# EMMY NOETHER HIGH SCHOOL MATHEMATICS DAY
Texas Tech University
May 15, 2013

SOLUTIONS.

1.) The closest and farthest points on the circle $C$ from the point $P$ are opposite endpoints of the diameter of the circle which, when extended, passes through $P$. Since the difference between these distances is 18, this is the length of the diameter of the circle and the radius is 9. The distance from $P$ to the center of the circle is then 15. The tangent line from $P$ to the the circle $C$ is perpendicular to a radius of the circle at the point of tangency. Thus, this tangent line forms one side of a right triangle with hypotenuse 15 and with other side of length 9. The length of this tangent line is thus $\sqrt{(15)^2 - (9)^2} = \boxed{12}$.

2.) Each non-leap year consists of $365 = 52 \cdot 7 + 1$ days, so the day of the week shifts one day for each non-leap year. In a leap year there are $366 = 52 \cdot 7 + 2$ and so the day of the week shifts two days in a leap year. Between 1863 and 2013 there was leap year every fourth year from 1864 through 2012 except for 1900 or 37leap years and 113 non-leap years. Thus the day of the week got shifted $1 \cdot 113 + 2 \cdot 37 = 187 = 7 \cdot 26 + 5$ days.

Thus the Emancipation Proclamation was signed on a date five days of week earlier than the Tuesday of January 1 this year, i. e. on $\boxed{\text{Thursday.}}$

3.) If we have three consecutive odd integers each greater than 3 and if the first number is a prime, then it is not a multiple of 3 and so is either of the form $n = 3k + 1$ or $n = 3k + 2$. If it is of the form $n = 3k + 1$, then the next term is of the form $n + 2 = 3k + 3 = 3(k + 1)$ and hence is a multiple of 3 and not prime. If the first term is of the form $n = 3k + 2$, then the last term is of the form $n + 4 = 3k + 6 = 3(k + 2)$ and hence is a multiple of 3 and not prime. In either case one cannot have three consecutive odd integers each greater than 3 with each a prime. (One can find many "twin primes," i.e. two consecutive odd integers each of which is prime. It is unknown if there are infinitely many such "twin primes.")

4.) Each dog always runs at right angles to the direction of the dog which pursues him and this dog always runs directly toward the dog which he is pursuing. Thus the motion of the pursued dog does not affect the relative speed of approach of the pursuing dog, only the direction of approach. Each dog thus starts 100 meters from the dog which he is pursuing and always approaches him directly at the rate of 10 meters per second. Thus, in $\boxed{\text{10 seconds}}$ he will meet the dog he is pursuing after having travelled $\boxed{\text{100 meters.}}$ (This can also be expressed by, instead of using a fixed coordinate system, using a moving coordinate system which is always centered on, say, dog B. In this coordinate system dog A will start 100 meters from dog B and approach him directly at 10 meters per second.) The dogs will be moving in curved paths (logarithmic spirals, though it is not necessary to determine this). At every moment the dogs form the corners of a square. The square is rotating and decreasing in area, the lengths of its sides decreasing at a constant rate of 10 meters per second. The dogs meet at the center of the original square. $\boxed{\text{The tracks of the dogs will not intersect before meeting at the center.}}$ If any dog stepped on the track of another dog it would mean that the other dog at a different time had been in the same spot. However, the distances of the dogs from the center of the square are at each time equal and are steadily decreasing. The assumption that the pursuing dog runs in the direction of the pursued dog at each moment is commonly used in "pursuit problems" such as this. However, some herding breeds of dogs, e.g. collies, move in the direction of the anticipated position rather than the current position of the pursued animal.

5.) The overlapping region of the two squares is an irregular quadrilateral. However, it is not necessary to directly compute this area. A line from the center of the upper square (the top corner of the bottom square) to the lower right corner of the top square divides this region of overlap into two triangles. The upper right of these two triangles $T_1$ (with vertices at the center of the upper square, the lower right corner of the upper square and the point on the right side of the upper square where the side of the lower triangle intersects it) is congruent to the triangle $T_2$ with vertices at the center of the upper square, the lower left corner of the upper square and the point on the bottom side of the upper square where the edge of the lower square intersects it. From the original quadrilateral of overlap of the two squares replacing triangle $T_1$ with triangle $T_2$ produces the triangle with vertices the center and two bottom corners of the upper square. This triangle has the same area as the original quadrilateral and is clearly $\frac{1}{4}$ the area of the top square. Since the top square has each side of length 1, it has area $(1)^2 = 1$. Thus the area of the quadrilateral of overlap is $\frac{1}{4}$ of this or $\boxed{\frac{1}{4}.}$

6.) There are three cases to consider, if all four letters of the word are different, if the word has two identical letters with the other two letters different and if the word has two pairs of identical letters.

If all four letters are different, there are seven choices for the first letter, six choices for the second letter, five choices for the third letter and four choices for the last letter, for $7 \cdot 6 \cdot 5 \cdot 4 = 840$ possible words.

If there are two identical letters, there are six choices for which spaces are identical (first and second, first and third, first and fourth, second and third, second and fourth or third and fourth), three choices for what is the repeated letter (N, O or U), six choices for the first non-repeated letter and five choices for the last non-repeated letter, for $6 \cdot 3 \cdot 6 \cdot 5 = 540$ possible words.

If there are two pairs of repeated letters, there are three choices for where the first letter is repeated (first and second, first and third or first and fourth) and three choices for what is the first letter (N, O or U). There are then two choices for what is the repeated letter not in first place, for $3 \cdot 3 \cdot 2 = 18$ possible words.

There is thus a total of $840 + 540 + 18 = \boxed{1398}$ possible words.