

EMMY NOETHER HIGH SCHOOL MATHEMATICS DAY
Texas Tech University
May 13, 2015

SOLUTIONS.

1.) If Deimos is at a given time directly overhead for an observer on the equator of Mars, it will next be directly overhead at the same point when Mars has made precisely one more rotation on its axis than Deimos has made orbits of Mars. Thus if Deimos has made x orbits of Mars (or parts of orbits) then Mars will have made $1+x$ rotations in the same time period.

Thus $30.3x = 24.6(1+x)$. Thus $x = \frac{24.6}{5.7}$ and the time is $\boxed{30.3 \left(\frac{24.6}{5.7} \right)} \approx 131$ hours.

2.) If x is a positive integer, let $R(x)$ denote the remainder when x is divided by 11. Thus, $R(x)$ is an integer in the range from 0 through 10, with $R(x) = 0$ when x is divisible by 11. Below is a table of values of $R(x)$ for small powers of 5, 4 and 3.

x	$R(x)$	x	$R(x)$	x	$R(x)$
5^0	1	4^0	1	3^0	1
5^1	5	4^1	4	3^1	3
5^2	3	4^2	5	3^2	9
5^3	4	4^3	9	3^3	5
5^4	9	4^4	3	3^4	4
5^5	1	4^5	1	3^5	1

If $R(x) = a$ and $R(y) = b$, then it is easily verified that $R(xy) = R(ab)$ and $R(x+y) = R(a+b)$. Thus, for every natural number k , $R(5^{5k}) = R(5^5 \cdot 5^5 \cdots 5^5 \cdot 5^5) = R(1 \cdot 1 \cdots 1 \cdot 1) = R(1) = 1$. Similarly, $R(4^{5k}) = 1$ and $R(3^{5k}) = 1$. Thus, $R(5^{5k+1}) = R(5^{5k} \cdot 5^1) = R(1 \cdot 5) = R(5) = 5$. Similarly, $R(4^{5k+2}) = R(4^{5k} \cdot 4^2) = R(1 \cdot 5) = R(5) = 5$. Thus $R(5^{5k+1} + 4^{5k+2} + 3^{5k}) = R(5 + 5 + 1) = R(11) = 0$. Hence,

for every natural number k , $5^{5k+1} + 4^{5k+2} + 3^{5k}$ is a multiple of 11.

3.) We consider cases. If the hour display is 1, then the minute display can be any number from 23 through 29, from 34 through 39, from 45 through 49 or from 56 through 59, for a total of $7 + 6 + 5 + 4 = 22$ possibilities. If the hour display is 2, then the minute display can be any number from 34 through 39, from 45 through 49 or from 56 through 59, for a total of $6 + 5 + 4 = 15$ possibilities. If the hour display is 3, then the minute display can be any number from 45 through 49 or from 56 through 59, for a total of $5 + 4 = 9$ possibilities. If the hour display is 4, then the minute display can be any number from 56 through 59, for a total of 4 possibilities. The hour display cannot be 5, 6, 7, 8, 9, 10 or 11. If the hour display is 12, then the minute display can be any number from 34 to 39, from 45 through 49 or from 56 through 59, for a total of $6 + 5 + 4 = 15$ possibilities. Considering all possible hour displays, there are a total of $22 + 15 + 9 + 4 + 15 = \boxed{65}$ possibilities.

4.) From the center of the circle draw lines to each of the vertices of the triangle. These divide the triangle into three equal subtriangles, each with an angle of 120° at the vertex at the center of the circle. If one draws an altitude from the vertex at the center of the circle to the midpoint of the opposite side of one of these triangles, one divides this triangle into

two equal $30^\circ - 60^\circ - 90^\circ$ right triangles. Each of these triangles has a hypotenuse which is a radius of the circle and thus of length 1. Thus the other sides of this triangle are of lengths $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$. Each of these right triangles thus has area $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{2} \right) = \frac{\sqrt{3}}{8}$.

There are a total of six such right triangles in the original equilateral triangle, so this triangle

has an area of $6 \frac{\sqrt{3}}{8}$. Each side of the original equilateral triangle has been subdivided into two of the longer sides of the right triangles. Thus, each side of the original equilateral triangle is of length $2 \left(\frac{\sqrt{3}}{2} \right) = \sqrt{3}$.

5.) Assume that there exist natural numbers x, y, z and n such that $x^n + y^n = z^n$ and $n \geq z$. Then $x < z, y < z$ and $x \neq y$. Without loss of generality we can assume that $x < y$. Then $x^n = z^n - y^n = (z - y)(1 \cdot z^{n-1} + y \cdot z^{n-2} + \dots + y^{n-2} \cdot z + y^{n-1} \cdot 1) \geq 1 \cdot nx^{n-1} > x^n$

The inequality $x^n > x^n$ is a contradiction. Note that there are two places where we use the assumption that x, y and z are natural numbers. This is used in concluding that $x \neq y$ and that $z - y \geq 1$. The assumption that n is a natural number is used in the factoring of $z^n - y^n$.

6.) There are eight distinct letters in M A T H E M A T I C S, not counting repetition. Thus, if the four letter word has four distinct letters without repetition there are eight choices for the first letter, then seven choices for the second letter, then six choices for the third letter and five choices for the last letter, for a total of $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ distinct words. If the four letter word has two repeated letters with the other two letter distinct, then there are three choices as to which letter is repeated (M, A or T), six choices as to the positions of the repeated letters in the four letter word (first and second letter, first and third letter, first and fourth letter, second and third letter, second and fourth letter or third and fourth letter), seven choices for the first nonrepeated letter and six choices for the other nonrepeated letter, for a total of $3 \cdot 6 \cdot 7 \cdot 6 = 756$ possible words. If the four letter word contains two pairs of repeated letters, there are three choices as to which pair of letters are repeated (M and A, M and T or A and T) and then six choices as to the position of the alphabetically first letter, for a total of $3 \cdot 6 = 18$ possible four letter words. There are thus a total of $1680 + 756 + 18 = \boxed{2454}$ distinct four letter words.