EMMY NOETHER HIGH-SCHOOL MATHEMATICS DAY
Texas Tech University
May 17, 2017

SOLUTIONS.

1.) If n is an integer, then n can be expressed in one of the three forms $n = 3k$, $n = 3k+1$ or $n = 3k+2$ for some integer k. In the first case, n is a multiple of 3. If n is not a multiple of 3, then it is of one of the two forms $n = 3k+1$ or $n = 3k+2$. In these cases, $n^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2)+1 = 3j+1$ or $n^2 = (3k+2)^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1 = 3m+1$. Thus, in either case, the square of an integer is always either a multiple of 3 or 1 more than a multiple of 3. If neither a nor b is a multiple of 3, then each of a^2 and b^2 is 1 more than a multiple of 3 and a^2+b^2 is 2 more than a multiple of 3. However, there is no integer c such that c^2 is 2 more than a multiple of 3. Thus, at least one of a or b must be a multiple of 3.

2.) During a nonleap year of 365 days, there are 52 weeks of seven days each plus one extra day. During a leap year of 366 days, there are 52 weeks of seven days each plus two extra days. From 1914 through 2017 is 103 years, of which 26 years (the years from 1916 through 2016 divisible by 4) are leap years. Thus the day of the week of a particular date advances 129 days from 1914 through 2017. 129 days is 18 weeks plus 3 days. $(129 = 18 \cdot 7 + 3.)$ Since May 14, 2017 is a Sunday, May 14, 1914 was a Thursday, three weekdays earlier. Thus the second Sunday in May in 1914 was 4 days earlier and the first Mother's Day was $\boxed{\text{Sunday May 10, 1914.}}$

3.) Each of the small circles in the corners of the square is of the same size. The large circle has diameter 2 (the same as the length of each side of the square) and hence radius 1. Consider the line segment going from the center of the large circle to the lower left corner of the square. Since this is at a 45° angle to either the horizontal or vertical side of the square, this line segment has length $\sqrt{2}$. If the radius of the smaller circle is r, then the length of the line segment from the center of the lower left small circle to the lower left corner of the square has length $\sqrt{2}r$. The line segment from the center of the large circle to the lower left corner of the square is made up of three parts, a radius of the large circle from the center of the large circle to its point of tangency with the small circle, a radius of the small circle from this point of tangency to the center of the small circle and the line segment from the center of the small circle to the lower left corner of the square. Adding the lengths of these line segments one has $1 + r + \sqrt{2}r = \sqrt{2}$. Solving for r, the radius of the small circle, gives $\boxed{r = \dfrac{\sqrt{2}-1}{\sqrt{2}+1} = (\sqrt{2}-1)^2 \approx 0.1716.}$ Students are not assumed to have access to a calculator during the competition and the exact answer with radicals is preferable to the decimal approximation.

4.) There are eight distinct letters in M A T H E M A T I C S, not counting repetition. Thus, if the four letter word has four distinct letters without repetition there are eight choices for the first letter, then seven choices for the second letter, then six choices for the third letter and five choices for the last letter, for a total of $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ distinct words. If the four letter word has two repeated letters with the other two letters distinct, then there are three choices as to which letter is repeated (M, A or T), six choices as to the positions of the repeated letters in the four letter word (first and second letter, first and third letter, first and fourth letter, second and third letter, second and fourth letter or third and fourth letter), seven choices for the first nonrepeated letter and six choices for the other nonrepeated letter, for a total of $3 \cdot 6 \cdot 7 \cdot 6 = 756$ possible words. If the four letter word contains two pairs of repeated letters, there are three choices as to which pair of letters are repeated (M and A, M and T or A and T) and then six choices as to the position of the alphabetically first letter, for a total of $3 \cdot 6 = 18$ possible four letter words. There are thus a total of $1680 + 756 + 18 =$ $\boxed{\text{2454 distinct four letter words.}}$

5.) Let t be the time which it takes Nancy to run back to the entrance of the train tunnel. We are given that Nancy runs at the same constant speed in each direction. Thus, if she runs towards the end of the tunnel, then after time t she has run two fifths of the length of the tunnel from her starting point. She starts running two fifths of the way through the tunnel from the entrance. Thus, after time t she is four fifths of the way through the tunnel from the entrance. This is the same time at which she would have been at the entrance if she ran in the other direction, so this is the time when the train reaches the entrance of the tunnel. If Nancy continues running to the end of the tunnel, she reaches it at the same time the train does. Thus the train passes the entire length of the tunnel in the same time it takes Nancy to run the last fifth of the length of the tunnel. Nancy thus runs one fifth as fast as the train, or $\boxed{\text{10 miles per hour}}$ and collapses just outside the tunnel as the train whizzes by her.

6.) We present two slightly different arguments. First consider the case of two consecutive odd numbers. These can be expressed as n and $n + 2$. The difference of their squares is $(n + 2)^2 - n^2 = (n^2 + 4n + 4) - n^2 = 4n + 4 = 4(n + 1)$. Since n is odd, $n + 1$ is even (a multiple of 2) and $4(n+1)$ is thus a multiple of 8. The difference between the squares of any two odd numbers is a sum of differences between the squares of consecutive odd numbers (e.g. $(n + 4)^2 - n^2 = \{(n + 4)^2 - (n + 2)^2\} + \{(n + 2)^2 - (n^2)\}$ and thus a sum of multiples of 8 (and hence itself a multiple of 8).

Without considering consecutive odd numbers, any two odd numbers differ by an even number and so can be expressed as n and $n+2k$ for some number k. The difference between their squares is then $(n + 2k)^2 - n^2 = (n^2 + 4nk + 4k^2) - n^2 = 4nk + 4k^2 = 4k(n + k)$. If k is odd, then $n + k$ is even (since n is also odd) and thus $4(n + k)$ is a multiple of 8 and so is $4k(n + k)$. If k is even then $4k$ is a multiple of 8 and so is $4k(n + k)$.