Integrability and Control of Figure Skating

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[2] J. of Nonlinear Sci (2019), Nonlinear Dynamics (2022)

Examples of figure skating trajectories

Figure: Left: Compulsory figure skating figure; Right: a cusp on ice (M. Hall).

Thanks to: D. V. Zenkov, J. Hocher, A. A. Bloch, D. D. Holm, . . .

Figure skating: general considerations

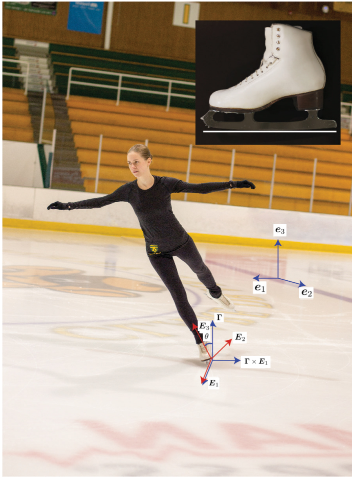

Figure: Figure skater (M. Hall) and coordinate axes: spatial $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_i\}$. Insert: a figure skate. Notice a consistent slight curve of the blade.

Review: Variational principles in mechanics

Consider the system of n variables $\boldsymbol{q} = (q_1, \ldots, q_n)$ and Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$. Equations of motion are given by using the Hamilton's critical action principle

$$\delta S = \delta \int_{t_0}^{t_1} L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mathrm{d}t = 0$$

on variations $\delta \boldsymbol{q}(t_0) = \delta \boldsymbol{q}(t_1) = 0$ The equations of motion are computed by assuming $\boldsymbol{q}(t) = \boldsymbol{q}_0(t) + \epsilon \delta \boldsymbol{q}(t)$ and selecting the first-order terms in ϵ gives Euler-Lagrange equations

$$\begin{aligned} \delta S &= \delta \int_{t_0}^{t_1} L(\boldsymbol{q}_0 + \epsilon \delta \boldsymbol{q}, \dot{\boldsymbol{q}}_0 + \epsilon \delta \dot{\boldsymbol{q}}) \mathrm{d}t \\ &= \epsilon \int \frac{\partial L}{\partial \boldsymbol{q}} \delta \boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \delta \dot{\boldsymbol{q}} \mathrm{d}t + O(\epsilon^2) = \epsilon \int \underbrace{\left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right)}_{=\mathbf{0}:\text{EL eqs}} \delta \boldsymbol{q} \mathrm{d}t + O(\epsilon^2) \end{aligned}$$

$$\text{Euler-Lagrange equations:} \quad - \frac{d}{dt} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} + \frac{\partial L}{\partial \boldsymbol{q}} = \mathbf{0}$$

Holonomic and non-holonomic constraints

1. Suppose there are m constraints that can be reduced to functions of coordinates and time, $f^i(\boldsymbol{q}, t) = 0$, $i = 1, \dots k$. Use the Lagrange multiplier method

$$\delta S = \delta \int_a^b L(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \lambda_i f^i(\boldsymbol{q}, t) \mathrm{d}t = \int \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} + \frac{\partial L}{\partial \boldsymbol{q}} + \lambda_i \frac{\partial f^i}{\partial \boldsymbol{q}} \right) \cdot \delta \boldsymbol{q} \mathrm{d}t$$

gives Euler-Lagrange equations with constraints:

$$\delta S = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = \lambda_i \frac{\partial f^i}{\partial \boldsymbol{q}}$$

assuming $\delta \boldsymbol{q}(a) = \delta \boldsymbol{q}(b) = \mathbf{0}$.

2. If the constraints are affine in velocities, $a^i_k(\boldsymbol{q}, t)\dot{q}^k = b^i(\boldsymbol{q}, t)$, and are not reducible to holonomic constraints, under the right physical assumptions we can use the Lagrange-d'Alembert's principle of non-holonomic mechanics:

$$\delta S = \delta \int L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mathrm{d}t \quad \text{on variations} \quad a^i_k(\boldsymbol{q}, t)\delta q^k = 0$$

Equations of motion of nonholonomic mechanics

1. Enforce the Lagrange-d'Alembert's constraints on variations using Lagrange multipliers $\lambda_i(t)$

$$\delta S = \delta \int_a^b L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mathrm{d}t + \int_a^b \lambda_i a^i_k(\boldsymbol{q}, t) \delta q^k \mathrm{d}t = 0$$

2. Equations of motion of non-holonomic mechanics are computed by collecting the terms proportional to δq^k:

$$\int \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} + \frac{\partial L}{\partial q^k} + \lambda_i a^i_k(\boldsymbol{q}, t) \right) \delta q^k \mathrm{d}t = 0$$

We obtain $n+m$ equations for $n+m$ variables $(q^1, \ldots, q^n, \lambda_1, \ldots \lambda_m)$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = \lambda_i a^i_k(\boldsymbol{q}, t) \,, \quad a^i_k(\boldsymbol{q}, t)\dot{q}^k = b^i(\boldsymbol{q}, t) \,.$$

A discussion of non-holonomic vs Vakonomic approaches

1. As an alternative, enforce $\delta \int L dt$ using the Lagrange multiplier methods for constraints (not variations!) as

$$\delta S_V = \delta \int_a^b L(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \lambda_i \left(a^i_k(\boldsymbol{q}, t)\dot{q}^k - b^i(\boldsymbol{q}, t) \right) \mathrm{d}t = 0$$

2. This procedure will, in general, give equations different from those obtained by the Lagrange-d'Alembert's princple. They are called Vakonomic equations (used e.g. for control theory).
3. Non-holonomic equations (Lagrange-d'Alembert's principle) are obtained in the limit of very large viscous force normal to the constraint.
4. Vakonomic equations are obtained in the limit of increasing some parts of moments of inertia/mass tensors to infinity [1]

[1] Karapetyan (1981), Kozlov (1982/83), Arnold, Kozlov, Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics (2006)*

Frames and coordinates

1. Physics Assume no friction along the skate's direction [2]
2. Physics As with rigid body, choose the variables expressed in the body frame
3. Math A figure skater is represented by a (articulated/pseudo[3]) rigid body moving in space.
4. Math Configuration manifold $G = SE(3)$ (rotations/translations). Variables are Λ (orientation) and $\mathbf{r}$ (position).
5. Physics/Math For dynamics, assume static skater. For control, skater can change its configuration.

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[2] See Rosenberg (2005), Lozowski et al. (2013), Berre and Pomeau (2015) for discussion of friction on the skate

[3] (e.g. Amstrong & Green, Graphics Interphace (1985) Holm, Schmah, Stoica (2009))

Equation of motion

1. Lagrangian is kinetic minus potential energy. In general, depends on $(\Lambda, \dot{\Lambda}, \mathbf{r}, \dot{\mathbf{r}})$ - Lots of variables and constraints!
2. Symmetry reduced Lagrangian depends on the angular $\boldsymbol{\Omega}$ and linear $\boldsymbol{Y}$ velocities in the body frame, vertical vector $\boldsymbol{\Gamma} = \Lambda^T \mathbf{e}_3$ and vector $\mathbf{A}$ from contact point to CM:

$$L = \frac{1}{2} \langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle + \frac{1}{2} m \left| \boldsymbol{\Omega} \times \mathbf{A} + \boldsymbol{Y} \right|^2 - mg \langle \mathbf{A}, \boldsymbol{\Gamma} \rangle$$

3. Holonomic constraints: a) pitch constancy (the blade does not tilt forward/backward) and b) continuous ice contact
4. Non-holonomic constraint: the blade cannot move normal to itself.

Equations of motion:
Lagrange-d'Alembert's method and symmetry reduction

Equations of motion

$$\left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\boldsymbol{\Pi} - mg\boldsymbol{\Gamma}\times\mathbf{A} + \boldsymbol{Y}\times\mathbf{P} = \kappa\big(\mathbf{E}_1\times\boldsymbol{\Gamma}\big)$$

$$\left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\mathbf{P} = \lambda\boldsymbol{\Gamma} + \mu\big(\mathbf{E}_1\times\boldsymbol{\Gamma}\big)$$

$$\mathbf{P} = \frac{\partial L}{\partial \boldsymbol{Y}} = \boldsymbol{Y} + \boldsymbol{\Omega}\times\mathbf{A} \quad \text{(linear momentum)}$$

$$\boldsymbol{\Pi} = \frac{\partial L}{\partial \boldsymbol{\Omega}} = \mathbb{I}\boldsymbol{\Omega} + \mathbf{A}\times\mathbf{P} \quad \text{(angular momentum)}$$

In addition, $\dot{\boldsymbol{\Gamma}} = -\boldsymbol{\Omega}\times\boldsymbol{\Gamma}$, and there are three constraints: pitch constancy, continuous contact and non-holonomic constraint :

$$\langle \boldsymbol{E}_1, \boldsymbol{\Gamma}\rangle = 0\,, \quad \langle \boldsymbol{R}, \boldsymbol{\Gamma}\rangle = 0\,, \quad \langle \boldsymbol{Y}, \boldsymbol{E}_1\times\boldsymbol{\Gamma}\rangle = 0\,.$$

Phase space and conservation laws

1. The phase space of reduced system is 4-dimensional: inclination angle θ wrt vertical, $\Omega_1 = \dot{\theta}$, $\Omega_2 = \langle \boldsymbol{\Omega}, \boldsymbol{\Gamma} \rangle$ (vertical angular velocity) and skate's velocity v.
2. Once these are known, the trajectory on the ice can be reconstructed
3. The energy (kinetic+potential energy) of the system is conserved, see, e.g. Bloch (2003):

$$E = \frac{1}{2}\langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle + \frac{1}{2} m \left| \boldsymbol{Y} + \boldsymbol{\Omega} \times \mathbf{A} \right|^2 + mg \langle \mathbf{A}, \boldsymbol{\Gamma} \rangle = \text{const.}$$

4. For general $\mathbb{I}$ and $\mathbf{A}$, conservation of energy is all the information one can obtain from the general principles.

Symmetry considerations

- The system (Lagrangian and constraints) is invariant with respect to (left) rotations and translations along the ice: symmetry group is $SE(2)$.
  - Additional symmetry of rotation about vertical axis if $\langle \mathbf{A}, \boldsymbol{E}_1 \rangle = 0$, and inertia tensor is diagonal
  - Nonholonomic Noether's theorem (Kozlov (2002), Fasso & Sansonetto (2005)), the vertical angular momentum $J_1 = \langle \mathbb{I}\boldsymbol{\Omega}, \boldsymbol{\Gamma} \rangle$ is conserved

- This is as far as you can get using standard procedure... Almost there. Need one more constant of motion for integrability.
- We can look for first integrals (constants of motion) as linear functions of $\langle \mathbf{E}_1, \mathbf{P} \rangle$, $\langle \boldsymbol{\Gamma}, \boldsymbol{\Pi} \rangle$ and $\langle \mathbf{E}_1, \boldsymbol{\Pi} \rangle$. They are known as the Gauge integrals [4]

[4]Bates & Snyaticki (1993), Cushman *et al* (1998), Fasso *et al* (2008), Balsero & Sansonetto (2016), Garcia-Naranjo and Montaldi (2017), . . .

Complete solution: integrability for $\langle \mathbf{A}, \boldsymbol{E}_1 \rangle = 0$

It turns out that we can look for a constant of motion in the form $J_2 = \beta(\theta) \langle \boldsymbol{\Gamma}, \boldsymbol{\Pi} \rangle$, and after some computations we get:

$$J_2 = \begin{cases} v + 2\Omega_2 \langle \mathbf{A}, \boldsymbol{E}_1 \times \boldsymbol{\Gamma} \rangle & \text{if } I_2 = I_3 \\ v + \Omega_2 \langle \mathbf{A}, \boldsymbol{E}_1 \times \boldsymbol{\Gamma} \rangle - \frac{J_1 A_2}{\sqrt{I_2 |\Delta I|}} \operatorname{arctanh}(\sqrt{\frac{|\Delta I|}{I_2}} \Gamma_3) & \\ \quad + \frac{J_1 A_3}{\sqrt{I_3 |\Delta I|}} \arctan(\sqrt{\frac{|\Delta I|}{I_3}} \Gamma_2), \quad \text{if } I_2 > I_3 & \\ v + \Omega_2 \langle \mathbf{A}, \boldsymbol{E}_1 \times \boldsymbol{\Gamma} \rangle - \frac{J_1 A_2}{\sqrt{I_2 |\Delta I|}} \arctan(\sqrt{\frac{|\Delta I|}{I_2}} \Gamma_3) & \\ \quad + \frac{J_1 A_3}{\sqrt{I_3 |\Delta I|}} \operatorname{arctanh}(\sqrt{\frac{|\Delta I|}{I_3}} \Gamma_2), \quad \text{if } I_3 > I_2 & \end{cases}$$

where $\Omega_2 = \langle \boldsymbol{\Omega}, \boldsymbol{\Gamma} \rangle$ and $\Delta I = I_2 - I_3$.

Examples of integrable non-holonomic systems

(Bates & Cushman 1999, Kozlov 2002, Bloch 2003, ...)

1. *Routh's sphere* A dynamically symmetric sphere with an off-set center of mass rolling without friction
2. *Chaplygin's sphere* An inhomogeneous sphere with the center of mass coinciding with the geometric center
3. *Chaplygin's sleigh*
4. *Suslov's problem*: Rigid body with the constraint $\langle \boldsymbol{\Omega}, \mathbf{a} \rangle = 0$, where $\mathbf{a}$ is a fixed vector in the body frame.
5. Suslov's top: same as above, with the addition of gravity $U(\boldsymbol{\Gamma})$ (Veselova problem).
6. Rolling vertical disk
7. A blade on an inclined plane
8. ...

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6. Rolling vertical disk
7. A blade on an inclined plane
8. ...
9. Figure skating!

Numerical simulations: parameters

1. Skater is model by a uniform rectangular block with the parameters:
   - $m = 50$ kg
   - $I_1 = 15.95 \text{kg}\cdot m^2$ (rotation axis along the skate),
   - $I_2 = 13.56 \text{ kg}\cdot m^2$ (rotation about the sideways axis),
   - $I_3 = 3.99 \text{ kg}\cdot m^2$ (rotation about the vertical body axis going from the point of contact through the ankle).
2. The center of mass is taken to be at $\mathbf{A} = (A_1, A_2, A_3)$ in the frame of the skate, with
   - $A_1 = 0\, m$ (integrable case) and $A_1 = 0.1\, m$ (non-integrable case),
   - $A_2 = 0.12\, m$ (sideways axis),
   - $A_3 = 0.875\, m$ (vertical body axis) and two cases.
3. The initial conditions are
   - $\Omega_1(0) = 0.01 s^{-1}$ (rotation about the skate's axis),
   - $\Omega_2(0) = 1.25 s^{-1}$ (rotation about the vertical)
   - $v(0) = 0.5 m/s$ (initial velocity).

Numerical results: Dynamic variables

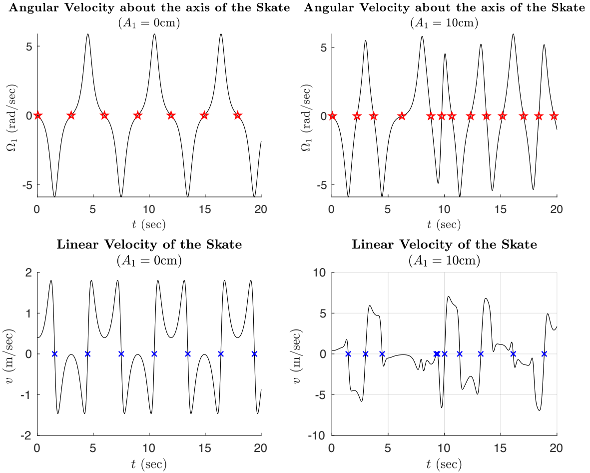

Figure: The Behavior of dynamic variables $\Omega_1 = \dot{\theta}$ and v as a function of time. Left: integrable case; right: chaotic case. Blue crosses: $v = 0$, red stars: $\Omega_1 = \dot{\theta} = 0$.

Numerical results: Phase space and ice trajectories

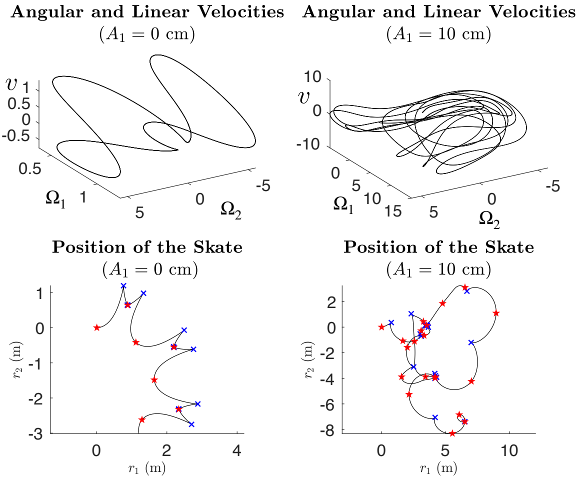

Figure: Phase space and ice trajectories. Left: integrable case; right: chaotic case. Blue crosses: $v = 0$, red stars: $\Omega_1 = \dot{\theta} = 0$.

Numerical results: Constants of motion

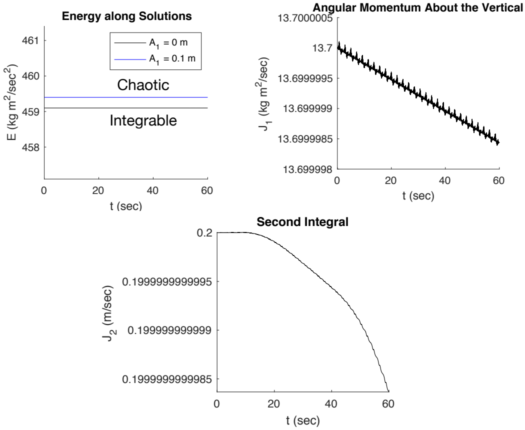

Figure: Energy (integrable/chaotic case) and integrals of motions J_1 and J_2 for $A_1 = 0$.

Numerical results: Chaotic behavior for $A_1 \neq 0$

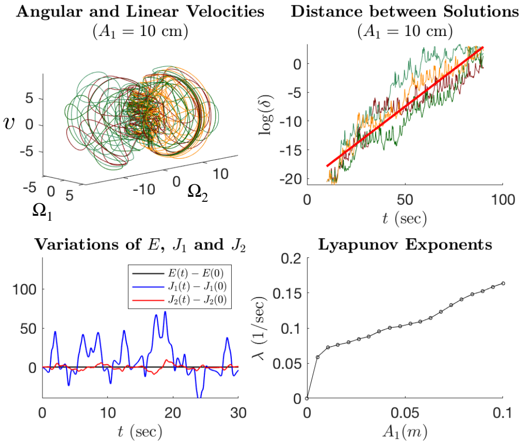

Figure: Trajectories, constants of motion and growth of distances between nearby trajectories on the same energy surface for $A_1 = 0.1m$

Numerical results: Bifurcation from the integrable case

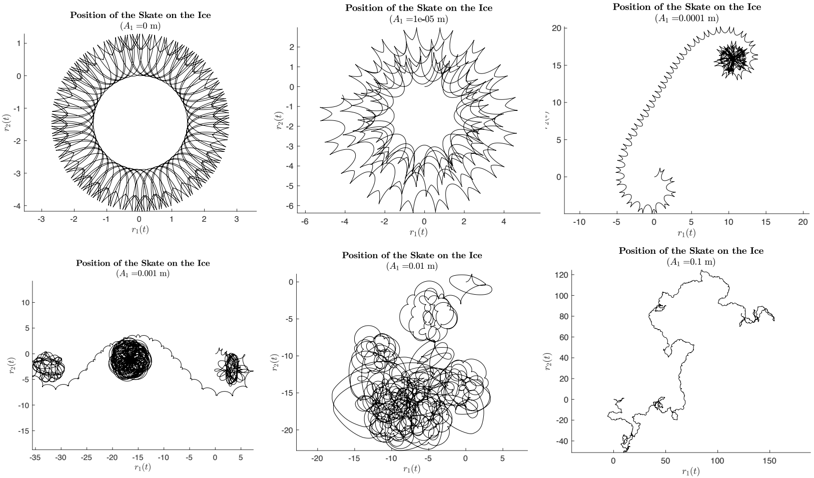

Figure: Ice trajectories for a variety of values of A_1.

Bifurcation from the integrable case, expanded

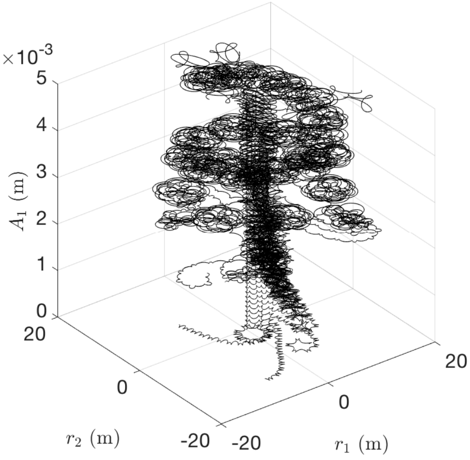

Figure: Ice trajectories shown for increasing values of A_1 (vertical line).

Trajectory tracing and control

How does a figure skater trace the desired trajectory on ice?

Preliminaries: Hamel's approach to mechanics

- Lagrange's mechanics: coordinates $(\boldsymbol{q}, \dot{\boldsymbol{q}})$
- Lagrange's momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$, eqs of motion $\frac{\mathrm{d} p_i}{\mathrm{d} t} - \frac{\partial L}{\partial q_i} = F_i$
- For constrained systems, awkward (Lagrange multipliers)
- Introduce quasivelocities $\boldsymbol{\xi}$ according to $\dot{\boldsymbol{q}} = \mathbb{A}(\boldsymbol{q})\boldsymbol{\xi}$
- Define the Lagrangian $\ell(\boldsymbol{q}, \boldsymbol{\xi}) = L(\boldsymbol{q}, \mathbb{A}(\boldsymbol{q})\boldsymbol{\xi})$
- Define vector fields u_i and their action on functions
$$u_i = A_i^j(\boldsymbol{q}) \frac{\partial}{\partial q^j} \quad \Rightarrow \quad u_i[\ell] = A_i^j(\boldsymbol{q}) \frac{\partial \ell}{\partial q^j}$$
- The commutator $[u_i, u_j]$ gives rise to functions $c_{ij}^m(\boldsymbol{q})$:
$$[u_i, u_j] = c_{ij}^k(\boldsymbol{q}) u_k \, , \quad c_{ij}^m(\boldsymbol{q}) = \left(\mathbb{A}^{-1}\right)_s^m \left(\frac{\partial A_j^s}{\partial q^p} A_i^p - \frac{\partial A_i^s}{\partial q^p} A_j^p \right)$$
- Hamel's equations of motion are [5]
$$\frac{dp_i}{dt} = c_{ji}^m \frac{\partial \ell}{\partial \xi^m} \xi^j + u_i[\ell] \, , \quad p_i := \frac{\partial \ell}{\partial \xi^i} \, , \quad \xi_i = \xi_i(\mathbf{p})$$
- Hamel's eqs reduce to Euler-Lagrange eqs if $\mathbb{A} = \mathrm{Id}$, *i.e.* $\dot{\boldsymbol{q}} = \boldsymbol{\xi}$.

[5] G. Hamel, *Z. Math. Phys*, (1904)

Hamel's equations for constrained mechanics

Hamel's approach is particularly useful for nonholonomic systems [6]

- Suppose there are m nonholonomic constraints for an n-dimensional system which are expressed as

$$a_j^k(\boldsymbol{q})\dot{q}^j = 0\,, \quad k = 1, \ldots, m\,.$$

- Define the last m quasivelocities to be exactly the constraints:

$$\xi^{n-k} = a_j^k(\boldsymbol{q})\dot{q}^j\,, \quad k = 1, \ldots, m\,.$$

- The first $n-m$ velocities are described in an arbitrary (non-degenerate) way
- The equations of motion are:

$$\begin{cases} \dfrac{dp_i}{dt} = c_{ji}^m p_m \xi^j + u_i[\ell]\,, \quad p_i := \frac{\partial \ell}{\partial \xi^i}\,, & i = 1, \ldots, n-m \\ a_j^k(\boldsymbol{q})\dot{q}^j = 0\,, & k = 1, \ldots, m \end{cases}$$

Only n equations and no Lagrange multipliers!

[6] Bloch, Marsden, Zenkov, *Dyn. Sys* (2009); Ball, Zenkov, *Geometry, Dynamics and Mechanics*, (2015); Shi, Zenkov, Bloch, *JNLS* (2017, 2020)

Trajectory tracing on ice

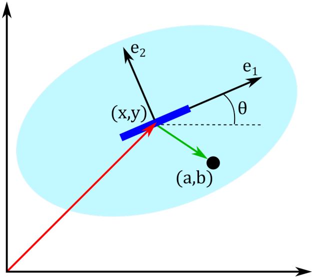

Figure: Chaplygin sleigh with added mass m

- Dynamics of Chaplygin sleigh with a moving mass: Bizyaev *et al.*, *Reg. Chaotic Dynamics* (2017), *Nonlinear Dynamics* (2019), *Nonlinearity* (2019); Fedonyuk & Tallapragada: Proc Am. Control Conf, IEEE (2017), *Nonlinear Dynamics* (2018)
- Control of Chaplygin's sleigh: Osborne & Zenkov (2005) (moving mass) , Fedonyuk & Tallapragada, *Am. Control Conf. IEEE* (2020) (trajectory tracing with a rotor)

Constraints, quasivelocities and Hamel's equations

- Configuration manifold $SO(2)$ with coordinates (x, y, θ)
- Constraint on velocity: $-\dot{x}\sin\theta + \dot{y}\cos\theta = 0$
- Therefore, choose the quasivelocities:

$$\begin{aligned} \xi^1 &= \dot{\theta}, \\ \xi^2 &= \dot{x}\cos\theta + \dot{y}\sin\theta\,, \\ \xi^3 &= -\dot{x}\sin\theta + \dot{y}\cos\theta \quad (=0) \end{aligned}$$

- Largangian is kinetic energy
- Nonholonomic momenta $p_i = \frac{\partial \ell}{\partial \xi_i}$, express $\xi^i = \xi^i(\mathbf{p})$
- Physical meaning of p_i for $b = 0$:
  1. p_1- angular momentum wrt contact point,
  2. p_2- projection of linear momentum on blade's direction.

Equations of motion of a Chaplygin's sleigh with a moving mass

Hamel's approach gives equations for momenta, velocity and trajectory tracing and constraint [7]

$$\begin{cases} \dot{p}_1 = -m\eta\xi^2 , \quad \dot{p}_2 = m\eta\xi^1 , \quad \dot{\theta} = \xi^1 \\ \dot{x} = \xi^2 \cos\theta , \quad \dot{y} = \xi^2 \sin\theta \end{cases}$$

where

$$\begin{cases} \xi^1 = \dfrac{1}{\gamma}\Big((M+m)(p_1 - ma\dot{b}) + mb(p_2 + M\dot{a}) \Big) , \\ \xi^2 = \dfrac{1}{\gamma}\Big(m[b(p_1 - ma\dot{b}) - (I + ma^2)\dot{a}] + [I + m(a^2+b^2)]p_2 \Big) , \\ \eta = \dfrac{1}{\gamma}\Big([Mmb^2 + I(M+m)]\dot{b} + a[(M+m)p_1 + mb(p_2 + M\dot{a})] \Big) , \\ \gamma = (M+m)(I + ma^2) + Mmb^2 . \end{cases}$$

[7] Osborne & Zenkov *Proc 44th IEEE Conf. on Decision and Control* (2005)

Formulation of figure skating control problem

There is no requirement on speed to follow the curve on ice. Therefore, the most general control procedure is formulated as

Problem (General statement of control procedure)

Suppose a given piecewise smooth plane curve $x = X(s), y = Y(s)$ forms a graph G on (x, y) plane. Find the initial conditions and controls $(a, b, \dot{a}, \dot{b})$ such that the graph G_s of the solution curve given by equations of motion minimizes the deviation from the curve in some norm.

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Very difficult

Maybe possible with application of AI algorithms such as reinforcement learning. We will use an alternative method.

Tracing circular arcs

Lemma (On tracing circular trajectories)

A trajectory is a circular arc of radius r if and only if the motion of control masses satisfies $\xi^2 = r\xi^1$, yielding an affine relationship between $(\dot{a}, \dot{b})$:

$$\begin{aligned}
&A\dot{a} + B\dot{b} = C\,, \quad \textit{where} \\
&A := mMrb + m(I + ma^2)\,, \\
&B := m^2ab - (M+m)mar\,, \\
&C := p_1\left[mb - r(M+m)\right] + p_2\left[I + m(a^2+b^2) - mbr\right]\,.
\end{aligned}$$

Proof:

1. Arclength changes as $ds = \sqrt{dx^2 + dy^2} = \xi^2 \mathrm{d}t$
2. Angle changes as $d\theta = \xi^1 \mathrm{d}t$
3. Equation for a circle is $\theta'(s) = 1/r$ giving $\xi^2 = r\xi^1$

Corollary: For a straight line, $r = \infty$ and the condition is $\xi^1 = 0$:

$$(M+m)(p_1 - ma\dot{b}) + mb(p_2 + M\dot{a}) = 0$$

System reduction for circular arcs

Integral of motion Any motion of the system on a circle of radius r also yields the first integral

$$p_1 + rp_2 = \mathrm{const}$$

Proof: Since $\dot{p}_1 = -m\eta\xi^2$ and $\dot{p}_2 = m\eta\xi^1$ we have $\dot{p}_1 + r\dot{p}_2 = 0$ if $\xi^2 = r\xi^1$.

Control mechanism: Given a circular trajectory, one could in principle select *e.g.* $\dot{b}$ and calculate $\dot{a}$, but this approach leads to singularities when $b = 0$ (infinite velocities)
Need to choose another control mechanism

Lazy figure skater

Choose the control tracing a circular trajectory of radius r and minimizing the kinetic-energy like quantity

$$(v_a, v_b) = \arg\min \frac{1}{2}(v_a^2 + v_b^2) \text{ s.t. } Av_a + Bv_b = C$$

with definition of (A, B, C) as before

$$\begin{aligned} A &:= mMrb + m(I + ma^2)\,, \\ B &:= m^2 ab - (M+m)mar\,, \\ C &:= p_1\left[mb - r(M+m)\right] + p_2\left[I + m(a^2+b^2) - mbr\right]\,. \end{aligned}$$

The solution of the optimization problem is

$$\dot{a} = v_a = \frac{AC}{A^2+B^2}\,, \quad \dot{b} = v_b = \frac{BC}{A^2+B^2}\,.$$

This control procedure yields a robust and stable system for simulations.

Approximation of a given curve by circular arcs

Lemma (On approximating smooth curves by circular arcs)

(Meek & Walton, J. Comp. Appl. Math, (1995)): If the bounding circular arcs enclose a given spiral segment of positive curvature [a] *$Q(s)$, $s_0 \leq s \leq s_1$, and a biarc that matches the same data as the bounding circular arcs is found, then the maximum distance between the biarc and the spiral is $\mathcal{O}(h^3)$, where $h = s_1 - s_0$.*

[a] A spiral arc has local radius of curvature varying with arclength

For a non-spiral curve, Meek & Walton separate the curve into arcs and apply the approximation.

We use the following control procedure:

1. Smooth parts of trajectories are approximated by circular arcs as per lemma above
2. At the cusps, a skater performs an instantaneous finite turn breaking nonhlonomic constraint
3. The linear velocity at the cusp needs to vanish: $\xi^2 = 0$

Control mechanism for trajectory tracing on ice

For each smooth part of the curve approximated by arcs, find the lazy figure skater control solution optimizing the 'relative kinetic energy' $\dot{a}^2 + \dot{b}^2$, and satisfying

1. tracing the arc of given radius and
2. vanishing of the linear velocity ξ^2 at the end of each smooth part of the trajectory.

In our work, each smooth part of the trajectory is composed of only one circular arc.

Inner trajectory of the figure skating pattern

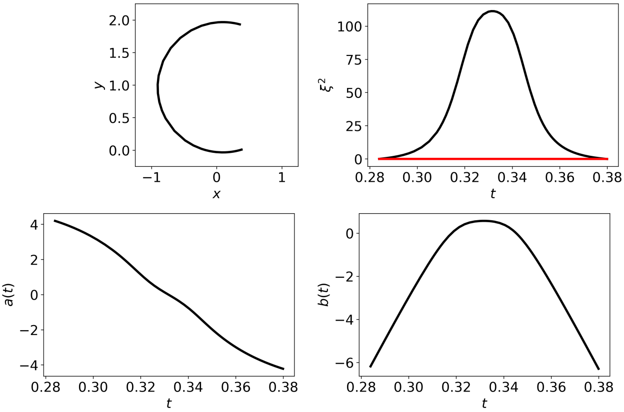

Figure: Inner pattern trajectory (top left), ξ^2 profile (top right), and optimized control functions, $a(t)$ (bottom left) and $b(t)$ (bottom right).

Outer trajectory of the figure skating pattern

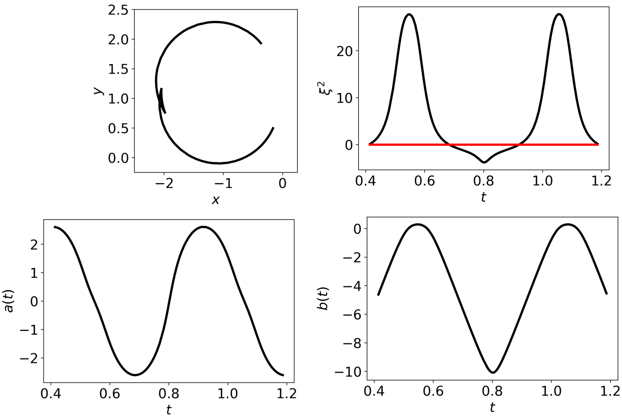

Figure: Outer pattern trajectory (top left), ξ^2 profile (top right), and optimized control functions, $a(t)$ (bottom left) and $b(t)$ (bottom right).

Combined figure

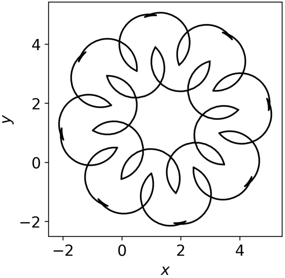

Figure: Full pattern reconstruction using the control procedure.

Conclusions

1. Do circles/spheres represent special trajectories for autonomous vehicles?
2. Application to the dynamics control of underwater vehicles (nonholonomic vs vakonomic?)
3. Extension of theory to coupled rigid bodies with elastic connection modelling body+leg up to ankle/knee to explain injuries caused by 'catching the blade'.
4. Dynamics and control of skating robots.

Thank you!