

EMMY NOETHER MIDDLE SCHOOL MATHEMATICS DAY  
Texas Tech University  
May 15, 2013

SOLUTIONS.

1.) There are four one-digit positive even integers. These are followed by the 45 two-digit even integers, from 10 through 98. Thus the first 94 digits in the sequence represent one- or two-digit numbers. The remaining ones come from three digit numbers. The next 906 digits come from the first 302 three-digit numbers, with the 1000th digit of the sequence being the last digit of the 302nd three-digit even number. The first three-digit even number is 100. The even number 702 ( $= 100 + 2 \cdot 301$ ) is thus the 302nd three-digit number and its last digit, i.e.  $\boxed{2}$ , is the 1000th digit in the sequence.

2.) There are two cases to consider, whether all four letters of the word are different or whether the word contains both letters E. If all four letters are all different, then there are seven choices for the first letter of the word, six choices for the second letter, five choices for the third letter of the word and four choices for the last letter, or  $7 \cdot 6 \cdot 5 \cdot 4 = 8400$  possible words. If the word includes both letters E, then there are six choices for which spaces the E's occur in the word (first and second letter, first and third letter, first and fourth letter, second and third letter), second and fourth letter or third and fourth letter) and for each of these choices six choices for the first of the remaining letters and five choices for the last remaining letter, or  $6 \cdot 6 \cdot 5 = 180$  possible words. There is thus a total of  $840 + 180 = \boxed{1020}$  possible words.

3.) The line joining the centers of the circles is perpendicular to their common chord and bisects it. Thus for each circle a right triangle is formed by the radius from the center to one endpoint of the chord, the half of the chord from that endpoint to where the line joining the two centers bisects it and the line from the center to the midpoint of the chord. For the smaller circle this triangle has a hypotenuse of length 10 and one side of length 8. Thus the other side has length  $\sqrt{(10)^2 - (8)^2} = 6$ . For the larger circle this triangle has hypotenuse of length 17 and one side of length 8. Thus the other side has length  $\sqrt{(17)^2 - (8)^2} = 15$ . The line segment joining the centers of the two circles is formed by the two sides of triangles whose lengths were just computed and thus has length  $6 + 15 = \boxed{21}$ .

4.) On the second day one can assume that the marked and unmarked fish are distributed throughout the lake so that each fish has the same likelihood of being caught, whether it is marked or unmarked. Of the fish caught on the second day, 2 out of 40 (or  $1/20$ th of those caught) are marked. Thus one can assume that  $1/20$ th of all the fish in the lake are marked. Since there are 30 marked fish, there are thus approximately  $30 \cdot 20 = \boxed{600}$  total fish in the lake.

5.) The largest circle fitting in the central hole will have center at the center of the square and be tangent to (just touch) each of the circles already inside the square. Thus its radius will be the distance from the center of the square to any of the circles. Dividing the square into four equal subsquares each containing one of the given circles, the radius of the circle in the central hole of the large square will be the distance from a vertex (corner) of a subsquare

to the circle contained in it. Since the original square has each subside of length 4, each smaller square will have each side of length 2. It will thus have a diagonal of length  $2\sqrt{2}$ . Thus the radius of the circle in the central hole will be  $\frac{1}{2}(2\sqrt{2} - 2)$  and its diameter will be

$$2\sqrt{2} - 2 \approx 0.8284.$$

6.) Let  $n$  be the number of nickels,  $d$  the number of dimes and  $q$  be the number of quarters which Alice has. Thus  $n + d + q = 14$  and  $5n + 10d + 25q = 145$ . Just from the two equations in three variables, we cannot uniquely solve for  $n$ ,  $d$  or  $q$ . However, we know that there must be at least one quarter, since if there is no quarter then 14 coins each worth either 5 cents or 10 cents, then the total value is at most \$1.40. There cannot be four or more quarters since, if there are four quarters, then the remaining 10 coins of five cents or more each plus the four quarters add to at least \$1.50. Considering each of the cases of there being either one, two or three quarters, we determine that there are either

$2$  nickels,  $11$  dimes and  $1$  quarter or  $5$  nickels,  $7$  dimes and  $2$  quarters  
or  $8$  nickels,  $3$  dimes and  $3$  quarters.