Homework 1

September 24, 2025

1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-fields. Prove that $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field. Present an example showing that $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field necessarily.

Proof. (1) Note that $\emptyset \in \mathcal{F}_1, \mathcal{F}_2$ and therefore $\emptyset \in \mathcal{F}_1 \cap \mathcal{F}_2$. If $A \in \mathcal{F}_1 \cap \mathcal{F}_2$, $A^c \in \mathcal{F}_1, \mathcal{F}_2$. This says $A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$. If $A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$, then $\cup A_i \in \mathcal{F}_1, \mathcal{F}_2$. This says $\cup A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$. By the definition, $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field.

(2) Let $\Omega = \{1, 2, 3\}$. Let $\mathcal{F}_1 = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{F}_2 = \{\emptyset, \{2\}, \{1, 3\}, \{1, 2, 3\}\}$. It is straightforward to check $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field.

2. (1) Let $\{A_i\}$ be a sequence of events with $A_i \subset A_{i+1}$ with $A = \cup_n A_n$. Prove that

$$P(A) = \lim_{n\to\infty} P(A_n).$$

(2) Let X be a random variable with $P(X > 0) > 0$. Prove that there is a $\delta > 0$ such that $P(X \geq \delta) > 0$.

Proof. (1) Let $B_1 = A_1$ and $B_i = A_i \cap A_{i-1}^c$ for $i = 2, 3, \cdots$. We see that $\{B_i : i = 1, 2, \cdots\}$ are disjoint with $\cup_{i=1}^n B_i = A_n$. Then

$$P(A) = P(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P(\cup_{i=1}^{n} B_i) = \lim_{n\to\infty} P(A_n).$$

(2) Note that $\{X > 0\} = \cup_{i=1}^{\infty}\{X \geq \frac{1}{i}\}$ and $\{X \geq \frac{1}{i}\}$ is increasing. By (1), we have

$$\lim_{i\to\infty} P(X \geq \frac{1}{i}) = P(X > 0) > 0.$$

There exists a i such that $P(X \geq \frac{1}{i}) > 0$.

3. Write the definition of the outer measure of P on a field $\mathcal{F}_0$. Prove that if A_i are disjoint and P^*-measurable, it follows that

$$P^*(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P^*(A_i).$$

Proof. From the subadditivity, it follows that

$$P^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P^*(A_i).$$

Now let us prove the other direction of inequality. By the definition of P^*-measurable sets, taking $E = A_1 \cup A_2$, we have

$$P^*(A_1 \cup A_2) = P^*((A_1 \cup A_2) \cap A_1) + P^*((A_1 \cup A_2) \cap A_1^c) = P^*(A_1) + P^*(A_2).$$

By induction, we have

$$P^*(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P^*(A_i).$$

Note that

$$P^*(\cup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{n} P^*(A_i) \uparrow \sum_{i=1}^{\infty} P^*(A_i).$$

It follows that

$$P^*(\cup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} P^*(A_i).$$

4. Write down the definitions of convergence in probability and w.p.1. (1) Prove that if X_n is decreasing and converges to X in probability, then $X_n \to X$ w.p.1.

(2) Suppose that X_n is decreasing and bounded from below, then X_n admits a limit w.p. 1.

Proof. (1) Because $X_n \to X$ in probability, we have

$$P(|X_n - X| \geq \varepsilon) \to 0$$

for any each $\varepsilon > 0$. Note that $P(|X_n - X| \geq \varepsilon) \geq P(X_n - X \leq -\varepsilon)$ and $P(X_n - X \leq -\varepsilon)$ is increasing in n. Therefore we must have $P(X_n - X \leq -\varepsilon) = 0$ for all n. Moreover $P(\cup_{n\geq 1}\{X_n - X \leq -\varepsilon\}) = 0$

Because X_n is decreasing, $\{X_n - X \geq \varepsilon\} \supset \{X_{n+1} - X \geq \varepsilon\}$. As a consequence, $\{X_n - X \geq \varepsilon\} \supset \cup_{k=n}^{\infty}\{X_k - X \geq \varepsilon\}$. Then

$$P(\cup_{k=n}^{\infty}\{|X_k - X| \geq \varepsilon\}) = P(\cup_{k=n}^{\infty}\{X_k - X \leq -\varepsilon\}) + P(\cup_{k=n}^{\infty}\{X_k - X \geq \varepsilon\}) = P(X_n - X \geq \varepsilon) \to 0.$$

This says that $X_n \to X$ w.p.1.

(2) For each ω, we know that $X_n(\omega)$ is decreasing and bounded from below. Therefore it admits a limit $X(\omega)$. Because $\{|X_n - X| \geq \varepsilon\} = \{X_n - X \geq \varepsilon\} \downarrow \emptyset$. By the continuity of P around $\emptyset$, we have that $X_n \to X$ with probability. By (1), our result is true.

5. Denote $A - B = A \cap B^c$. Prove that

$$\limsup_{n\to\infty} A_n - \liminf_{n\to\infty} A_n = \limsup_{n\to\infty}(A_n \cap A_{n+1}^c).$$

Proof. If $w \in$ RHS, there exists $n_k \uparrow \infty$ such that $w \in A_{n_k} \cap A_{n_k+1}^c$. Since $w \in A_{n_k}$ for all $k \geq 1$, $w \in A_n$, i.o. ($w \in \limsup_{n\to\infty} A_n$). Since $w \notin A_{n_k+1}$ for all $k \geq 1$, we have $w \notin \liminf_{n\to\infty} A_n$. Therefore $w \in$ LHS.

If $w \in$ LHS, there exists $n_k \uparrow \infty$ and $m_k \uparrow \infty$ such that $w \in A_{n_k}$ but $w \notin A_{m_k}$. We take $j_1 = n_1$, $j_2 = \inf\{m_k : m_k > j_1\}$, $j_3 = \inf\{n_k : n_k > j_2\}$, and so on. We notice that $w \in A_{j_1}, A_{j_3}, \cdots$ but $w \notin A_{j_2}, A_{j_4}, \cdots$. Because $w \in A_{j_{2k+1}}$ and $w \notin A_{j_{2k+2}}$, there must exist an $l_k \in [j_{2k+1}, j_{2k+2} - 1]$ such that $w \in A_{l_k}$ but $w \notin A_{l_k+1}$, i.e. $w \in A_{l_k} \cap A_{l_k+1}^c$. Let $k \to \infty$, this says that $w \in A_{l_k} \cap A_{l_k+1}^c$ happens i.o., i.e. $w \in$RHS.

6. Prove that

$$\mathcal{A} = \{\cup_{i=1}^n (a_i, b_i] : 0 \le a_i < b_i < a_{i+1} < b_{i+1} \le 1 \text{ for all } i \text{ and } n\}$$

is a field on $(0, 1]$ but not a sigma-field.

Proof. It is simple to show $\mathcal{A}$ is a field. Now we show that it is not a σ-field. Consider $A_n = (1/2 - 1/n, 1/2] \in \mathcal{A}$ for $n \ge 3$. While

$$\bigcap_{n \ge 3} A_n = \{1/2\} \notin \mathcal{A}.$$

7. State and prove the 0-1 law.

Proof. Suppose $\{A_n\}$ is a sequence of independent events. Define

$$\mathcal{T} = \bigcap_{n \geq 1} \sigma(A_n, A_{n+1}, \cdots).$$

Then for any $A \in \mathcal{T}$, we have $P(A) = 0$ or 1.

Fist we know that $\sigma(A_1, \cdots, A_{n-1})$ is independent of $\sigma(A_n, A_{n+1}, \cdots)$. Because $A \in \mathcal{T} \subset \sigma(A_n, A_{n+1}, \cdots)$. Therefore, A is independent of $\sigma(A_1, \cdots, A_{n-1})$. By the arbitrariness of n, A is independent of all A_i, and therefore A is independent of $\sigma(A_1, A_2, \cdots)$. Note that $A \in \sigma(A_1, A_2, \cdots)$. We then have A is independent of A, that is

$$P(A) = P(A \cap A) = P(A)^2.$$

This is to say $P(A) = 0$ or 1.