EMMY NOETHER MIDDLE SCHOOL MATHEMATICS DAY
Texas Tech University
May 13, 2015

SOLUTIONS.

1.) If the box is sitting upright, there are four horizontal levels of wooden cubes within it. Every cube on the lowest level touches the bottom of the box. It is given that the box has no lid and one is only to determine the number of small wooden cubes touching the sides or the bottom of the box, not the ones touching the top. Each of the three horizontal levels except for the lowest has all cubes in the outside ring touching the sides of the box, but with the four in the middle 2×2 square not touching the sides or the bottom. There are four such small wooden cubes on each of three horizontal levels which do not touch the sides or bottom of the box, for a total of 12 such cubes. Each of the remaining $64 - 12 = \boxed{52}$ small wooden cubes either touches the bottom or one of the sides of the box.

2.) One could try to list all possible such three letter words, but the list would be extensive. It is preferable to determine the number of such three letter words without having to individually count each of them. There are six distinct letters in A L G E B R A without repetition. If the word has three distinct letters without repetition, there are six choices for the first letter, five choices for the second letter and four choices for the third letter. Thus there are $6 \cdot 5 \cdot 4 = 120$ possible such words. If the word contains both A's and a third distinct letter, then there are three possible combinations of places where the A's can occur (first and second letter, first and third letter or second and third letter). For each of these cases there are five possibilities for what the third, nonrepeated, letter is. There are thus $3 \cdot 5 = 15$ possible such words. These include all possible three letter words, so there are thus a total of $120 + 15 = \boxed{135}$ possible three letter words.

3.) By drawing lines from the center of the hexagon to each of its vertices (corners) one divides it into six equal triangles. The 360° circle at the center is divided into six equal angles of 60° in each of the triangles. The distance from the center of the hexagon to each of the vertices is the same, so each of the six triangles is an isosceles triangle. Since the angle of each triangle at the center of the hexagon is 60°, each triangle is an equilateral triangle with all three sides equal and with each angle equal to 60°. For an equilateral triangle, the height is $\frac{\sqrt{3}}{2}$ times the length of the base. The base of each of these triangles is thus 1 and the height is $\frac{\sqrt{3}}{2}$. The area of a triangle $\frac{1}{2}$(base)(height). Thus the area of each of the six triangles is $\frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$ and the combined area of all six triangles, i. e. the area of the hexagon is $\boxed{\frac{6\sqrt{3}}{4}}$.

4.) Each of 3, 5 and 7 is a prime number, a positive integer greater than 1 which is divisible only by 1 and itself. Thus 3, 5 and 7 are a prime triple. We now show that they are the only such by showing that if a, b and c are three consecutive odd integers then one of them is a multiple of 3. If each is a prime number, then one of them must be 3. Since 3 is the smallest prime number which is odd, this would force the other two prime numbers to be 5 and 7.

To see that one of a, b or c is a multiple of 3, there are three cases to consider. Assume that $a < b < c$. If a is a multiple of 3, then we are done. If not, then either $a = 3k + 1$ for some integer k or $a = 3k + 2$ for some integer k, i.e. when a is divided by 3, then the remainder is either 1 or 2. If a and b are consecutive odd integers with $a < b$, the $b = a + 2$. Thus, if $a = 3k+1$ then $b = 3k+1+2 = 3k+3 = 3(k+1)$. In this case b is a multiple of 3. If b and c are consecutive odd integers with $b < c$, then $c = b+2 = a+4$. Thus, if $a = 3k+2$ for some integer k, then $c = a+4 = 3k+2+4 = 3k+6 = 3(k+2)$. In this case, c is a multiple of 3. In either of the three possible cases, $\boxed{\text{one of } a,\ b \text{ or } c \text{ is a multiple of 3.}}$

5.) The minute hand of an analog clock moves 12 times as rapidly as the hour hand. Between 12:00 noon and 12:00 midnight the hour hand makes one complete circuit around the clock face, while the minute hand makes twelve complete circuits around the clock face, eleven more circuits than the hour hand. At 12:00 noon the hour hand and the minute hand start out pointing in the same direction. They will next point in the same direction when the minute hand has made exactly one more complete circuit around the clock face than the hour hand has, i. e. if the hour hand has moved x circuits (or part of a circuit), then the minute hand, which has moved 12 times as fast, has moved $12x = 1 + x$ circuits, one more than the hour hand. This means that the hour hand has moved $\frac{1}{11}$ of a circuit around the clock face while the minute hand has move $\frac{12}{11} = 1\frac{1}{11}$ circuit around the clock face. This is at approximate time 1:05.4545... . The hands will next point in the same direction when the hour hand has moved $\frac{2}{11}$ of the circuit around the clock face and the minute hand has moved $12\left(\frac{2}{11}\right) = 2\frac{2}{11}$ of the circuit around the clock face, at approximate time 2:10.9090.... This pattern will continue until the hour hand has moved $\frac{10}{11}$ of the circuit around the clock face and the minute hand has moved $12\left(\frac{10}{11}\right) = 10\frac{10}{11}$ of the way around the clock face, at approximate time 10:54.545... . The next time that the hands are pointing in the same direction will be when the hour hand has made one complete circuit of the clock face at 12:00 midnight. There are thus $\boxed{\text{10 times}}$ when the hour hand and minute hand are pointing in the same direction between 12:00 noon and 12:00 midnight, not including either noon or midnight.

6.) Each nonleap year has $365 = (52) \cdot 7 + 1$ days. Thus the day of the week corresponding to a particular date shifts one day each nonleap year. In a leap year it shifts an additional day for a total shift of two days. From 1970 through 2015 was 45 years including 11 leap years Thus the day of the week corresponding to April 22 shifted $45 + 11 = 56 = (8) \cdot 7$ days during this period. This is an exact number of 8 weeks with no extra days. Hence April 22, 1970 was on the same day of the week that April 22, 2015 was, i.e. on $\boxed{\text{Wednesday}}$.