Homework 1

Due 9/22/2025 Monday

1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-fields. Prove that $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field. Present an example showing that $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field necessarily.

2. (1) Let $\{A_i\}$ be a sequence of events with $A_i \subset A_{i+1}$ with $A = \cup_n A_n$. Prove that

$$P(A) = \lim_{n\to\infty} P(A_n).$$

(2) Let X be a random variable with $P(X > 0) > 0$. Prove that there is a $\delta > 0$ such that $P(X \geq \delta) > 0$.

3. Write the definition of the outer measure of P on a field $\mathcal{F}_0$. Prove that if A_i are disjoint and P^*-measurable, it follows that

$$P^*(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P^*(A_i).$$

4. Write down the definitions of convergence in probability and w.p.1. (1) Prove that if X_n is decreasing and converges to X in probability, then $X_n \to X$ w.p.1.

(2) Suppose that X_n is decreasing and bounded from below, then X_n admits a limit w.p. 1.

5. Denote $A - B = A \cap B^c$. Prove that

$$\limsup_{n\to\infty} A_n - \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} (A_n \cap A_{n+1}^c).$$

6. Prove that

$$\mathcal{A} = \{\cup_{i=1}^{n} (a_i, b_i] : 0 < a_i \leq b_i \leq a_{i+1} \leq b_{i+1} \leq 1 \text{ for all } i \text{ and } n\}$$

is a field on $(0, 1]$ but not a sigma-field.

7. State and prove the 0-1 law.