The Algebraic Surfaces of Enneper's Maximal Surfaces in 3D Minkowski Space

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thanks ...

First and foremost, I would like to begin by thanking Christos Konaxis for his support with some Maple codes.

danke schön ...

Alfred Enneper
(1830 – 1885)

source:
http://www.math.uni-hamburg.de/home/grothkopf/fotos/math-ges/

minimal surfaces

- In general, it is accepted that research on the theory of *minimal surfaces* began with the works of Leonhard Euler in 1744 and Joseph Louis Lagrange in 1760.

Definition (Lagrange)

A **minimal surface** is a surface of vanishing mean curvature in three dimensional Euclidean space $\mathbb{E}^3$.

for some books ...

Darboux [1, 2], Dierkes [3], Fomenko and Tuzhilin [4], Gray et al. [5], Nitsche [6], Osserman [7], Spivak [8],

for some papers ...

Lie [9], Schwarz [10], Small [11, 12], and Weierstrass [13, 14] related to minimal surfaces in Euclidean geometry

minimal surfaces

- There are many classical and modern minimal surfaces.

classical minimal surfaces

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- Plane (trivial),

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- Henneberg (1875),
- Richmond (1901).

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  - The 2000s: Fujimori, Shoda, Traizet, Weber, etc.

algebraic minimal surfaces

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- See also Nitsche [6], Enneper [15], Güler [16], and Ribaucour [17] for further reading.

minimal & maximal surfaces

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- Weierstrass [13] revealed a representation for minimal surfaces in $\mathbb{E}^3$.
- Nearly a century later, Kobayashi [18] provided a similar Weierstrass-type representation for conformal spacelike surfaces with zero mean curvature, known as **maximal surfaces**, in three-dimensional Minkowski space $\mathbb{E}^{2,1}$

outline

- In this talk, we consider the Enneper's maximal surfaces $\mathcal{E}_m$ for positive integers $m \geq 1$ by using Weierstrass data $(1, \zeta^m)$ for $\zeta \in \mathbb{C}$, and then show that these surfaces are algebraic in $\mathbb{E}^{2,1}$.

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- serve degree and class numbers of $\mathcal{E}_m(u, v)$,
- finally, summarize all findings in Table 1, and in Table 2, then give some open problems.

notions of 3D Minkowski geometry

- A vector is identified with its transpose.

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- Let $\mathbb{E}^{2,1} := (\{\alpha = (\alpha_1, \alpha_2, \alpha_0) \mid \alpha_i \in \mathbb{R}\}, \langle \cdot, \cdot \rangle)$ be the 3-dimensional Minkowski space with Lorentzian metric

$$\langle \alpha, \beta \rangle = \alpha_1 \beta_1 + \alpha_2 \beta_2 - \alpha_0 \beta_0,$$

where $\beta = (\beta_1, \beta_2, \beta_0)$, $\beta_i \in \mathbb{R}$.

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- A vector $\alpha \in \mathbb{E}^{2,1}$ is called
  - space-like if $\langle \alpha, \alpha \rangle > 0$,
  - time-like if $\langle \alpha, \alpha \rangle < 0$,
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  where $\beta = (\beta_1, \beta_2, \beta_0)$, $\beta_i \in \mathbb{R}$.
- A vector $\alpha \in \mathbb{E}^{2,1}$ is called
  - space-like if $\langle \alpha, \alpha \rangle > 0$,
  - time-like if $\langle \alpha, \alpha \rangle < 0$,
  - light-like if $\alpha \neq 0$ and $\langle \alpha, \alpha \rangle = 0$.
- A surface in $\mathbb{E}^{2,1}$ is called space-like (resp., time-like, light-like) if the induced metric on the tangent planes is a Riemannian (resp., Lorentzian, degenerate) metric.

notions of 3D Minkowski geometry

- Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A **maximal curve** is an analytic function $\vartheta : \mathcal{U} \longrightarrow \mathbb{C}^{2,1}$ such that $\left\langle \vartheta' \left(\zeta\right), \vartheta' \left(\zeta\right) \right\rangle = 0$, where $\zeta \in \mathcal{U}$, and $\vartheta' := \frac{\partial \vartheta}{\partial \zeta}$.

holomorphic and meromorphic functions

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- A function f is **holomorphic** on a doman if it is complex differentiable at every point in that domain.
- A function g is **meromorphic** on a doman if it is holomorphic throughout the domain except at isolated points, which are poles of g. At these poles, g can be expressed as $h(\omega)/k(\omega)$, where h and k are holomorphic and $\omega \in \mathbb{C}$, $k \neq 0$.

Kobayashi's Weierstrass type representation

Theorem (Kobayashi)

Let $g(\omega)$ be a meromorphic function and let $f(\omega)$ be a holomorphic function, $\omega \in \mathbb{C}$, fg^2 is analytic, defined on a simply connected open subset $U \subset \mathbb{C}$ such that $f(\omega)$ does not vanish on U except at the poles of $g(\omega)$ in $\mathbb{C}^{2,1}$. Then,

$$\mathbf{x}(u,v) = \mathrm{Re} \int^{\zeta} \begin{pmatrix} 1+g^2 \\ i\left(1-g^2\right) \\ -2g \end{pmatrix} f d\omega \quad (\zeta = u + iv) \tag{1}$$

is a space-like conformal immersion with mean curvature identically 0 (i.e., a maximal surface) in $\mathbb{E}^{2,1}$. Conversely, any maximal surface can be described in this manner.

Weierstrass data

Definition

A pair of a meromorphic function g and a holomorphic function f, (f, g) is called **Weierstrass data** *for a maximal surface.*

Enneper's maximal curve

Lemma

The curve of Enneper of order m:

$$\varepsilon_m(\zeta) = \left(\zeta + \frac{\zeta^{2m+1}}{2m+1},\ i\left(\zeta - \frac{\zeta^{2m+1}}{2m+1} \right),\ -\frac{2\zeta^{m+1}}{m+1} \right) \tag{2}$$

is a maximal curve in $\mathbb{C}^{2,1}$, $\zeta \in \mathbb{C} - \{0\}$, $i = \sqrt{-1}$, $m \neq -1, -1/2$.

Enneper's maximal surface

- Therefore, we have $\langle \varepsilon_m, \varepsilon_m \rangle = 0$ by using (2).

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- Hence, in $\mathbb{E}^{2,1}$, the Enneper's maximal surface is given by

$$\mathcal{E}_m (u, v) = \mathrm{Re} \int \varepsilon_m (\zeta) \, d\zeta, \tag{3}$$

where $\zeta = u + iv$.

Enneper's maximal surface

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- Hence, in $\mathbb{E}^{2,1}$, the Enneper's maximal surface is given by

$$\mathcal{E}_m (u, v) = \operatorname{Re} \int \varepsilon_m (\zeta) \, d\zeta, \tag{3}$$

  where $\zeta = u + iv$.
- $\operatorname{Im} \int \varepsilon_m (\zeta) \, d\zeta$ determines the adjoint (i.e., conjugate) maximal surface $\mathcal{E}_m^* (u, v)$ of $\mathcal{E}_m (u, v)$.

Weierstrass data

- The Weierstrass data $(1, \zeta^m)$ of (3) is a representation of the Enneper maximal surface, where integer $m \geq 1$.

Enneper's real parametric maximal surfaces

- Considering $\zeta = re^{i\theta}$, Enneper's real parametric maximal surfaces is given by

$$\mathcal{E}_m(r,\theta) = \begin{pmatrix} r\cos(\theta) + \frac{1}{2m+1} r^{2m+1} \cos[(2m+1)\theta] \\ -r\sin(\theta) + \frac{1}{2m+1} r^{2m+1} \sin[(2m+1)\theta] \\ -\frac{2}{m+1} r^{m+1} \cos[(m+1)\theta] \end{pmatrix}, \qquad (4)$$

where $m \neq -1, -1/2$.

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  where $m \neq -1, -1/2$.
- See Figure 1 for Enneper maximal surfaces $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$.

graphics ...

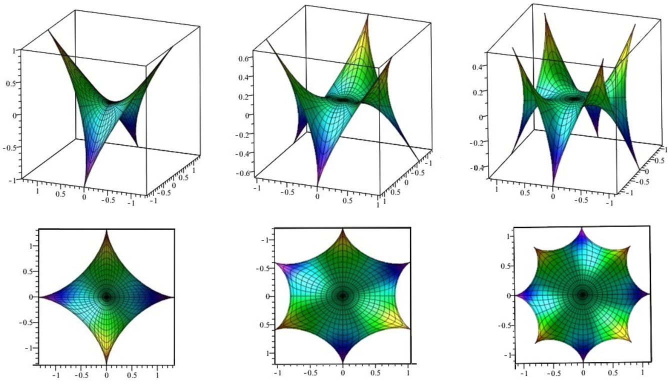

Figure 1. Enneper's real parametric maximal surfaces
left: $\mathcal{E}_1(r,\theta)$, middle: $\mathcal{E}_2(r,\theta)$, right: $\mathcal{E}_3(r,\theta)$

parametric equations

$\mathcal{E}_m(u,v)$ is determined by

$$
\begin{aligned}
x(u,v) &= \operatorname{Re}\left[u+iv+\tfrac{1}{2m+1}\sum_{k=0}^{2m+1}\binom{2m+1}{k}u^{2m+1-k}(iv)^k\right],\\
y(u,v) &= \operatorname{Re}\left[iu-v+\tfrac{i}{2m+1}\sum_{k=0}^{2m+1}\binom{2m+1}{k}u^{2m+1-k}(iv)^k\right],\\
z(u,v) &= \operatorname{Re}\left[-\tfrac{2}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k}u^{m+1-k}(iv)^k\right].
\end{aligned}
\tag{5}
$$

in Cartesian coordinates ...

- We research surface $\mathcal{E}_m(u, v)$ for $m = 1, 2, \ldots, 5$, taking $\zeta = u + iv$ in Cartesian coordinates x, y, z, and in inhomogeneous tangential coordinates a, b, c, by using Weierstrass' representation equation.

Enneper maximal surface

Theorem

The surface

$$\mathcal{E}_1(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + u \\ -\frac{1}{3}v^3 + u^2 v - v \\ -u^2 + v^2 \end{pmatrix} \tag{6}$$

which has Weierstrass data $(1, \zeta)$, *is an Enneper maximal surface in* $\mathbb{E}^{2,1}$.

Gauss map

- The Gauss map $e_1(u,v)$ of $\mathcal{E}_1$ is as follows

$$e_1 = \left(-\frac{2u}{\lambda - 1}, -\frac{2v}{\lambda - 1}, \frac{\lambda^2 + 1}{\lambda - 1} \right), \tag{7}$$

where $\lambda \neq 1$.

- The mean curvature and the Gaussian curvature of the surface $\mathcal{E}_1$:

$$\begin{aligned} H &= 0, \\ K &= \frac{4\left(3\lambda + 1\right)^2}{\left(\lambda - 1\right)^6}, \end{aligned}$$

respectively.

higher order parametric equations

- We then reveal the parametric equations of the Enneper's higher order maximal surfaces $\mathcal{E}_m(u,v) = (x(u,v), y(u,v), z(u,v))$ (see Figure 2 middle for $\mathcal{E}_2$, and Figure 2 right for $\mathcal{E}_3$):

m=2,3,4

$$\mathcal{E}_2(u,v) = \begin{pmatrix} \frac{1}{5}u^5 - 2u^3v^2 + uv^4 + u \\ \frac{1}{5}v^5 - 2u^2v^3 + u^4v - v \\ -\frac{2}{3}u^3 + 2uv^2 \end{pmatrix}, \qquad (8)$$

$$\mathcal{E}_3(u,v) = \begin{pmatrix} \frac{1}{7}u^7 - 3u^5v^2 + 5u^3v^4 - uv^6 + u \\ -\frac{1}{7}v^7 + 3u^2v^5 - 5u^4v^3 + u^6v - v \\ -\frac{1}{2}u^4 + 3u^2v^2 - \frac{1}{2}v^4 \end{pmatrix}, \qquad (9)$$

$$\mathcal{E}_4(u,v) = \begin{pmatrix} \frac{1}{9}u^9 - 4u^7v^2 + 14u^5v^4 - \frac{23}{3}u^3v^6 + uv^8 + u \\ \frac{1}{9}v^9 - 4u^2v^7 + 14u^4v^5 - \frac{23}{3}u^6v^3 + u^8v - v \\ -\frac{2}{5}u^5 + 4u^3v^2 - 2uv^4 \end{pmatrix}, \qquad (10)$$

m=5

$$\mathcal{E}_5\left(u,v\right)=$$

$$\begin{pmatrix} \frac{1}{11}u^{11}-5u^{9}v^{2}+30u^{7}v^{4}-42u^{5}v^{6}+15u^{3}v^{8}+uv^{10}+u \\ -\frac{1}{11}v^{11}+5u^{2}v^{9}-30u^{4}v^{7}+42u^{6}v^{5}-15u^{8}v^{3}+u^{10}v-v \\ -\frac{1}{3}u^{6}+5u^{4}v^{2}-5u^{2}v^{4}+\frac{1}{3}v^{6} \end{pmatrix}. \quad (11)$$

graphics ...

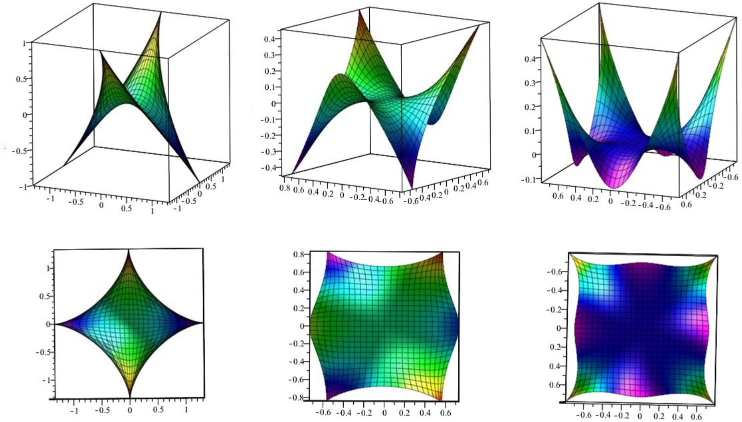

Figure 2. Enneper's real parametric maximal surfaces
left: $\mathcal{E}_1(u, v)$, middle: $\mathcal{E}_2(u, v)$, right: $\mathcal{E}_3(u, v)$

algebraic surface equation, degree and class

- Next, using some elimination techniques, we derive the algebraic surface equation, degree and class numbers of Enneper maximal surfaces $\mathcal{E}_m(u, v)$ for integers $1 \leq m \leq 5$ in $\mathbb{E}^{2,1}$.
- Let us see some basic notions of the surfaces.

degree number

Definition

The set of roots of a polynomial gives an algebraic surface equation $Q(x, y, z) = 0$. Maximum degree $\max(\alpha + \beta + \gamma)$ of the term $x^{\alpha} y^{\beta} z^{\gamma}$ of Q is called **degree** number **d**, where $\alpha, \beta, \gamma \in \mathbb{Z}$.

in inhomogeneous tangential coordinates

- At a point (u, v) on a surface $\mathbf{s}(u, v) = (x(u, v), y(u, v), z(u, v))$, the tangent plane is given by

$$Xx + Yy - Zz + P = 0, \tag{12}$$

where $e = (X(u, v), Y(u, v), Z(u, v))$ indicates the Gauss map, and $P = P(u, v)$ is just a function.

- Then, in inhomogeneous tangential coordinates a, b, c, we have the surface

$$\widehat{\mathbf{s}}(u, v) = (a(u, v), b(u, v), c(u, v)) = (X/P,\ Y/P,\ Z/P). \tag{13}$$

class number

Definition

The maximum degree of the real algebraic equation $\hat{Q}(a, b, c) = 0$ of $\widehat{\mathbf{s}}(u, v)$ gives the **class** number.

algebraic surface equation

We compute the algebraic surface equation $Q_1(x, y, z) = 0$ (see Figure 3, left) of Enneper's maximal surface $\mathcal{E}_1(u, v)$ in (6) by using some elimination techniques.

algebraic surface equation

- We then obtain algebraic equation of the Enneper's maximal surface

$$
\begin{aligned}
Q_1(x,y,z) \quad = \quad & 64z^9 + 432x^2z^6 - 432y^2z^6 - 1215x^4z^3 \\
& -6318x^2y^2z^3 + 3888x^2z^5 - 1215y^4z^3 \\
& +3888y^2z^5 - 1152z^7 + 729x^6 - 2187x^4y^2 \\
& -4374x^4z^2 + 2187x^2y^4 + 6480x^2z^4 \\
& -729y^6 + 4374y^4z^2 - 6480y^2z^4 \\
& +729x^4z - 1458x^2y^2z - 3888x^2z^3 \\
& +729y^4z - 3888y^2z^3 + 5184z^5
\end{aligned}
\tag{14}
$$

- Hence, its degree number is 9.

algebraic equations

Next, we continue our computations to find $Q_m(x, y, z) = 0$ for integers $m = 2, 3$. We compute the following algebraic surface equations $Q_2(x, y, z) = 0$ (see Figure 3, middle) and $Q_3(x, y, z) = 0$ (see Figure 3, right) of the surfaces $\mathcal{E}_2(u, v)$ and $\mathcal{E}_3(u, v)$, respectively,

algebraic equation for m=2

$$
\begin{aligned}
Q_2(x,y,z) = {} & 847\,288\,609\,443 z^{25} \\
& -4358\,480\,501\,250 x^3 z^{20} \\
& +13\,075\,441\,503\,750 x y^2 z^{20} \\
& -131\,157\,978\,046\,875 x^6 z^{15} \\
& -474\,186\,536\,015\,625 x^4 y^2 z^{15} \\
& +107 \text{ other lower degree terms}
\end{aligned}
$$

algebraic equation for m=3

$$
\begin{aligned}
Q_3(x,y,z) \;=\;& 2475\,880\,078\,570\,760\,549\,798\,248\,448\; z^{49} \\
&+5079\,604\,062\,565\,768\,134\,821\,675\,008\; x^4 z^{42} \\
&-30\,477\,624\,375\,394\,608\,808\,930\,050\,048\; x^2 y^2 z^{42} \\
&+5079\,604\,062\,565\,768\,134\,821\,675\,008\; y^4 z^{42} \\
&-633\,850\,350\,654\,216\,217\,766\,624\,493\,568\; x^8 z^{35} \\
&+446 \text{ other lower degree terms}
\end{aligned}
$$

algebraic equations and degrees for m=2,3

- Therefore, Q_2 and Q_3 are the algebraic maximal surfaces supplying $Q_m(x, y, z) = 0$ of the surfaces $\mathcal{E}_m\left(u, v\right)$, where $m = 2, 3$.
- Q_2 and Q_3 have also degree numbers 25 and 49, respectively.

graphics ...

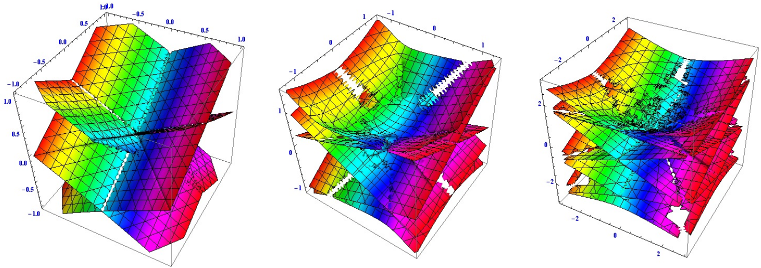

Figure 3. Enneper's algebraic maximal surfaces
left: $Q_1(x, y, z)$, middle: $Q_2(x, y, z)$, right: $Q_3(x, y, z)$

Gauss maps for m

- Next, we introduce the class numbers of the surfaces $\mathcal{E}_m(u, v)$ for integers $1 \leq m \leq 4$.
- The case $m = 5$, marked with "$*$" in Table 2.
- To compute the algebraic surface equations $\hat{Q}_m(a, b, c) = 0$, we obtain the Gauss maps $e_m(u, v)$ (see Figure 4 for e_1, e_2, e_3) for integers $1 \leq m \leq 5$ of the surfaces $\mathcal{E}_m(u, v)$, and generalize them.

Gauss maps for positive integers m

$$
\begin{aligned}
e_1 &= \left(-2\frac{u}{\lambda-1}, -2\frac{v}{\lambda-1}, \frac{\lambda+1}{\lambda-1}\right), \\
e_2 &= \left(-2\frac{u^2-v^2}{\lambda^2-1}, -2\frac{2uv}{\lambda^2-1}, \frac{\lambda^2+1}{\lambda^2-1}\right), && (15) \\
e_3 &= \left(-2\frac{u^3-3uv^2}{\lambda^3-1}, -2\frac{3u^2v-v^3}{\lambda^3-1}, \frac{\lambda^3+1}{\lambda^3-1}\right), && (16) \\
e_4 &= \left(-2\frac{u^4-6u^2v^2+v^4}{\lambda^4-1}, -2\frac{4u^3v-4uv^3}{\lambda^4-1}, \frac{\lambda^4+1}{\lambda^4-1}\right), && (17)
\end{aligned}
$$

Gauss maps for positive integers m

$$e_5 = \left(-2\frac{u^5 - 10u^3v^2 + 5uv^4}{\lambda^5 - 1}, -2\frac{5u^4v - 10u^2v^3 + v^5}{\lambda^5 - 1}, \frac{\lambda^5 + 1}{\lambda^5 - 1}\right) \tag{18}$$

$$e_m = \left(-2\frac{\mathrm{Re}\,(\zeta^m)}{|\zeta|^m - 1}, -2\frac{\mathrm{Im}\,(\zeta^m)}{|\zeta|^m - 1}, \frac{|\zeta|^m + 1}{|\zeta|^m - 1}\right), \tag{19}$$

where $\zeta = u + iv$, $|\zeta| = \lambda$.

inhomogeneous tangential coordinates

Using (6), (7), (12) and (13), with $P_1(u,v)=\frac{(\lambda-3)\left(-u^2+v^2\right)}{3(\lambda-1)}$, we get the following surface $\widehat{\mathcal{E}}_1\,(u,v)$ (see Figure 5, left) in inhomogeneous tangential coordinates:

$$
\begin{aligned}
a(u,v) &= \frac{6u}{\left(-u^2+v^2\right)(\lambda-3)}, \\
b(u,v) &= \frac{6v}{\left(-u^2+v^2\right)(\lambda-3)}, \\
c(u,v) &= -\frac{3(\lambda+1)}{\left(-u^2+v^2\right)(\lambda-3)},
\end{aligned}
$$

where $\lambda=u^2+v^2$, $\lambda\neq 3$, $u,v\neq 0$.

inhomogeneous tangential coordinates ...

- Therefore, algebraic surface equation $\hat{Q}_1(a, b, c) = 0$ (see Figure 6, left) of the surface $\widehat{\mathcal{E}}_1(u, v)$, is determined by

$$\begin{aligned}\hat{Q}_1(a, b, c) \;&=\; 4a^6 - 4a^4 b^2 - 3a^4 c^2 - 4a^2 b^4 + 6a^2 b^2 c^2 \\ &\quad +4b^6 - 3b^4 c^2 - 18a^4 c + 12a^2 c^3 + 18b^4 c \\ &\quad -12b^2 c^3 + 9a^4 + 18a^2 b^2 + 9b^4\end{aligned}$$

- So, the class number is 6.

graphics ...

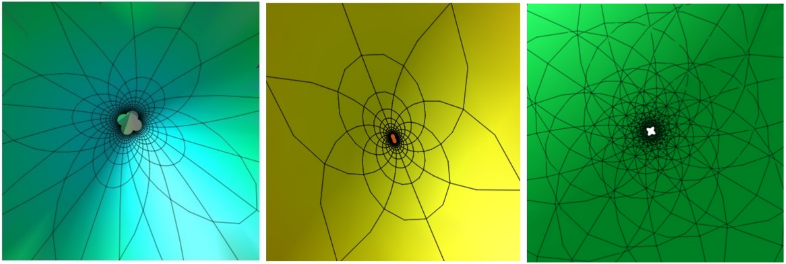

Figure 4. Top views of Gauss maps of the surfaces $\mathcal{E}_m(u,v)$
left: $e_1(u,v)$, middle: $e_2(u,v)$, right: $e_3(u,v)$

inhomogeneous tangential coordinates

- Next, we continue our computations to find $\hat{Q}_m$ for integers $2, 3, 4$.
- To find the class number of surface $\mathcal{E}_2\,(u, v)$ (see Figure 5, middle), we use (9), (12), (13) and (16).

inhomogeneous tangential coordinates

- We compute $P_2(u,v) = -\frac{4\left(u^3-3uv^2\right)\left(\lambda^2-5\right)}{15\left(\lambda^2-1\right)}$,
- then get surface $\widehat{\mathcal{E}}_2(u,v)$ in inhomogeneous tangential coordinates:

$$
\begin{aligned}
a &= \frac{15\left(u^2-v^2\right)}{2\left(u^3-3uv^2\right)\left(\lambda^2-5\right)}, \\
b &= \frac{15uv}{\left(u^3-3uv^2\right)\left(\lambda^2-5\right)}, \\
c &= -\frac{15\left(\lambda^2+1\right)}{4\left(u^3-3uv^2\right)\left(\lambda^2-5\right)},
\end{aligned}
$$

where $\lambda = u^2+v^2$, $\lambda^2 \neq 5$, $u, v \neq 0$.

inhomogeneous tangential coordinates

- In the inhomogeneous tangential coordinates a, b, c, we find the algebraic surface equation $\hat{Q}_2(a,b,c) = 0$ (see Figure 6, middle) of the surface $\widehat{\mathcal{E}}_2(u,v)$:

$$\begin{aligned}\hat{Q}_2(a,b,c) \quad = \quad & 2176\,782\,336\ a^{16}b^4 + 5804\,752\,896\ a^{14}b^6 \\ & -4837\,294\,080\ a^{14}b^4c^2 + 2902\,376\,448\ a^{12}b^8 \\ & -8062\,156\,800a^{12}b^6c^2 \\ & +120 \text{ other lower degree terms}\end{aligned}$$

- Hence, $\hat{Q}_2(a,b,c) = 0$ is the algebraic surface of the surface $\widehat{\mathcal{E}}_2(u,v)$, and has class number 20.

inhomogeneous tangential coordinates

Using similar ways, we compute algebraic surface equation $\hat{Q}_3(a, b, c) = 0$ (see Figure 6, right) of surface $\widehat{\mathcal{E}}_3(u, v)$ (see Figure 5, right):

inhomogeneous tangential coordinates

$$
\begin{aligned}
\hat{Q}_3(a,b,c) \;=\;& 26\,623\,333\,280\,885\,243\,904\ a^{42} \\
& -718\,829\,998\,583\,901\,585\,408\ a^{40} b^{2} \\
& -104\,829\,374\,793\,485\,647\,872\ a^{40} c^{2} \\
& +6868\,819\,986\,468\,392\,927\,232\ a^{38} b^{4} \\
& +2935\,222\,494\,217\,598\,140\,416\ a^{38} b^{2} c^{2} \\
& +774 \text{ other lower degree terms}
\end{aligned}
$$

and has class number 42.

inhomogeneous tangential coordinates

Algebraic surface equation $\hat{Q}_4(a, b, c) = 0$ of the surface $\widehat{\mathcal{E}}_4(u, v)$ is determined by
$\hat{Q}_4(a, b, c) =$

$$
\begin{aligned}
&429496729600000000000000000000000000000 a^{64} b^{8} \\
&-2473901162496000000000000000000000000000 a^{62} b^{8} c^{2} \\
&-9620726743040000000000000000000000000000 a^{60} b^{12} \\
&+2473901162496000000000000000000000000000 a^{60} b^{10} c^{2} \\
&+6679533138739200000000000000000000000000 a^{60} b^{8} c^{4} \\
&+2604 \text{ other lower degree terms}
\end{aligned}
$$

and has class number 72.

graphics ...

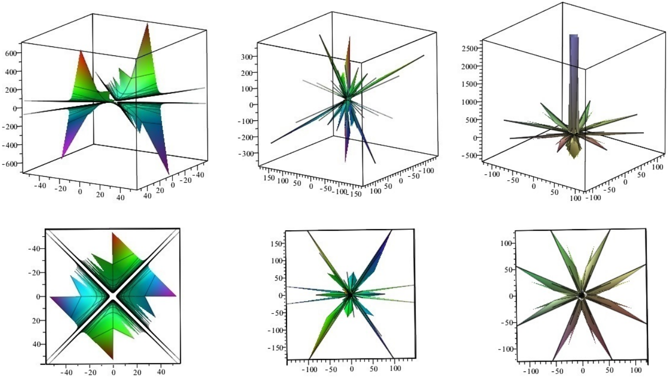

Figure 5. Enneper's surfaces in inhomogeneous tangential coordinates
left: $\widehat{\mathcal{E}}_1(u, v)$, middle: $\widehat{\mathcal{E}}_2(u, v)$, right: $\widehat{\mathcal{E}}_3(u, v)$

graphics ...

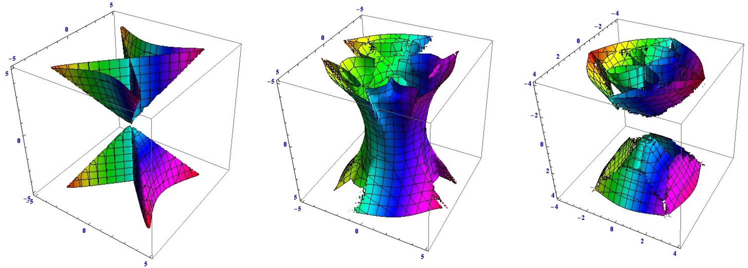

Figure 6. Algebraic surfaces in inhomogeneous tangential coordinates left: $\hat{Q}_1(a, b, c) = 0$, middle: $\hat{Q}_2(a, b, c) = 0$, right: $\hat{Q}_1(a, b, c) = 0$

functions

- We obtain the functions $P_i = P_i(u, v)$:

$$P_1 = -\frac{\left(u^2 - v^2\right)(\lambda - 3)}{3(\lambda - 1)},$$

$$P_3 = -\frac{3\left(u^4 - 6u^2v^2 + v^4\right)(\lambda^3 - 7)}{14\left(\lambda^3 - 1\right)},$$

$$P_5 = -\frac{5\left(u^6 - 15u^4v^2 + 15u^2v^4 - v^6\right)(\lambda^5 - 11)}{33\left(\lambda^5 - 1\right)},$$

functions

- $$P_2 = -\frac{4\left(u^3-3uv^2\right)(\lambda^2-5)}{15\left(\lambda^2-1\right)},$$

$$P_4 = -\frac{8\left(u^5-10u^3v^2+5uv^4\right)\left(\lambda^4-9\right)}{45\left(\lambda^4-1\right)},$$

$$P_6 = -\frac{12\left(u^7-21u^5v^2+35u^3v^4-7uv^6\right)\left(\lambda^6-13\right)}{91\left(\lambda^6-1\right)}.$$

- Then, generalize the functions P_i :

generalization

Corollary

The functions $P_{m\geq 1}$ for integers m, are described by

$$P_{2k-1} = \left(-\frac{2\left(2k-1\right)\ \left[\lambda^{2k-1}-\left(2k+1\right)\right]}{\left(k+1\right)\left(2k+1\right)\left(\lambda^{2k-1}-1\right)} \right) \mathrm{Re}\left(\zeta^{2k}\right),$$

$$P_{2k} = \left(-\frac{4k\ \left[\lambda^{2k}-\left(2k+1\right)\right]}{\left(2k+1\right)\left(4k+1\right)\left(\lambda^{2k}-1\right)} \right) \mathrm{Re}\left(\zeta^{2k+1}\right),$$

where integers $k \geq 1$, $\zeta = u + iv$ and $|\zeta| = \lambda$.

inhomogeneous tangential coordinates

So far, we find surfaces $\widehat{\mathcal{E}}_1$ and $\widehat{\mathcal{E}}_2$. By using $\mathcal{E}_3 - \mathcal{E}_5$, $e_3 - e_5$, and also (12), (13), we obtain the following surfaces: $\widehat{\mathcal{E}}_m(u, v) = (a, b, c)$:

inhomogeneous tangential coordinates

$$\widehat{\mathcal{E}}_1 = -\frac{3}{\left(u^2-v^2\right)\left(\lambda-3\right)}\begin{pmatrix} -2u \\ -2v \\ \lambda+1 \end{pmatrix},$$

$$\widehat{\mathcal{E}}_2 = -\frac{15}{4\left(u^3-3uv^2\right)\left(\lambda^2-5\right)}\begin{pmatrix} -2\left(u^2-v^2\right) \\ -4uv \\ \lambda^2+1 \end{pmatrix},$$

$$\widehat{\mathcal{E}}_3 = -\frac{14}{3\left(u^4-6u^2v^2+v^4\right)\left(\lambda^3-7\right)}\begin{pmatrix} -2\left(u^3-3uv^2\right) \\ -2\left(u^2v-v^3\right) \\ \lambda^3+1 \end{pmatrix},$$

inhomogeneous tangential coordinates

$$\widehat{\mathcal{E}}_4 = -\frac{45\begin{pmatrix} -2\left(u^4-6u^2v^2+v^4\right) \\ -2\left(4u^3v-4uv^3\right) \\ \lambda^4+1 \end{pmatrix}}{8\left(u^5-10u^3v^2+5uv^4\right)\left(\lambda^4-9\right)},$$

$$\widehat{\mathcal{E}}_5 = -\frac{33\begin{pmatrix} -2\left(u^5-10u^3v^2+5uv^4\right) \\ -2\left(5u^4v-10u^2v^3+v^5\right) \\ \lambda^5+1 \end{pmatrix}}{5\left(u^6-15u^4v^2+15u^2v^4-v^6\right)\left(\lambda^5-11\right)}.$$

- We also generalize the above functions, and find the following results:

inhomogeneous tangential coordinates

Corollary

In $\mathbb{E}^{2,1}$, the surfaces $\widehat{\mathcal{E}}_{m\geq 1}(u,v)$ for odd and even integers m, are determined by

$$\widehat{\mathcal{E}}_{2k-1}(u,v) = -\frac{k(4k-1)\begin{pmatrix} -2\mathrm{Re}\left(\zeta^{2k-1}\right) \\ -2\mathrm{Im}\left(\zeta^{2k-1}\right) \\ |\zeta|^{2k-1}+1 \end{pmatrix}}{(2k-1)\left[\lambda^{2k-1}-(2k+1)\right]\mathrm{Re}\left(\zeta^{2k}\right)},$$

$$\widehat{\mathcal{E}}_{2k}(u,v) = -\frac{(2k+1)(4k+1)}{4k\left[\lambda^{2k}-(4k+1)\right]\mathrm{Re}\left(\zeta^{2k+1}\right)}\begin{pmatrix} -2\mathrm{Re}\left(\zeta^{2k}\right) \\ -2\mathrm{Im}\left(\zeta^{2k}\right) \\ |\zeta|^{2k}+1 \end{pmatrix},$$

where integers $k\geq 1$, $\zeta = u+iv$ and $|\zeta| = \lambda$.

inhomogeneous tangential coordinates

Corollary

In $\mathbb{E}^{2,1}$, the relations between the surfaces $\widehat{\mathcal{E}}_{m\geq 1}(u,v)$ and the Gauss maps $e_{m\geq 1}(u,v)$ of the surfaces $\mathcal{E}_{m\geq 1}(u,v)$ are given by

$$\widehat{\mathcal{E}}_{2k-1} = \left(-\frac{k\,(4k-1)\left(\lambda^{2k-1}-1\right)}{(2k-1)\left[\lambda^{2k-1}-(2k+1)\right]\operatorname{Re}\left(\zeta^{2k}\right)} \right) e_{2k-1},$$

$$\widehat{\mathcal{E}}_{2k} = \left(-\frac{(2k+1)\,(4k+1)\left(\lambda^{2k}-1\right)}{4k\left[\lambda^{2k}-(2k+1)\right]\operatorname{Re}\left(\zeta^{2k+1}\right)} \right) e_{2k},$$

where integers $k \geq 1$, $\zeta = u + iv$ and $|\zeta| = \lambda$.

conclusions

To reveal the algebraic surface equations of the Enneper maximal surfaces $\mathcal{E}_m(u, v)$ in $\mathbb{E}^{2,1}$, we have tried a series of standard techniques in elimination theory: only Sylvester by hand for $Q_1(x, y, z) = 0$, and then projective (Macaulay) and sparse multivariate resultants implemented in the Maple software [19] package multires for $Q_m(x, y, z) = 0$ and $\hat{Q}_m(a, b, c) = 0$.

conclusions

Maple's native implicitization command Implicitize, and implicitization based on Maple's native implementation of the Groebner Basis. For the latter, we implemented in Maple the method in [20] (Chapter 3, p. 128). Under reasonable time, we only succeed for $m = 1, 2$ in all above methods.

conclusions

For $m = 3$, the successful method we have tried was to compute the equation defining the elimination ideal using the Groebner Basis package FGb of Faugere in [21].
The time required to output the irreducible algebraic surface equations $Q_m(x, y, z) = 0$ (resp. $\hat{Q}_m(a, b, c) = 0$) for integers $1 \leq m \leq 3$ and polynomials defining the elimination ideal was under reasonable seconds determined by the following Table 1 (resp. Table 2).

conclusions

For the degree (resp. class) of the irreducible algebraic surface equation $Q_4(x, y, z) = 0$ (resp. $\hat{Q}_5(a, b, c) = 0$) of the surface $\mathcal{E}_4(u, v)$ (resp. $\widehat{\mathcal{E}}_5(u, v)$), marked with "$*$" in Table 1 (resp., Table 2), was rejected (i.e., "out of memory") by Maple 17 on a laptop Pentium Core i5-4310M 2.00 GHz, 4 GB RAM, with the time given in CPU seconds.
Hence, we propose the following:

Proposition

For integers $m \geq 1$, *degree number of the irreducible algebraic surfaces* $Q_m(x, y, z) = 0$ *in the Cartesian coordinates is of* $(2m+1)^2$, *and class number of irreducible algebraic surfaces* $\hat{Q}_m(a, b, c) = 0$ *in inhomogeneous tangential coordinates is of* $2m(2m+1)$ *of the* $(1, \zeta^m)$*-type real Enneper maximal surfaces* $\mathcal{E}_m(u, v)$.

all findings

Finally, we give all findings on Table 1, and Table 2.

all findings via tables

Algebraic Surface	Degree of Surface	Number of Terms	Gröbner Time (s)	FGb Time (s)
Q_1	9	23	0.266	0.041
Q_2	25	112	321.953	0.835
Q_3	49	451	*	266.854
Q_4	81	*	*	*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Q_m	$(2m+1)^2$	*	*	*

Table 1. Results for the Enneper algebraic maximal surfaces $Q_m(x,y,z)=0$.

Algebraic Surface	Class of Surface	Number of Terms	Gröbner Time (s)	FGb Time (s)
$\hat{Q}_1$	6	14	0.94	0.030
$\hat{Q}_2$	20	125	61.152	0.114
$\hat{Q}_3$	42	779	*	125.904
$\hat{Q}_4$	72	2609	*	1306.718
$\hat{Q}_5$	110	*	*	*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\hat{Q}_m$	$2m(2m+1)$	*	*	*

Table 2. Results for the Enneper algebraic surfaces $\hat{Q}_m(a,b,c)=0$.

problem 1

Problem

Find the irreducible Enneper algebraic maximal surface eq. $Q_{m\geq 4}(x, y, z) = 0$ in Cartesian coordinates by using the parametric equation of Enneper maximal surface $\mathcal{E}_{m\geq 4}(u, v)$.

problem 2

Problem

Find the irreducible Enneper algebraic surface eq. $\hat{Q}_{m\geq 5}(a, b, c) = 0$ *in inhomogeneous tangential coordinates by using the parametric equation of Enneper surface* $\widehat{\mathcal{E}}_{m\geq 5}(u, v)$.

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... for your patience!