EMMY NOETHER HIGH SCHOOL MATHEMATICS DAY
Texas Tech University
May 14, 2014

SOLUTIONS.

1.) Any integer (whole number) n which is 40 or greater can be expressed in one of the forms $5k$, $5k+1$, $5k+2$, $5k+3$ or $5k+4$, where k is an integer greater than or equal to 8. If $n = 5k$, then using k 5-cent stamps produces $n-$cents postage. If $n = 5k+1$, then using $k-2$ 5-cent stamps and one 11-cent stamp produces $n-$cents postage. If $n = 5k+2$, then using $k-4$ 5-cent stamps and two 11-cent stamps produces $n-$cents postage. If $n = 5k+3$, then using $k-6$ 5-cent stamps and three 11-cent stamps produces $n-$cents postage. If $n = 5k+4$, then using $k-8$ 5-cent stamps and four 11-cent stamps produces $n-$cents postage. Note that, since $k \geq 8$, then $k-8 \geq 0$. One cannot produce 39 cents postage using only 5 cent and 11 cent stamps. What is significant here is that 5 and 11 are relatively prime, i.e. have no common divisor other than 1. A similar problem can be stated for any two relatively prime positive integers x and y. If x and y are relatively prime positive integers, each greater than 1, then there exists a positive integer z such that, for every integer w greater than or equal to z there exist nonnegative integers a and b such that $w = ax + by$.

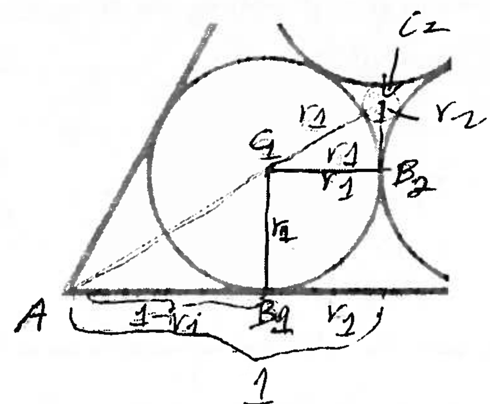
2.) Notice that $2^3 = 8 < 9 = 3^2$. Thus $x = 2^{7132} < 2^{7200} = (2^3)^{2400} < (3^2)^{2400} = 3^{4800} < 3^{4803} = y$. Thus $x < y$. Also note that $5^3 = 125 < 128 = 2^7$. Thus $z = 5^{2998} < 5^{3000} = (5^3)^{1000} < (2^7)^{1000} = 2^{7000} < 2^{7132} = x$. Thus $z < x$.

Hence $\boxed{z = 5^{2998} < x = 2^{7132} < y = 3^{4803}.}$

3.) The minute hand of a clock moves at 6° per minute (a complete circuit of 360° in one hour, or 60 minutes). The hour hand of a clock moves at 0.5° degrees per minute (one-twelfth of a complete circuit, i.e. 30°, in one hour, or 60 minutes). At 12:00 noon the hour hand and minute hand are together, both pointing straight upward. Each minute the minute hand moves 5.5° more than the hour hand.

The second hand moves at 360° per minute, a complete circuit every minute. It gains 354° on the minute hand each minute, or 120° in $\frac{120}{354} \approx 0.338983$ minutes and gains 240° on the minute hand in twice this time. During a 12 hour (720 minute) period, between when the hands are all straight up and when they are again all together and straight up, there are $\frac{720}{\frac{120}{5.5}} = 33$ times when the hour hand and the minute hand are an exact multiple of 120° apart. During the same 12 hour period, there are $\frac{720}{\frac{120}{354}} = 2108$ times when the minute hand and the second hand are an exact multiple of 120° apart. The numbers 33 and 2108 have no common integer divisors other than one, i.e. are relatively prime. There is thus no time shorter than 12 hours when all three hands are exact multiples of 120° apart and hence **NO TIME** when the three hands divide the face of the clock into equal thirds.

4.) Consider two triangles, $\triangle AC_1B_1$ with vertex A from the original equilateral triangle and with vertex C_1 the center of one of the original circles and $\triangle C_1C_2B_2$ with vertices C_1 and C_2 the centers of one of the original circles and of the central circle respectively. Since the original triangle is equilateral with each angle equal to 60°, each of these two triangles is a $30° - 60°-$right triangle. Thus the sides of each of these triangles are in the ratio short side :: long side :: hypotenuse $= 1 :: \sqrt{3} :: 2$. Let r_1 be the radius of the original circle with center C_1 and let r_2 be the radius of the central circle with center C_2.



Triangle $\triangle AC_1B_1$ has short side of length r_1 and long side of length $1 - r_1$. Thus $1 - r_1 = \sqrt{3}r_1$ or $r_1 = \frac{1}{\sqrt{3}+1}$.

Triangle $\triangle C_1C_2B_2$ has hypotenuse of length $r_1 + r_2$ and long side of length r_1. Thus $r_1 + r_2 = \frac{2}{\sqrt{3}}r_1$ or $r_2 = \left(\frac{2}{\sqrt{3}} - 1\right) r_1 = \left(\frac{2-\sqrt{3}}{\sqrt{3}}\right) r_1$.

Thus $r_2 = \left(\frac{2-\sqrt{3}}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}+1}\right) = \boxed{\frac{2-\sqrt{3}}{3+\sqrt{3}} = \frac{2-\sqrt{3}}{3+\sqrt{3}} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{9-5\sqrt{3}}{6} \approx 0.056624.}$ (It is not necessary to give the decimal approximation and it is not assumed that students have access to a calculator. The exact answer is preferred, with the approximation provided only to indicate magnitude.)

5.) First, consider the case where $\triangle ABC$ is equilateral. This is illustrated on the left of the diagram on the competition problem sheet. Let O be the center of the inscribed circle and let O' be the center of the circumscribed circle. Let M be the midpoint of side BC, N be the midpoint of side AB and R be the midpoint of side AC. Note that $AB = AC$, $AM = AM$ and $BM = MC$. Thus $\triangle ABM$ is congruent to $\triangle AMC$. Thus angles $\angle BMA$ and $\angle AMC$ are both right angles and AM is the perpendicular bisector of side BC. Also, AM is the bisector of $\angle BAC$. Similar results hold for BR and CN. Point O is the point of intersection of the bisectors of the angles of $\triangle ABC$. Point O' is the point of intersection of the perpendicular bisectors of the sides of $\triangle ABC$. Thus, point O coincides with point O'.

Now consider the converse. This is illustrated in the right of the diagram on the competition problem sheet. Let O, O', M, N and R be defined as previously. Assume that O and O' coincide. Then $ON = ON$ and $OA = OB$, since we are assuming that O and O' coincide and O' is the center of the circumscribed circle with OA and OB radii of this circle. Thus right $\triangle ONA$ is congruent to right $\triangle ONB$ since they have corresponding legs and hypotenuses equal. Therefore $AN = BN$. Also $BN = BM$ since they are tangents to the inscribed circle from the same point. A similar argument gives that $\triangle BOM$ is congruent to $\triangle MOC$ and that sides $BM = MC$ and $MC = CR$. Similarly, $CR = RA$ and $RA = AN$. Therefore $AN + NB = BN + BM = BM + MC$. Therefore $AB = BC$. Similarly, $BC = AC$. Thus $\triangle ABC$ is an equilateral triangle.

6.) There are eight distinct letters not counting repetition. If a word has no repeated letters, there are eight choices for the first letter, then seven choices for the second letter, then six choices for the third letter, then five choices for the fourth letter. There are thus, $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ such words. If the word has two repeated letters, with the other two letters distinct, then the repeated letter can be either E or M. There are six places where the repeated letters can occur (first and second, first and third, first and fourth, second and third, second and fourth or third and fourth). There then seven choices for the first nonrepeated letter and six choices for the remaining letter. There are thus $2 \cdot 6 \cdot 7 \cdot 6 = 504$ such words. The word could contain three E's. There are four places where these can occur and seven choices for the remaining letter. Thus there are $4 \cdot 7 = 28$ such words. Finally the word can contain two E's and two M's. There are six places where the two E's occur, with the two M's automatically occurring in the remaining places. There are thus 6 such words. Considering all case, there are a total of $1680 + 504 + 28 + 6 = \boxed{2218}$ possible words.