# EMMY NOETHER MIDDLE SCHOOL MATHEMATICS DAY
Texas Tech University
May 17, 2017

SOLUTIONS.

1.) Going upstream, Anne and Betty are paddling against the current. We are told that they paddle at twice the rate of the current. If the rate of the current is $c$, then Anne and Betty paddle at a rate of $2c$. Going upstream, their rate relative to the land is $c - 2c = -c$. (Subtraction since they are paddling in the direction opposite the current.) Going downstream, their rate relative to the land is $c + 2c = 3c$. (Addition since they are paddling with the current.) Thus, relative to the land Anne and Betty travel downstream three times as fast as they travel upstream. On the first two days, they travel upstream for a total of 15 hours, 7 hours the first day and 8 hours the second day. Thus, it will take them one-third as long, or 5 hours to travel the same distance downstream. They must start downstream on the third day 5 hours before they are to meet Karen at 3:00 pm, hence by $\boxed{\text{10:00am.}}$

2.) There are two cases to consider, whether all three letters of the word are different or whether the word contains two identical letters. If all three letters are different, then there are six choices for the first letter of the word, five choices for the second letter and four choices for the third letter, or $6 \cdot 5 \cdot 4 = 120$ possible words. If the word includes two identical letters, then there are two choices for which letter, D or I, is repeated, then there are three choices for which spaces the repeated letters occur in the word (first and second letter, first and third letter or second and third letter) and for each of these choices five choices for the remaining letter, or $2 \cdot 3 \cdot 5 = 30$ possible words. There is thus a total of $120 + 30 = \boxed{150}$ possible words.

3.) Rather than simply trying a large number of cases and hoping to discover a solution, one can do some analysis to reduce the number of case to be considered. Assume that there exist integers $a$ and $b$ such that $a^2 + b^2 = 2017$, and let $a$ be the larger of the integers. Without loss of generality one can assume that each integer is positive. Then $1008 < a^2 < 2017$. Since $a$ is a positive integer, this implies that $31 < a < 45$. This is a small enough range of possibilities that one could try each possible value for $a$ to determine if there is a corresponding value for $b$. However, further analysis can reduce the number of cases to check. If $a$ is an integer, then $a^2$ ends in the digit 1, 4, 9, 6, 5 or 0. The same is the case for $b^2$. The only combination of these which will produce a number ending in 7 is for one of $a^2$ or $b^2$ to end in 1 and the other in 6. If $a$ is an integer, then $a^2$ ends in 1 if and only if $a$ ends in either 1 or 9. Similarly, if $a$ is an integer, then $a^2$ ends in 6 if and only if $a$ ends in either 4 or 6. Combining these with the results which we had earlier, the only possibilities for $a$ are 34, 36, 39, 41 and 44. A quick check of these five cases shows that the only case where $a^2 + b^2 = 2017$ for positive integers $a$ and $b$ is when $\boxed{a = 44 \text{ and } b = 9}$ in which case $a^2 + b^2 = (44)^2 + (9)^2 = 1936 + 81 = 2017$.

4.) At 12:00 noon, the hour hand and minute hand both point in the same direction, pointing directly at 12. Between 12:00 noon and 12:00 midnight the hour hand makes one complete circuit while the minute hand makes 12 complete circuits, 11 more than the hour hand. The two hands will point in directly opposite directions when the minute hand has made an integer plus $\frac{1}{2}$ number of circuits more than the hour hand, i.e. when the number of circuits more made by the minute hand compared to the hour hand is one of $\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \cdots, 10\frac{1}{2}$. Thus there are $\boxed{\text{11 times}}$ when the hands are directly opposite each other.

5.) We are not given the dimensions of the circles or the hexagon. However, we are to determine the fraction of the area of the hexagon included in the circular discs. This ratio is independent of scale. For convenience, assume that each circle has radius 1. Each side of the hexagon consists of the radii of two adjacent circles and so is of length 2. Each circle has area $\pi \cdot (1)^2 = \pi$. Inside the hexagon is the area of the central circle and 1/3 the area of each of the outer circles. Thus, the area of the hexagon included in the circular discs is $1 \cdot \pi + 6 \cdot \frac{1}{3}\pi = 3\pi$. The hexagon can be divided into six equilateral triangles, each having base along one side of the hexagon and opposite vertex at the center of the central circle. The area of each such triangle is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$. Thus the area of the hexagon is $6 \cdot \sqrt{3}$. The fraction of the area of the hexagon included in the circular discs is thus $\boxed{\frac{3\pi}{6\sqrt{3}} \approx 0.9069}$.

6.) During a nonleap year of 365 days, there are 52 weeks of seven days each plus one extra day. During a leap year of 366 days, there are 52 weeks of seven days each plus two extra days. From 1914 through 2017 is 103 years, of which 26 years (the years from 1916 through 2016 divisible by 4) are leap years. Thus the day of the week of a particular date advances 129 days from 1914 through 2017. 129 days is 18 weeks plus 3 days. $(129 = 18 \cdot 7 + 3.)$ Since May 14, 2017 is a Sunday, May 14, 1914 was a Thursday, three weekdays earlier. Thus the second Sunday in May in 1914 was 4 days earlier and the first Mother's Day was $\boxed{\text{Sunday May 10, 1914.}}$