumulated value of \$2000 invested at the same rate of compound interest for three years.



### Example (Example 1.6)

Find the accumulated value of \$8000 at the end of 6 years and 7 months invested at a rate of 8.5% compound interest per annum

- (1) Assuming compound interest throughout.
- (2) Assuming simple interest during the final fractional period.

# Present Value I

To find how much an initial invest must be so that the balance will be  $B$  at the end of the investment, we just need to multiply  $B$  by the corresponding discount function:

$$A(0) = Bb(t) = \frac{B}{a(t)}.$$



For compound interest, the present value (PV) can be easily solved by a TVM solver.

## Discount Functions

Discount Function for Simple Interest:  $b(t) = \frac{1}{1 + it}$ ,

Discount Function for Compound Interest:  $b(t) = \frac{1}{(1 + i)^t} = v^t$ ,

where

$$v = \frac{1}{1 + i} = \frac{1}{u}$$

is the discount factor per period.

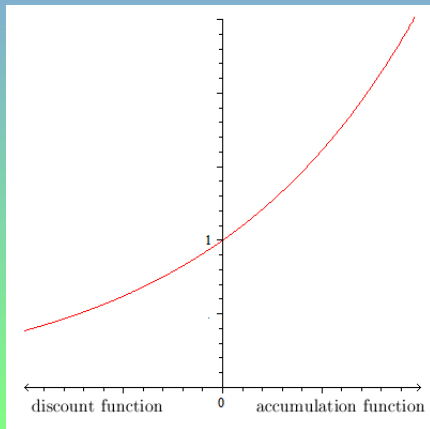
### Example (Exercise 1.19)

It is known that an investment of \$500 will increase to \$4,000 at the end of 30 years. Find the sum of the present values of three payments of \$10,000 each which will occur at the end of 20, 40, and 60 years.

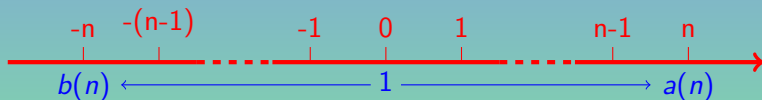
Note that the discount function for compound interest is

$$b(t) = \frac{1}{(1+i)^t} = (1+i)^{-t} = a(-t)$$

i.e. for compound interest, the discount function is the accumulation function with negative time.



We can visualize the accumulation and discount functions as following:



# The Effective Rate of Discount I

There are two ways to borrow money:

If a person borrows \$100 for one year at an effective rate 7% of interest, he/she receives \$100 and pay back \$107 one year later.

If a person borrows \$100 for one year at an effective rate 7% of discount, he/she receives \$93 and pay back \$100 one year later.  
The effective rate of interest is actually

$$\frac{100 - 93}{93} = 7.53\%.$$

## Effective Rate of Discount

The effective rate of discount during the  $n$ th period is the ratio of the interest earned to the investment amount at the **end** of the period, i.e.

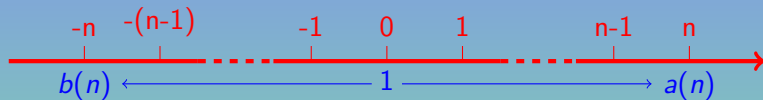
$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{a(n) - a(n-1)}{a(n)} = \frac{\text{interest}}{\text{ending amount}}.$$

Since  $a(t) = \frac{1}{b(t)}$ , we also have

$$d_n = \frac{\frac{1}{b(n)} - \frac{1}{b(n-1)}}{\frac{1}{b(n)}} = \frac{b(n-1) - b(n)}{b(n-1)}$$



Under this picture



for both  $f(t) = a(t)$  and  $f(t) = b(t)$ , we have

$$i_n = \frac{f(\text{right}) - f(\text{left})}{f(\text{left})}, \quad d_n = \frac{f(\text{right}) - f(\text{left})}{f(\text{right})}.$$

For compound interest,  $a(n) = u^n$  and  $a(n-1) = u^{n-1} = u^n v = a(n)v$ . so

$$d = \frac{a(n) - a(n-1)}{a(n)} = \frac{a(n) - a(n)v}{a(n)} = 1 - v$$

We have

$$i = u - 1 = u(1 - v) = ud, \quad \text{and} \quad d = \frac{1}{u}i = vi$$

accumulation factor:

$$u = 1 + i = \frac{1}{1 - d}$$

Discount factor:

$$v = 1 - d = \frac{1}{1 + i}$$

Relationship between  $u$  and  $v$ :

$$u = \frac{1}{v}, \quad v = \frac{1}{u}$$

Relationship between  $i$  and  $d$ :

$$i = \frac{d}{v}, \quad d = \frac{i}{u}.$$

People also use the simple discount (**it is not the discount function for simple interest!**), where the discount function is the linear function which has the same values at  $t = 0$  and  $t = 1$  as the discount function of the compound interest  $b(t) = v^t = (1 - d)^t$

**Simple discount function:**  $b(t) = 1 - dt$  for  $0 \leq t < \frac{1}{d}$

(in order to keep the present value positive). Simple discount is only used for short time transactions and as an approximation for compound discount over fractional period. It is not as widely used as simple interest.

### Example (Exercise 1.20)

- a) Find  $d_4$  if the rate of simple interest is 12%.
- b) Find  $d_4$  if the rate of simple discount is 12%.

### Example (Exercise 1.21)

Find the annual effective rate of discount at which a payment of \$200 immediately and \$300 one year from today will accumulate to \$600 two years from today.

### Example (Exercise 1.22)

The amount of interest earned on  $A$  is \$336, while the equivalent discount on  $A$  is \$300. Find  $A$ .

# Nominal Rates of Interest and Discount I

## Definition

For compound interest or discount payable (or compound, or convertible)  $m$  times per year, the given rate per year is called the nominal rate of interest, and is denoted by

$$i^{(m)}.$$

The  $\frac{1}{m}$ th of a year used to compute the interest is called interest conversion period.

For compound interest or discount payable  $m$  times per year, the interest used in each of the interest conversion period is

$$\frac{\text{nominal rate of interest}}{m} = \frac{i^{(m)}}{m}.$$



# Nominal Rates of Interest and Discount II

## Definition

The accumulation function for nominal rate of interest  $i^{(m)}$  compound  $m$  times per year is given by

$$a(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{mt}.$$

## Nominal Rates of Interest and Discount III

So the accumulation factor per year is given by

$$u = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

and the accumulation factor per conversion period is given by

$$u = 1 + \frac{i^{(m)}}{m}$$

## Nominal Rates of Interest and Discount IV

### Definition

The discount function for nominal rate of discount  $d^{(m)}$  convertible  $m$  times per year is given by

$$b(t) = \left(1 - \frac{d^{(m)}}{m}\right)^{mt}.$$

# Nominal Rates of Interest and Discount $V$

So the discount factor per year is given by

$$v = \left(1 - \frac{d^{(m)}}{m}\right)^m$$

and the discount factor per conversion period is given by

$$v = 1 - \frac{d^{(m)}}{m}$$

# Nominal Rates of Interest and Discount VI

We can easily find the relationship between different nominal rate of interests and/or discounts through a common accumulation or discount factor.

For examples, for an effective interest rate  $i$  per year,

$$1 + i = u = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

is the accumulation factor per year.

For an effective discount rate  $d$  per year,

$$1 - d = v = \left(1 - \frac{d^{(m)}}{m}\right)^m$$

is the discount factor per year.

## Nominal Rates of Interest and Discount VII

For any  $m$  and  $k$ ,

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = \frac{1}{\left(1 - \frac{d^{(k)}}{k}\right)^k}$$

because both of them are the accumulation factor per year.

## Nominal Rates of Interest and Discount VIII

Since  $\frac{i^{(m)}}{m}$  and  $\frac{d^{(m)}}{m}$  are the effective rates of interest and discount per conversion period, we have  $\frac{i^{(m)}}{m} = \frac{\frac{d^{(m)}}{m}}{v}$  and  $\frac{d^{(m)}}{m} = \frac{\frac{1^{(m)}}{m}}{u}$ , i.e.,

$$i^{(m)} = \frac{d^{(m)}}{v}, \quad \text{and} \quad d^{(m)} = \frac{i^{(m)}}{u}$$

where  $u = 1 + \frac{i^{(m)}}{m}$  and  $v = 1 - \frac{d^{(m)}}{m}$  are the accumulation and discount factors per conversion period, respectively.

$$i^{(m)} = \frac{d^{(m)}}{\text{discount factor per conversion period}}.$$

$$d^{(m)} = \frac{i^{(m)}}{\text{accumulation factor per conversion period}}$$

## Nominal Rates of Interest and Discount IX

Note that for any  $n$ ,  $PVu^n = \left(1 + \frac{i^{(m)}}{m}\right)^{kn} = FV$ . We can write nominal rate of discount to the similar form:

$$PV \frac{1}{\left(1 - \frac{d^{(m)}}{m}\right)^{mn}} = PV \left(1 + \frac{-d^{(m)}}{m}\right)^{m(-n)} = FV$$

**If there are no periodic payments**, then a TVM solver can be used to calculate values involve nominal rate of discount directly without changing to the nominal rate of interest, just enter the negative of the discount rate in the field of the “Interest Rate” and negative of the “Number of Payments”.



### Example (Exercise 1.26)

- a) Express  $d^{(4)}$  as a function of  $i^{(3)}$ .
- b) Express  $i^{(6)}$  as a function of  $d^{(2)}$ .

### Example (Exercise 1.28)

Find the accumulated value of \$100 at the end of two years:

- a) If the nominal annual rate of interest is 6% convertible quarterly.
- b) If the nominal annual rate of discount is 6% convertible once every 4 years.

### Example (Exercise 1.29)

Given that  $i^{(m)} = 0.1844144$  and  $d^{(m)} = 0.1802608$ , find  $m$ .

# Forces of Interest I

## Definition

The force of interest  $\delta(t)$  is defined by

$$\delta(t) = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)}.$$

$\delta(t)$  is the relative rates of change of the accumulation function  $a(t)$ .

From the definition we can see that

$$\delta(t) = \frac{d}{dt} \ln a(t).$$

So

$$\int_0^t \delta(r) \, dr = \ln a(t) - \ln a(0) = \ln a(t)$$

## Forces of Interest II

and

$$a(t) = e^{\int_0^t \delta(r) dr}.$$

$$\delta(t) = \frac{d}{dt} \ln a(t) = -\frac{d}{dt} \ln b(t)$$

$$a(t) = e^{\int_0^t \delta(r) dr}$$

$$b(t) = e^{-\int_0^t \delta(r) dr}$$

## Forces of Interest III

The force of interest is a constant for the compound interest because

$$\delta = \frac{d}{dt} \ln(1+i)^t = \frac{d}{dt} t \ln(1+i) = \ln(1+i).$$

For compound interest or discount:

$$\delta = \ln(1+i) = -\ln(1-d).$$

$$u = 1+i = e^{\delta}, \quad a(t) = u^t = e^{\delta t}.$$

$$v = \frac{1}{1+i} = e^{-\delta}, \quad b(t) = v^t = e^{-\delta t}.$$

## Forces of Interest IV

Note that  $\lim_{n \rightarrow \infty} \left(1 + \frac{i}{n}\right)^{nt} = e^{it}$ . So  $\delta$  is also called the nominal rate of interest compound continuously.

# Forces of Interest V

For simple interest

$$\delta(t) = \frac{d}{dt} \ln(1 + it) = \frac{i}{1 + it}$$

and for simple discount

$$\delta(t) = -\frac{d}{dt} \ln(1 - dt) = \frac{d}{1 - dt}.$$



### Example (Example 1.15)

Find the accumulation function if the force of interest is

$$\delta(t) = \frac{1}{1+t^2}$$

### Example (Exercise 1.34)

Fund  $A$  accumulates at a simple interest rate of 10%. Fund  $B$  accumulates at a simple discount rate of 5%. Find the point in time  $t$  at which the forces of interests on the two funds are equal.

### Example (Exam FM Sample Question 135)

At time 0, Cheryl deposits  $X$  into a bank account that credits interest at an annual effective rate of 7%. At time 3, Gomer deposits 1000 into a different bank account that credits simple interest at an annual rate of  $y\%$ . At time 5, the annual forces of interest on the two accounts are equal, and Gomer's account has accumulated to  $Z$ . Calculate  $Z$ .

### Example (Exam FM Sample Question 13)

Ernie makes deposits of 100 at time 0, and  $X$  at time 3. The fund grows at a force of interest

$$\delta_t = \frac{t^2}{100}, \quad t > 0.$$

The amount of interest earned from time 3 to time 6 is also  $X$ . Calculate  $X$ .

# Varying Interest

## Example (Exercise 1.37)

Find the level effective rate of interest over a three-year period which is equivalent an effective rate of discount 8% the first year, 7% the second year, and 6% the third year.

### Example (Exercise 1.39)

An investor makes a deposit today and earns an average continuous return (force of interest) of 6% over the next 5 years. What average continuous return must be earned over the subsequent 5 years in order to double the investment at the end of 10 years?