Homework 1

1. Let X and Y be independent binomial random variables having parameters (N, p) and (M, p), respectively. Find the distribution of $Z = X + Y$.

Proof.

$$\begin{aligned}
\mathbb{P}(Z=k) &= \sum_{i=0}^{k} \mathbb{P}(X=i, Y=k-i) \\
&= \sum_{i=0}^{k} \binom{N}{i} p^i (1-p)^{N-i} \binom{M}{k-i} p^{k-i} (1-p)^{M-k+i} \\
&= p^k (1-p)^{N+M-k} \sum_{i=0}^{k} \binom{N}{i} \binom{M}{k-i} \\
&= p^k (1-p)^{N+M-k} \binom{N+M}{k}
\end{aligned}$$

∎

2. Let $X_1, X_2, \cdots, X_n$ be independent random variables that are exponentially distributed with respective parameters $\lambda_1, \cdots, \lambda_n$. Identify the distribution of the minimum $V = \min(X_1, X_2, \cdots, X_n)$.

Proof. We first let $n = 2$.

$$\begin{aligned}
&\mathbb{P}(V \geq x) = \mathbb{P}(X_1 \geq x, X_2 \geq x) \\
&= e^{-\lambda_1 x} e^{-\lambda_2 x} = e^{-(\lambda_1+\lambda_2)x}
\end{aligned}$$

V is an exponential distribution with $\lambda = \lambda_1 + \lambda_2$. For the case n, it is easy to see that V is an exponential distribution with $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. ∎

3. Let N have a Poisson distribution with parameter $\lambda = 1$. Conditioned on $N = n$, let X have a uniform distribution over $\{0, \cdots, n+1\}$. What is the marginal distribution for X?

Proof. Note that

$$\begin{aligned}
\mathbb{P}(X=x) &= \sum_{n=0}^{\infty} \mathbb{P}(X=x|N=n)\mathbb{P}(N=n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n+2} I(x \in \{0, \cdots, n+1\}) e^{-\lambda} \frac{\lambda^n}{n!}
\end{aligned}$$

$$= e^{-\lambda} \sum_{n=x-1}^{\infty} \frac{\lambda^n}{n!(n+2)}.$$

∎

4. A card is picked at random from N cards labeled $1, \cdots, N$, and the number that appears is X. A second card is picked at random from cards numbered $1, \cdots, X$ and its number is Y. Determine the conditional distribution of X given Y.

Proof. Note that X is uniform on $\{1, \cdots, N\}$ and given X, Y is uniform on $\{1, \cdots, X\}$. Therefore

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(Y = y|X = x)\mathbb{P}(X = x) = \frac{1}{Nx} I(y \leq x).$$

Now we have

$$\begin{aligned}
\mathbb{P}(X = x|Y = y) &= \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)} \\
&= \frac{\mathbb{P}(X = x, Y = y)}{\sum_x \mathbb{P}(X = x, Y = y)} = \frac{1}{Nx * \sum_{z=y}^{N} \frac{1}{Nz}} I(y \leq x) \\
&= \frac{\frac{1}{x} I(x \geq y)}{\sum_{x=y}^{N} \frac{1}{x}}
\end{aligned}$$

∎

5. Let $D_1, D_2, \cdots$ be independent random variables each uniformly distributed over the interval $(0, 1]$. Show that $X_0 = 1$ and $X_n = 2^n D_1 D_2 \cdots D_n$ for $n = 1, 2, \cdots$ defines a martingale.

Proof. Note that

$$\mathbb{E}[X_{n+1}|X_n] = \mathbb{E}[2D_n X_n|X_n] = X_n \mathbb{E}[2D_n] = X_n.$$

∎

6. Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ be the sum of n independent random variables, each exponentially distributed with mean $E[\varepsilon] = 1$. Show that

$$X_n = 2^n \exp(-S_n), n \geq 0$$

defines a martingale.

Proof. Note that

$$\mathbb{E}[X_{n+1}|X_n] = \mathbb{E}[2X_n \exp(-\varepsilon_{n+1})|X_n] = X_n \mathbb{E}[2 \exp(-\xi_{n+1})] = X_n.$$

∎

7. Suppose that X and Y are independent random variables, each having the same exponential distribution with parameter α. What is the conditional probability density function for X, given that $X + Y = z$?

Proof. Let $U = X + Y$ and $W = X$. Then $X = W$ and $Y = U - W$. The Jacobian is 1. Therefore the joint density (U, W) is

$$f_{U,W}(u, w) = \alpha^2 e^{-\alpha w} e^{-\alpha(u-w)} I(u \geq w) = \alpha^2 e^{-\alpha u} I(u \geq w)$$

Then

$$f_{W|U}(w|u) = \frac{\alpha^2 e^{-\alpha u} I(u \geq w)}{\int_0^\infty \alpha^2 e^{-\alpha u} I(u \geq w) dw} = \frac{I(w \leq u)}{u}$$

∎

8. Let N have a Poisson distribution with parameter $\lambda > 0$. Suppose that, conditioned on $N = n$, the random variable X is binomially distributed with parameters (n, p). Set $Y = N - X$. Show that X and Y have Poisson distributions with respective parameters λp and $\lambda(1 - p)$ and that X and Y are independent.

Proof. Note that

$$\mathbb{P}(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}, \ \mathbb{P}(X = x|N = n) = \binom{n}{x} p^x (1 - p)^{n-x} I(x \in \{0, \cdots, n\}).$$

Then it follows that

$$\begin{aligned}
\mathbb{P}(X = x) &= \sum_{n=0}^{\infty} \mathbb{P}(N = n)\mathbb{P}(X = x|N = n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \binom{n}{x} p^x (1 - p)^{n-x} I(x \in \{0, \cdots, n\}) \\
&= \frac{e^{-\lambda} p^x (1 - p)^{-x}}{x!} \sum_{n=x}^{\infty} \frac{[\lambda(1 - p)]^n}{(n - x)!} \\
&= \frac{e^{-\lambda} p^x (1 - p)^{-x}}{x!} * e^{\lambda(1-p)} [\lambda(1 - p)]^x \\
&= \frac{e^{-\lambda p} (\lambda p)^x}{x!}.
\end{aligned}$$

This suggests that X is a Poisson distribution with parameter λp. Following the same procedure, noting that Y is a Binomial distribution with parameter $(n, 1-p)$ given $N = n$, Y is also a Poisson distribution with parameter $\lambda(1 - p)$.

Note that

$$\begin{aligned}
\mathbb{P}(X = x, Y = y) &= \sum_{n=0}^{\infty} \mathbb{P}(N = n)\mathbb{P}(X = x, Y = y|N = n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \binom{n}{x} p^x (1 - p)^{n-x} I(x \in \{0, \cdots, n\}) I(n = x + y) \\
&= e^{-\lambda} \frac{\lambda^{x+y}}{(x + y)!} \binom{x + y}{x} p^x (1 - p)^y \\
&= e^{-\lambda} \frac{\lambda^{x+y}}{x!y!} p^x (1 - p)^y
\end{aligned}$$

$$= e^{-\lambda}\frac{[\lambda p]^x}{x!}\frac{[\lambda(1-p)]^y}{y!}.$$

This shows that X and Y are independent. ∎

9. Suppose that $\{\xi_i\}$ are independent and identically distributed with $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = 1/2$. Let N be independent of $\{\xi_i\}$ and follow the geometric probability with parameter α. Let $Z = \xi_1 + \cdots + \xi_N$. Find $\mathbb{E}[Z^3]$.

Proof. Note that $\mathbb{E}[N^2] = (2-\alpha)/\alpha^2$ and $\mathbb{E}N = 1/\alpha$ where $\alpha = 1/2$. We also see $\mathbb{E}\xi_1 = \mathbb{E}\xi_1^3 = 0$ and $\mathbb{E}\xi_1^2 = \mathbb{E}\xi_1^4 = 1$.

$$\mathbb{E}[Z^3] = \mathbb{E}[\mathbb{E}[Z^3|N]] = 0.$$

$$\mathbb{E}[Z^4] = \mathbb{E}[\mathbb{E}[Z^4|N]] = \mathbb{E}[N + 6N(N-1)] = 6(2-\alpha)/\alpha^2 - 5/\alpha.$$

∎