Fourth Fundamental Form and Curvatures of Rotational Hypersurfaces in 4D Euclidean Space

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Submanifolds ...

- An isometric immersion $x\colon\ M \longrightarrow \mathbb{E}^m$ of a submanifold M in Euclidean m-space is called **finite type**, if x identified with the position vector field of M in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, i.e., $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, x_2, \ldots, x_k$ non-constant maps, and $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$.

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- If λ_i are different, M is called **k-type**. See [9] for details.
- Referring to Chen $[8 - 11]$, geometers have been studying finite type submanifolds, whose immersion into Euclidean space $\mathbb{E}^m$ (or pseudo-Euclidean space $\mathbb{E}^m_\nu$) is achieved using a finite number of eigenfunctions of their Laplacian, for nearly half a century.

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- Garay [24] served an extended Takahashi's theorem in $\mathbb{E}^m$.

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- Dursun [19] focused on hypersurfaces with pointwise 1-type Gauss maps in $\mathbb{E}^{n+1}$ (A submanifold of a Euclidean space is called **pointwise 1-type Gauss map** if its Gauss map satisfies $\Delta \mathbf{G} = f(\mathbf{G} + C)$ for some smooth function f on M and some constant vector C.).

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- Chen et al. [12] provided a survey on the developments of 1-type submanifolds and those with 1-type Gauss maps.

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- Garay [23] studied a specific class of finite type surfaces of revolution.

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- Senoussi and Bekkar [38] explored helicoidal surfaces that are of finite type with respect to the fundamental forms $\mathbb{I}, \mathbb{II}$ and $\mathbb{III}$, where their position vector field $r(u,v)$ satisfies the condition $\Delta^{J} r = Ar$, $J = \mathbb{I}, \mathbb{II}, \mathbb{III}$, where $A \in Mat(3,3)$.

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- Kim et al. [32] focused on the Cheng–Yau operator and the Gauss map of surfaces of revolution.

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- In recent years, the definition of L_k-finite type hypersurface has been given by changing the Laplace operator Δ in the definition of finite type hypersurfaces with the sequence of operators $L_0, L_1, ..., L_{n-1}$, such that $L_0 = -\Delta$.

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- Arslan et al. [2] studied Vranceanu surfaces M parametrized by

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- Güler and Turgay [28] introduced the Cheng–Yau operator and Gauss map of RHSs.
- Güler [25] specifically studied RHSs satisfying $\Delta^{\mathbb{I}} R = AR$, where $A \in Mat(4,4)$.

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- Turgay [41] gave some classifications of Lorentzian surfaces with finite type Gauss map,
- Dursun and Turgay [20] studied space-like surfaces in with pointwise 1-type Gauss map.

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- Bektaș, Canfes, and Dursun [7] classified surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in $\mathbb{E}_2^5$.

Summary

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- We introduce the fundamental concepts of four-dimensional Euclidean geometry.
- Next, we define the fourth fundamental form and the curvatures for hypersurfaces, and calculate $\mathfrak{C}_i$ and the fourth fundamental form for RHS.
- Finally, we investigate the RHS that satisfies $\Delta^{\mathbb{IV}}\mathbf{x} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 4×4 matrix.

Notions of Euclidean Geometry

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- Consider an n-dimensional Riemannian submanifold M within $\mathbb{E}^m$.
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- Consider an n-dimensional Riemannian submanifold M within $\mathbb{E}^m$.
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- We use the letters X, Y, Z, W to denote vector fields tangent to M, and ξ, η to denote vector fields normal to M.

Gauss and Weingarten

- The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$
$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2}$$

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  where X, Y indicate the vector fields, h, D and A represent the second fundamental form, the normal connection, and the shape operator of M, respectively.

- For each $\xi \in T_p^{\perp} M$, the shape operator A_ξ is a symmetric endomorphism (is a map $T : V \longrightarrow V$ is a linear transformation between a vector space V and itself.) of the tangent space $T_p M$ at a point $p \in M$.

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- The relationship between the shape operator and the second fundamental form is described by the Gauss and Codazzi equations, which are as follows:

$$\begin{aligned} \langle R(X, Y,)Z, W\rangle &= \langle h(Y, Z), h(X, W)\rangle \\ &\quad -\langle h(X, Z), h(Y, W)\rangle, \end{aligned} \tag{3}$$

$$\widetilde{\nabla}_X h(Y, Z) = \widetilde{\nabla}_Y h(X, Z), \tag{4}$$

where R represents the curvature tensor associated with connection ∇.

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- Let the dual basis of this frame field be $\{\theta^1, \theta^2, \ldots, \theta^n\}$.

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- Then, the first structural equation of Cartan is given by

$$d\theta^i = \sum_{j=1}^{n} \theta^j \wedge \omega_j^i, \quad i, j = 1, 2, \ldots, n, \tag{5}$$

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- Then, from the Codazzi equation (4), we have

$$e_i(k_j) = \omega_i^j(e_j)(k_i - k_j), \tag{6}$$
$$\omega_{ij}(e_l)(k_i - k_j) = \omega_i^l(e_j)(k_i - k_l) \tag{7}$$

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- The first equation relates the derivative of a principal curvature k_j in the direction of a vector field e_i to the difference between two curvatures, scaled by a connection 1-form (or covector field on a differentiable manifold is a differential form of degree one, that is, a smooth section of the cotangent bundle.).

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- We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

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- We would like to note that functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean curvature and Gauss–Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M.

Notions ...

We use the characteristic polynomial $P_{\mathbf{S}}(\lambda) = 0$ of $\mathbf{S}$. That is,

$$\det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^{n-k} s_k \lambda^{n-k} = 0, \tag{8}$$

where I_n denotes the identity matrix of order n. See[1] and [34] for details.

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- k-th fundamental form of M is defined by $\mathbb{I}\left(\mathbf{S}^{k-1}(X), Y\right) = \left\langle \mathbf{S}^{k-1}(X), Y\right\rangle$. Then,

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- In particular, one can get classical result

$$\mathfrak{C}_0 \mathbb{III} - 2\mathfrak{C}_1 \mathbb{II} + \mathfrak{C}_2 \mathbb{I} = 0$$

of surface theory for $n = 2$.

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- A rotational hypersurface $M \subset \mathbb{E}^{n+1}$ generated by a curve ℓ around an axis $\mathcal{C}$ that does not meet $\mathcal{C}$ is obtained by taking the orbit of $\mathcal{C}$ under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves ℓ pointwise fixed (See [18, Remark 2.3]).

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- In $\mathbb{E}^4$, an axis $\ell = (0, 0, 0, 1)$ and a matrix

$$\mathbf{Z}(v, w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad v, w \in \mathbb{R},$$

supply $\mathbf{Z} \cdot \ell^T = \ell^T$, where $\mathbf{Z} \in SO(4)$.

RHSs

- Therefore, the parametrization of the RHS generated by a curve $\mathcal{C}$ around an axis ℓ is given by

$$\mathbf{x}(u, v, w) = \mathbf{Z}(v, w) \cdot \gamma^T(u). \tag{11}$$

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- Triple vector product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ of $\mathbb{E}^4$ is defined by

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

RHSs

For a hypersurface $\mathbf{x}$ in 4-space, we have

$$\mathbb{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \ \mathbb{II} = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}, \ \mathbb{III} = \begin{pmatrix} X & Y & O \\ Y & Z & S \\ O & S & U \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle, & F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle, & G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \\
A &= \langle \mathbf{x}_u, \mathbf{x}_w \rangle, & B &= \langle \mathbf{x}_v, \mathbf{x}_w \rangle, & C &= \langle \mathbf{x}_w, \mathbf{x}_w \rangle, \\
L &= \langle \mathbf{x}_{uu}, \mathbf{G} \rangle, & M &= \langle \mathbf{x}_{uv}, \mathbf{G} \rangle, & N &= \langle \mathbf{x}_{vv}, \mathbf{G} \rangle, \\
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RHSs

- In addition, we have the following determinants

$$\begin{aligned}
\det \mathbb{I} &= \left(EG-F^2\right)C - EB^2 + 2FAB - GA^2, \\
\det \mathbb{II} &= \left(LN-M^2\right)V - LT^2 + 2MPT - NP^2, \\
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- Here, $\langle\ ,\ \rangle$ denotes the four-dimensional Euclidean inner product of two vectors,

$$\mathbf{G} = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\left\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\right\|} \tag{13}$$

determines the unit normal (i.e., the Gauss map) of hypersurface $\mathbf{x}$, and $\|\|$ indicates the norm of a vector in $\mathbb{E}^4$.

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  determines the unit normal (i.e., the Gauss map) of hypersurface $\mathbf{x}$, and $\| \|$ indicates the norm of a vector in $\mathbb{E}^4$.
- On the other hand, $\mathbb{I}^{-1} \cdot \mathbb{II}$ gives shape operator matrix $\mathbf{S}$ of hypersurface $\mathbf{x}$ in 4-space. See $[26 - 28]$ for details.

Curvatures

- Considering (4), and taking $n = 3$, we use characteristic polynomial of $\mathbf{S}$

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda I_3) = 0$$

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- Therefore, we reveal curvature formulas depend on the coefficients of $\mathbb{I}$ and $\mathbb{III}$ fundamental forms in 4-space:

Curvatures

Theorem 1. *Any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4$ *has following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\begin{array}{c}[(EN+GL-2FM)C+(EG-F^2)V-LB^2 \\ -NA^2-2(APG-BPF-ATF+BTE-ABM)]\end{array}}{3\left[(EG-F^2)C-EB^2+2FAB-GA^2\right]}, \quad (14)$$

$$\mathfrak{C}_2 = \frac{\begin{array}{c}[(EN+GL-2FM)\,V+\left(LN-M^2\right)C-ET^2 \\ -GP^2-2\,(APN-BPM-ATM+BTL-PTF)]\end{array}}{3\left[(EG-F^2)C-EB^2+2FAB-GA^2\right]}, \quad (15)$$

$$\mathfrak{C}_3 = \frac{\left(LN-M^2\right)V-LT^2+2MPT-NP^2}{(EG-F^2)C-EB^2+2FAB-GA^2}. \quad (16)$$

Fourth Fundamental Form

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- (c) *the fourth fundamental form* $(f_{ij}) = \mathbb{IV}$ *is given by* $\mathbb{IV} = \mathbb{III} \cdot \mathbf{S}$.

Fourth Fundamental Form

- **Corollary 1.** *For any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4$, *the fundamental forms and the curvatures are related by*

$$\mathbb{IV} - 3\mathfrak{C}_1\mathbb{III} + 3\mathfrak{C}_2\mathbb{II} - \mathfrak{C}_3\mathbb{I} = \mathcal{O}, \tag{17}$$

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- **Corollary 2.** *For any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4$, *the first fundamental form matrix, curvatures, and the shape operator matrix have following relation*

$$\mathbb{I}\cdot\left(\mathbf{S}^3 - 3\mathfrak{C}_1\mathbf{S}^2 + 3\mathfrak{C}_2\mathbf{S} - \mathfrak{C}_3\right) = \mathcal{O},$$

where $\mathcal{O}$ *detemines the zero matrix of order* 3.

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- **Corollary 3.** *In* $\mathbb{E}^4$, *the Gauss-Kronecker curvature and the fundamental forms of any hypersurface* $\mathbf{x}$ *are related by*

$$\mathfrak{C}_3 = \frac{\det \mathbb{II}}{\det \mathbb{I}} = \frac{\det \mathbb{III}}{\det \mathbb{II}} = \frac{\det \mathbb{IV}}{\det \mathbb{III}}.$$

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- **Corollary 4.** *For any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4$, *the fourth fundamental form* $\mathbb{IV} = (f_{ij})$ *is given by*

$$\mathbb{IV} = \begin{pmatrix} \zeta & \eta & \delta \\ \eta & \phi & \sigma \\ \delta & \sigma & \xi \end{pmatrix},$$

where

Fourth Fundamental Form

$$\zeta = \frac{1}{\det \mathbb{I}} \left\{ \begin{array}{c} CLM^2 - CL^2N + 2BL^2T + GLP^2 - B^2LX - A^2NX \\ -GL^2V - F^2VX - NP^2E - M^2VE + CNXE \\ -2BTXE + 2MPTE + GVXE + 2ABMX + 2ALNP \\ -2BLMP - 2ALMT - 2CFMX + CGLX - 2AGPX \\ +2BFPX + 2AFTX + 2FLMV - 2FLPT \end{array} \right\},$$

$$\eta = \frac{1}{\det \mathbb{I}} \left\{ \begin{array}{c} CM^3 - 2BM^2P - 2AM^2T - FNP^2 + GMP^2 - FLT^2 \\ -B^2LY - A^2NY + FM^2V - F^2VY + MT^2E \\ +CNYE - 2BTYE - MNVE + GVYE + 2ABMY \\ -CLMN + 2AMNP + 2BLMT - 2CFMY + CGLY \\ -2AGPY + 2BFPY + 2AFTY + FLNV - GLMV \end{array} \right\},$$

Fourth Fundamental Form

$$\delta = \frac{1}{\det \mathbb{I}} \left\{ \begin{array}{c} GP^3 - B^2LO - A^2NO + ANP^2 - 2BMP^2 + CM^2P \\ -ALT^2 - AM^2V - 2FP^2T - F^2OV + PT^2E \\ +CNOE - 2BOTE + GOVE - NPVE + 2ABMO \\ -2CFMO + CGLO - 2AGOP + 2BFOP + 2AFOT \\ -CLNP + ALNV + 2BLPT + 2FMPV - GLPV \end{array} \right\},$$

$$\phi = \frac{1}{\det \mathbb{I}} \left\{ \begin{array}{c} -CLN^2 + CM^2N + 2AN^2P - GLT^2 - B^2LZ - A^2NZ \\ -GM^2V - F^2VZ + NT^2E - N^2VE + CNZE \\ -2BTZE + GVZE + 2ABMZ - 2BMNP - 2AMNT \\ +2BLNT - 2CFMZ + CGLZ - 2AGPZ + 2BFPZ \\ +2AFTZ + 2FMNV - 2FNPT + 2GMPT \end{array} \right\},$$

Fourth Fundamental Form

$$\sigma = \frac{1}{\det \mathbb{I}} \left\{ \begin{array}{c} T^3E - BNP^2 - B^2LS - 2AMT^2 + BLT^2 - A^2NS \\ +CM^2T - BM^2V - 2FPT^2 + GP^2T - F^2SV \\ +CNSE - 2BSTE + GSVE - NTVE + 2ABMS \\ -2CFMS + CGLS - 2AGPS + 2BFPS + 2AFST \\ -CLNT + BLNV + 2ANPT + 2FMTV - GLTV \end{array} \right\},$$

$$\xi = \frac{1}{\det \mathbb{I}} \left\{ \begin{array}{c} -CNP^2 - CLT^2 - B^2LU - A^2NU + 2FMV^2 \\ -GLV^2 + GP^2V - F^2UV - NV^2E + T^2VE \\ +CNUE - 2BTUE + GUVE + 2ABMU - 2CFMU \\ +CGLU - 2AGPU + 2BFPU + 2AFTU + 2CMPT \\ +2ANPV - 2BMPV - 2AMTV + 2BLTV - 2FPTV \end{array} \right\},$$

and $\det \mathbb{I} = (EG - F^2)C - EB^2 + 2FAB - GA^2.$

Curvatures and the RHS

- We consider the i-th curvatures of the RHS (11), that is

$$\mathbf{x}(u,v,w) = (f(u)\cos v\cos w, f(u)\sin v\cos w, f(u)\sin w, \varphi(u)), \tag{18}$$

where $f \neq 0$ and $0 \leq v, w < 2\pi$, and the range of the parameter w must satisfy $w \neq \frac{\pi}{2}, \frac{3\pi}{2}$, otherwise the first fundamental form $\mathbb{I}$ is degenerated.

Curvatures and the RHS

- Using the first derivatives of RHS (18) with respect to u, v, w, we get the first quantities

$$\mathbb{I} = \operatorname{diag}\left(W, f^2 \cos^2 w, f^2\right), \tag{19}$$

where $W = f'^2 + \varphi'^2$, $f = f(u)$, $f' = \frac{df}{du}$, $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$.

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- The Gauss map of the RHS is determined by

$$\mathbf{G} = \left(\frac{\varphi'}{W^{1/2}} \cos v \cos w, \frac{\varphi'}{W^{1/2}} \sin v \cos w, \frac{\varphi'}{W^{1/2}} \sin w, -\frac{f'}{W^{1/2}}\right). \tag{20}$$

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$$\mathbb{II} = \operatorname{diag}\left(-\frac{f'\varphi'' - f''\varphi'}{W^{1/2}}, -\frac{f\varphi'}{W^{1/2}}\cos^2 w, -\frac{f\varphi'}{W^{1/2}}\right). \qquad (21)$$

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- Taking the first derivatives of (20) with respect to u, v, w, we find the third fundamental form matrix

$$\mathbb{III} = \operatorname{diag}\left(\frac{(f'\varphi'' - f''\varphi')^2}{W^2}, \frac{\varphi'^2}{W}\cos^2 w, \frac{\varphi'^2}{W}\right). \tag{22}$$

Curvatures and the RHS

- We calculate $\mathbb{I}^{-1} \cdot \mathbb{II}$, then obtain shape operator matrix

$$\mathbf{S} = \operatorname{diag}\left(-\frac{f'\varphi'' - f''\varphi'}{W^{3/2}}, -\frac{\varphi'}{fW^{1/2}}, -\frac{\varphi'}{fW^{1/2}} \right). \tag{23}$$

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- Finally, we obtain curvatures of the RHS (18).

Curvatures and the RHS

Theorem 2. *RHS* (18) *has following curvatures*

$$\begin{aligned}
\mathfrak{C}_0 &= 1 \text{ (by definition)}, \\
\mathfrak{C}_1 &= \frac{(ff'' - 2W)\,\varphi' - ff'\varphi''}{3fW^{3/2}}, \quad (24) \\
\mathfrak{C}_2 &= \frac{\varphi'^2 W - 2f\varphi'\,(\varphi' f'' - f'\varphi'')}{3f^2W^2}, \quad (25) \\
\mathfrak{C}_3 &= \frac{(\varphi' f'' - f'\varphi'')\,\varphi'^2}{f^2 W^{5/2}}, \quad (26)
\end{aligned}$$

where $W = f'^2 + \varphi'^2 \neq 0$, and $f = f(u) \neq 0$.

Curvatures and the RHS

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- **Corollary 6.** *RHS* (18) *is 1-minimal iff*

$$\varphi = \mp i c_1^{-1/4} \text{EllipticF}\left[i \sinh^{-1}\left(i c_1^{1/4} f \right), -1 \right] + c_2,$$

where $i = (-1)^{1/2}$, $\text{EllipticF}[\phi, m] = \int_0^{\phi} \left(1 - m \sin^2 \theta\right)^{-1/2} d\theta$ *is elliptic integral,* $\phi \in [-\pi/2, \pi/2]$, $0 \neq c_1, c_2$ *are constants.*

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- Here, as obtaining analytical solutions manually is highly challenging, we utilize software to solve the ODE $2\varphi' W + f\left(f'\varphi'' - f''\varphi'\right) = 0$.

Curvatures and the RHS

- **Corollary 7.** *RHS* (18) *is 2-minimal iff*

$$\varphi = c_1 \text{ or } \varphi = \mp \int \frac{e^{\int \frac{f''}{f} du}}{f^{1/2} \left(\int \frac{e^{\int \frac{2f''-f'}{f} du}}{ff'} du + c_1 \right)^{1/2}} du + c_2,$$

where $f \neq 0$, c_1, c_2 *are constants.*

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- Next, one can see some examples about RHS in $\mathbb{E}^4$.

Curvatures and the RHS

- **Example 1.** *Catenoidal-type Hypersurface. Taking* $f(u) = a\cosh u$ *and* $\varphi(u) = au$, *where* $-\infty < u < \infty$, $0 \leq v, w \leq 2\pi$, *we get*

$$\mathbf{x}(u, v, w) = (a\cosh u \cos v \cos w, a\cosh u \sin v \cos w, a\cosh u \sin w, au). \tag{27}$$

$\mathbf{x}$ *verifies* $\mathfrak{C}_1 = -\frac{1}{3a\cosh^2 u}$, $\mathfrak{C}_2 = -\frac{1}{3a^2\cosh^4 u}$, $\mathfrak{C}_3 = \frac{1}{a^3\cosh^6 u}$.

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- **Example 2.** *Hypersphere. Considering* $f(u) = r\cos u$ *and* $\varphi(u) = r\sin u$*, where* $r > 0$, $0 < u < \pi$, $0 \leq v, w \leq 2\pi$*, we have*

$$\mathbf{x}(u,v,w) = (r\cos u \cos v \cos w, r\cos u \sin v \cos w, r\cos u \sin w, r\sin u). \tag{28}$$

$\mathbf{x}$ *supplies* $\mathfrak{C}_i = \left(-\frac{1}{r}\right)^i$, where $i = 1, 2, 3$.

Curvatures and the RHS

- **Example 3.** *Right Spherical Hypercylinder. Taking* $f(u) = r > 0$ *and* $\varphi(u) = u$*, where* $0 < u < \pi$, $0 \leq v, w \leq 2\pi$, *we obtain*

$$\mathbf{x}(u, v, w) = (r \cos v \cos w, r \sin v \cos w, r \sin w, u) . \tag{29}$$

$\mathbf{x}$ *has* $\mathfrak{C}_1 = -\frac{2}{3r}$, $\mathfrak{C}_2 = \frac{1}{3r^2}$, $\mathfrak{C}_3 = 0$. *So, it is 3-minimal.*

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- Let us see some results of the fourth fundamental form of the RHS (18).

- **Corollary 9.** *The fourth fundamental form matrix* (f_{ij}) *of RHS* (18) *is determined by*

$$\mathbb{IV} = \mathrm{diag}\left(-\frac{(f'\varphi'' - f''\varphi')^3}{W^{7/2}}, -\frac{\varphi'^3}{fW^{3/2}}\cos^2 w, -\frac{\varphi'^3}{fW^{3/2}}\right). \qquad (30)$$

Curvatures and the RHS

- When $W = 1$, the curvatures (24), (25), and (26) of the RHS (18) reduce to

$$\mathfrak{C}_1 = \frac{ff'^2 f'' + (ff'' - 2)\left(1 - f'^2\right)}{3f\varphi'}, \tag{31}$$

$$\mathfrak{C}_2 = \frac{-f'^2\,(2ff'' + 1) + 1 - 2ff''\varphi'^2}{3f^2},$$

$$\mathfrak{C}_3 = \frac{f''\varphi'}{f^2},$$

where $f \neq 0$, $\varphi' \neq 0$.

Curvatures and the RHS

- **Corollary 10.** *When the curve* (10) *of* (18) *is parametrized by the arc length (i.e., $W=1$), then the curvatures of* (18) *have the relations*

$$0 = 9f^2\mathfrak{C}_1^2\left[-3f^2\mathfrak{C}_2 - f'^2\left(2ff''+1\right)+1\right] -2ff''\left[ff'^2f''+\left(ff''-2\right)\left(1-f'^2\right)\right]^2, \quad (32)$$

$$0 = 3f^3\mathfrak{C}_1\mathfrak{C}_3 - f''\left[ff'^2f''+\left(ff''-2\right)\left(1-f'^2\right)\right], \quad (33)$$

$$0 = f^2\left(3f''\mathfrak{C}_2+2f^3\mathfrak{C}_3^2\right) - f''\left[1-f'^2\left(2ff''+1\right)\right]. \quad (34)$$

- **Corollary 11.** *When* $f = u \neq 0$, $\varphi' \neq 0$ *in the previous corollary, then* (18) *has the following*

$$\mathfrak{C}_i = 0.$$

where $i = 1, 2, 3$. That is, the hypersurface (18) is i-minimal.

Fourth Laplace–Beltrami Operator

- **Definition 2.** The fourth Laplace–Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3)_{|\mathbf{D}}$ $(\mathbf{D} \subset \mathbb{R}^3)$ of class C^3 with respect to the fourth fundamental form of hypersurface $\mathbf{x}$ is the operator $\Delta^{\mathbb{IV}}$, defined by

$$\Delta^{\mathbb{IV}}\phi = \frac{1}{\mathfrak{f}^{1/2}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} \left(\mathfrak{f}^{1/2} f^{ij} \frac{\partial \phi}{\partial x^j} \right). \tag{35}$$

where $\left(f^{ij}\right) = \left(f_{ij}\right)^{-1}$ and

$$\begin{aligned} \mathfrak{f} &= \det\left(f_{ij}\right) \\ &= f_{11}f_{22}f_{33} - f_{11}f_{23}f_{32} - f_{12}f_{21}f_{33} + f_{12}f_{31}f_{23} + f_{21}f_{13}f_{32} - f_{13}f_{22}f_{31}. \end{aligned}$$

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- Here, the Laplace–Beltrami operator with respect to the metric $\mathbb{IV}$ is defined only when the fourth fundamental form $\mathbb{IV}$ is non-degenerated. The righ side of the operator (35) looks like the regular Laplace–Beltrami, but it depends on fourth fundamental form $\mathbb{IV} = (f_{ij})$.

Fourth Laplace–Beltrami Operator on RHSs

Theorem 3. *The fourth Laplace–Beltrami operator of RHS* (18) *is related by* $\Delta^{\mathbb{IV}}\mathbf{x} = \mathbf{A}\mathbf{x}$, *where* $\mathbf{A} = \mathrm{diag}(\Omega_1, \Omega_2, \Omega_3, \Phi)$, *and*

$$\Omega_i = \frac{W^{3/2}}{2f\varphi'^3\psi^4}\mathcal{P}_i, \tag{36}$$

$$\Phi = \frac{W^{3/2}}{2f\varphi'^3\psi^4}\mathcal{P}_4, \tag{37}$$

where $W = f'^2 + \varphi'^2$, $\psi = f'\varphi'' - f''\varphi'$, $i = 1, 2, 3$, *and also*

Fourth Laplace–Beltrami Operator on RHSs

$$\begin{aligned}\mathcal{P}_i = {} & 2f'^2W^2\varphi'^3\left(f'\varphi'' - f''\varphi'\right) + 3ff'^4f''\varphi'^3\left(f'\varphi'' + f''\varphi'\right) \\ & + 5ff'f''\varphi'^6\left(f'f'' + \varphi'\varphi''\right) - 16f^3f'f''\varphi'\varphi''\left(f'^2\varphi''^2 + f''^2\varphi'^2\right) \\ & + 4f^3\left(f''^4\varphi'^4 + f'^4\varphi''^4\right) + 3ff'W^2\varphi'^3\left(f'\varphi''' - \varphi'f'''\right) \\ & - 13ff'^4\varphi'^4\varphi''^2 + 8ff'^3f''\varphi'^5\varphi'' - 6ff'^3f'''\varphi'^6 \\ & - 7ff'^2\varphi'^6\varphi''^2 + 24f^3f'^2f''^2\varphi'^2\varphi''^2 + 2ff''^2\varphi'^8,\end{aligned}$$

$$\begin{aligned}\mathcal{P}_4 = {} & W\varphi'^3[2f'^4\varphi'\varphi'' - 8ff'^3\varphi''^2 + 2f'^2\varphi'^2\left(\varphi'\varphi'' - f'f''\right) \\ & + 7ff'^2f''\varphi'\varphi'' + ff'f''^2\varphi'^2 - 9f\varphi'^2\varphi''\psi - 2f'f''\varphi'^4 \\ & + 3fW\varphi'\left(f'\varphi''' - f'''\varphi'\right)].\end{aligned}$$

Fourth Laplace–Beltrami Operator on RHSs

- **Example 4.** *Considering* $f(u) = a\cosh u$ *and* $\varphi(u) = au$ *as in Example 1, we have*

$$\Delta^{\mathbb{IV}}\mathbf{x} = \frac{a^2\cosh^3 u}{2}\begin{pmatrix} (5+\cosh 2u)\cos v\cos w \\ (5+\cosh 2u)\sin v\cos w \\ (5+\cosh 2u)\sin w \\ -4\sinh u \end{pmatrix}. \tag{38}$$

Fourth Laplace–Beltrami Operator on RHSs

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- **Example 5.** *Taking $f(u) = r\cos u$ and $\varphi(u) = r\sin u$ as in Example 2, we obtain*

$$\Delta^{\mathbb{IV}}\mathbf{x} = 3r^2\begin{pmatrix} \cos u\cos v\cos w \\ \cos u\sin v\cos w \\ \cos u\sin w \\ \sin u \end{pmatrix}. \qquad (39)$$

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thanks ...

thank you very much!