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WHAT IS A TYPICAL QUANTUM STATE?

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Abstract. There is much interest in the understanding of non-classical correlations of observables in quantum systems, such as entanglement. We consider this question for a finite pair of coupled systems, the simplest setting of this problem. Now the dimension of the Hilbert spaces generically grows exponentially with the number of components and only the smallest cases are completely understood. With this in mind we adopt a statistical approach and pose the question - what is a typical quantum state and how can one best (i.e. uniformly) sample these from the state space? Consideration of this question has lead to a particular joint probability distribution governing the eigenvalues of a certain density matrix, known as the Bures-Hall ensemble. Some of the essential results in this understanding will be derived in a simple pedagogical manner, without assuming a prior background and suitable for students in mathematics, statistics and physics.

This then leads to a further question, namely, how do the largest and smallest eigenvalues of the density matrix behave as the system size (number of components) grows? Through recent work by the speaker the answer to this question has been found using cutting-edge tools from integrable probability and approximation theory. Time permitting a brief snapshot of some results will be revealed.

2000 *Mathematics Subject Classification.* ,

1. Essential References for Further Exploration

- Quantum Information Background and Online Notes: [20], [8], [1], [18], [6]
- Quantum Information Monograph: [4]
- Fubini-Study Metrics: [14],[22]
- Original Bures & Hall: [7], [15]
- Understanding Bures Metric: [16], [10]
- Selberg Integrals: [19], [13]
- Random Matrix Theory Monographs: [17], [11]
- Random Matrix Theory Extras: [23], [21]
- Approximation Theory Background: [9]
- Cauchy-Laguerre Matrix Models: [5], [12]
- Contemporary Issues in QIT: [2], [3]

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