Homework 2

Due 10/13/2023 Friday

1. Suppose f and g are two measurable functions on $(\Omega, \mathcal{F})$. Prove that $f+g$ is also measurable.

Answer: Suppose f_n and g_n are simple functions with limits f and g respectively. Then it is easy to see that $f_n + g_n$ has the limit $f + g$. Therefore it suffices to show that $f_n + g_n$ is also measurable. This is obvious because f_n and g_n are both simple functions.

2. State Fatou's lemma. Give an example that Fatou's lemma fails if $f_n \geq 0$ is not satisfied.

Answer: Consider $f_n(x) = -1 \times I(x \in (n, n+1))$. It is easy to see that $f_n \to 0$ a.s. While

$$\int f_n dx = -1 < \int f dx = 0,$$

which violates the Fatou's lemma.

3. State the dominant convergence theorem. Give an example that the theorem fails if g is not integrable.

Answer: Consider $f_n(x) = -1 \times I(x \in (n, n+1))$. It is easy to see that $f_n \to 0$ a.s. with $|f_n| \le 1$, While $\int 1dx = \infty$.

$$\int f_n dx = -1 < \int f dx = 0,$$

which violates the dominant convergence theorem.

4. In the probability measure space $(\Omega, \mathcal{F}, P)$ (i.e. $P(\Omega = 1)$). Prove that if $X_n \to X$ in probability and $|X_n|$ is uniformly bounded, the $\mathbb{E}X_n \to \mathbb{E}X$.

Answer: Note that for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}|X_n - X| &= \mathbb{E}|X_n - X| I(|X_n - X| \geq \varepsilon) + \mathbb{E}|X_n - X| I(|X_n - X| \leq \varepsilon) \\ &\leq K\mathbb{P}(|X_n - X| \geq \varepsilon) + \varepsilon\end{aligned}$$

By the convergence in probability, we have

$$\limsup_{n\to\infty} \mathbb{E}|X_n - X| \leq \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we know $\mathbb{E}|X_n - X| \to 0$. Consequently, $\mathbb{E}X_n \to \mathbb{E}X$.

5. Suppose X_n is a uniformly bounded sequence of real-valued random variables. If $X_n \Longrightarrow X$, then $\mathbb{E}X_n \to \mathbb{E}X$.

Answer: Without loss of generality, we suppose $X_n, X \geq 0$. Otherwise, we work with $X_n + M$ for some large $M > 0$.

In this case, for some $M > 0$, (we select such that $F_n(M) = F(M) = 1$)),

$$\mathbb{E}X_n = \int_0^\infty (1 - F_n(x))dx = \int_0^M (1 - F_n(x))dx$$

Since $F_n(x)$ converges to $F(x)$ on the continuous points of $F(x)$ and F has at most countable many discontinuous points, $F_n \to F$ a.s. (with respect Lebesgue measure). Note that $|1 - F_n| \leq 1$ and $\int_0^M 1dx < \infty$, dominant convergence theorem yields that

$$\mathbb{E}X_n = \int_0^M (1 - F_n(x))dx \to \int_0^M (1 - F(x))dx = \mathbb{E}X.$$

The proof is complete.

6. Present an example that $\{f_n\}$ is uniformly integrable, while $\{f_n\}$ can not be bounded by an integrable function.

Let $s_n = \sum_{k=1}^{n-1} \frac{1}{k^2}$ and $f_n(x) = nI(x \in [s_n, s_n + \frac{1}{n^2}])$. Because

$$\int f_n^2(x)ds = 1$$

for all n, f_n is uniformly integrable. While $g = \sup_n f_n = \sum_n f_n$ with

$$\int gdx = \sum \int f_n dx = \sum \frac{1}{n} = \infty.$$

7. Prove that if $X_n \to^P X$ and $Y_n \Longrightarrow Y$, then $X_n + Y_n \Longrightarrow X + Y$.

The problem has a typo. The original problem has the following counter example.

Let X be a standard normal ε_1 and $X_n = \varepsilon_1$. It is obvious that $X_n \to X$ in probability. Let $Y_n = -\varepsilon_1$ for n being odd and $Y_n = \varepsilon_2$ for n being even. Then $X_n + Y_n = 0$ for n being odd and $X_n + Y_n$ is $\varepsilon_1 + \varepsilon_2$ which is a normal distribution with mean 0 and variance 2. Such a sequence has no limit in distribution.

7. Prove that if $X_n \to^P a$ and $Y_n \Longrightarrow Y$, then $X_n + Y_n \Longrightarrow a + Y$.

The above is the so-called Slutsky's theorem

$$\limsup_n \mathbb{P}(X_n + Y_n \le z) = \limsup_n \Big(\mathbb{P}(X_n + Y_n \le z, |X_n - a| \le \varepsilon) + \mathbb{P}(X_n + Y_n \le z, |X_n - a| > \varepsilon)\Big)$$
$$\le \limsup_n \mathbb{P}(|X_n - a| \le \varepsilon) + \limsup_n \mathbb{P}(Y_n \le z - a + \varepsilon) \le \mathbb{P}(Y \le z + a + \varepsilon)$$

Here we note that $(-\infty, z + a]$ is a closed set.

$$\liminf_n \mathbb{P}(X_n + Y_n > z) - \liminf_n \Big(\mathbb{P}(X_n + Y_n > z, |X_n - a| \le \varepsilon) + \mathbb{P}(X_n + Y_n > z, |X_n - a| > \varepsilon)\Big)$$
$$\ge \liminf_n \mathbb{P}(Y_n > z - a - \varepsilon) \ge \mathbb{P}(Y > z - a - \varepsilon)$$

Here we note that $(z - a - \varepsilon, \infty)$ is an open set.

When $z - a$ is a continuous point of the distribution of Y (i.e. z is a continuous point of the distribution function of $Y + a$). we have

$$\lim_n \mathbb{P}(X_n + Y_n \le z) = \mathbb{P}(Y + a \le z).$$

The proof is complete

8. Provide an example that $X_n \Longrightarrow X$ and $Y_n \Longrightarrow Y$, while $X_n + Y_n \Longrightarrow X + Y$ fails.
Solution: Suppose ε_i are i.i.d standard normal variable.
Let $X_i = \varepsilon_1$ and $Y_i = \varepsilon_2$. $X = Y = \varepsilon_3$. $X_i + Y_i$ is $N(0, 2)$ while $X + Y = 2\varepsilon_3$ is $N(0, 4)$.