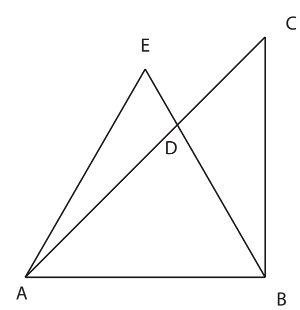
EMMY NOETHER MATHEMATICS DAY
Texas Tech University
May 16, 2018

SOLUTIONS.

1.) The one-digit numbers 1 through 9 take up the first 9 spaces in the printing. The 90 two-digit numbers from 10 through 99 take up the next 180 spaces in printing. Thus 189 spaces are used to print the one-digit and two-digit numbers. The next 810 spaces are used in printing the first 270 three-digit numbers, from 100 through 369. At this point, 999 spaces have been used to print the numbers so far and the next, 1000th, digit printed is the first digit of the next number, 370, i.e. $\boxed{3}$.

2.) The number can not be a one-digit number, since it is then the sum of its digits. If the number is a three-digit number, then the sum of the digits can be at most 27 (the sum if all three digits are 9) and three times the sum of the digits can be at most 81, a two-digit number. The discrepancy becomes even greater if the number has four or more digits. Thus the number being considered must be a two-digit number. The number must also be a multiple of 3. The sum of the digits can be at most 18, so the number can be at most 54. However, if the number is a multiple of 3, then the sum of its digits is also a multiple of 3. Since the number is 3 times the sum of its digits, it must then be a multiple of 9. If the number is a multiple of 9, then the sum of its digits is also a multiple of 9. Since the number is 3 times the sum of its digits, it must be a multiple of 27. The numbers 27 and 54 are the only remaining numbers satifying these conditions. A quick check shows that the number must be $\boxed{27 = 3(2{+}7)}$ as the only solution. The property that if m is a multiple of n, then the sum of the digits of m is also a multiple of n, only holds if n is 1, 3 or 9. Thus, our analysis above could not have been extended further.

3.) Rather than directly trying to compute the area of triangle $\triangle ADE$ it is probably easier to compute the areas of triangles $\triangle ABE$ and $\triangle ABD$ and take the difference. The area of a triangle is given by area $= \frac{1}{2}$ (base) (height). Each of triangles $\triangle ABE$ and $\triangle ABD$ has base AB of length 1. Triangle $\triangle ABE$ is an equilateral triangle of base 1 and so has height of $\dfrac{\sqrt{3}}{2}$. (This is a standard fact for equilateral triangles or can be derived from properties of $30° - 60°$ right triangles and the Pythagorean theorem.) The area of triangle $\triangle ABE$ is thus $\dfrac{1}{2}(1)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{3}}{4}$.



To determine the area of triangle $\triangle ADE$ we need to determine its height and hence the location of the point D. The point D is at the intersection of line segment AC with line segment EB. Consider a coordinate system with origin at the point A, $x-$axis horizontal and $y-$axis vertical. Line segment AC has equation $y = x$. Line segment EB has equation $y = \sqrt{3} - \sqrt{3}x$. (Consider its slope and $y-$intercept.) Solving this system of two equations for the point of intersection D, one has $y = \dfrac{\sqrt{3}}{1+\sqrt{3}}$, giving the area of triangle $\triangle ABD$ as $\dfrac{1}{2}(1)\left(\dfrac{\sqrt{3}}{1+\sqrt{3}}\right)$. The area of triangle $\triangle ADE$ is thus $\boxed{\dfrac{\sqrt{3}}{4} - \dfrac{1}{2}\dfrac{\sqrt{3}}{1+\sqrt{3}} = \dfrac{2\sqrt{3}-3}{4}.}$

4.) We do not know how many students are in the class, but this information is not needed. Let x be the percentage of students who passed the exam and $100-x$ be the percentage of students who failed the exam. From the given information $\dfrac{(x)(75)+(100-x)(45)}{100} = 63$. The solution of this equation is $x = 60$ and hence $\boxed{60\%}$ of the students passed the exam. (Note that though we know that over half of the students passed the exam and the average grade of all students was 63, we cannot conclude that 63 was a passing grade. The above averages are possible with a passing grade of 70 and at least 60% of students doing above average.)

5.) This problem is complicated by the fact that water is continuing to flow into the ocean liner while the pumps work to pump it out. There are three quantities to consider. Let w be the amount of water present in the ship when the pumps begin to operate. Let x be the rate of inflow of water into the ocean liner per hour. Let y be the amount of water pumped out of the ship by a single pump per hour. From the information given we can form two equations, $w + 3x - (12)(3)y = 0$ and $w + 10x - (5)(10)y = 0$, reflecting the fact that with 12 pumps the amount of water in the ship after 3 hours is 0 and with 5 pumps the amount of water in the ship after 10 hours is 0. Subtracting the first equation from the second equation gives $7x - 14y = 0$ or $x = 2y$. Substituting this for x in either of the two equations gives $w - 30y = 0$ or $w = 30y$.

If n is the number of pumps required to pump all of the water out of the ship in 2 hours, then $w + 2x - (n)(2)y = 0$. Substituting $x = 2y$ and $w = 30y$ gives $34y - (n)(2)y = 0$, or $\boxed{n = 17}$ pumps are needed to pump all of the water out of the ship in 2 hours.

(Note that though we had only two equations and three variables, w, x and y, we never solved for the three variables, only for the ratios between them. We do not have enough information to determine the value of each, but since each equation is a linear expression equal to 0, i.e. this is a homogeneous system of equations, with the equations independent, the ratios are uniquely determined.)

6.) Let n be the number of nickels which Debra has, d the number of dimes which she has and q the number of quarters which she has. From the given information we have that $n + d + q = 12$ (from the number of coins) and $5n + 10d + 25q = 180$ (from the monetary value of the coins). Without any other restrictions on the values of n, d and q these equations have infinitely many solutions (allowing for non-integer and non-positive values of the variables). However, we are only interested in solutions where each of n, d and q is a positive integer. Rather than trying to distinguish which of the infinitely many numerical solutions satisfy these additional constraints, we consider cases.

If there is only one quarter, the remaining eleven coins (each worth at most ten cents) have a total value of \$1.55, an impossibility.

Similarly, if there are two quarters, the remaining ten coins (each worth at most ten cents) have a total value of \$1.30, also an impossibility.

If there are three quarters, then the remaining nine coins (each worth at most ten cents) have a total value of \$1.05, still an impossibility.

If there are four quarters, the remaining eight coins have a total value of \$0.80. This is possible if each of the eight remaining coins is a dime, but is not a valid answer since there are no nickels.

If there are five quarters, the remaining seven coins have a total value of \$0.55, which is only possible with 3 nickels and four dimes.

If there are six quarters, the remaining six coins have a total value of \$0.30. As indicated in the statement of the problem, this is possible if each of the remaining coins is a nickel, but it is not a valid answer since there are no dimes.

If there are seven or more quarters, the total value of the twelve coins must exceed the indicated amount of \$1.80.

Thus, the only valid solution, with at least one of each type coin, is

$\boxed{\text{3 nickels, 4 dimes and 5 quarters.}}$