EMMY NOETHER HIGH SCHOOL MATHEMATICS DAY
Texas Tech University
May 17, 2012

SOLUTIONS.

1.) Since n is an odd integer, it is of the form $n = 2k + 1$ for some integer k. Thus $n^2 - 1 = (2k + 1)^2 - 1 = (4k^2 + 4k + 1) - 1 = 4k^2 + 4k = 4(k^2 + k) = 4(k)(k + 1)$. One of k or $k + 1$ is even and hence a multiple of 2. Thus $n^2 - 1$ is a multiple of $4 \cdot 2 = 8$.

2.) One can separately consider the case of whether the year is a leap year or not. First, consider the special case of a non-leap year, with January 1 on Sunday. In successive months the 13th occurs on Friday in January, Monday in February, Monday in March, Thursday in April, Saturday in May, Tuesday in June, Thursday in July, Sunday in August, Wednesday in September, Friday in October, Monday in November and Wednesday in December. Throughout the year it has occurred on every day of the week and a maximum of three times on Monday. If the year begins on a day other than Sunday, the dates for the 13th of each month are all shifted by the same amount and again there is some month when it occurs on a Friday and it can still occur on the same day of the week a maximum of three times. It will occur three times on a Friday when the year begins on a Thursday, four days later that for the listing above.

On a leap year, the dates for March through December are all shifted one day from the above. For a leap year with January 1 on a Sunday, the 13th occurs on Friday in January, Monday in February, Tuesday in March, Friday in April, Sunday in May, Wednesday in June, Friday in July, Monday in August, Thursday in September, Saturday in October, Tuesday in November and Thursday in December. Again, throughout the year it occurs on every day of the week and a maximum on three times on Friday. If the leap year begins on a day other than Sunday, the dates for the 13th of each month are all shifted by the same amount and again there is some month when it occurs on a Friday and it can still occur on the same day of the week a maximum of three times. It will occur three times on Friday when the leap year begins on Sunday, as with the listing above and as is the case this year 2012.

3.) We do not know how many students are in the class, but this information is not needed. Let x be the percentage of students who passed the exam and $100 - x$ be the percentage of students who failed the exam. From the given information $\dfrac{(x)(78) + (100 - x)(38)}{100} = 64$. The solution of this equation is $x = 65$ and hence $\boxed{65\%}$ of the students passed the exam. (Note that though we know that over half of the students passed the exam and the average grade of all students was 64, we cannot conclude that 64 was a passing grade. The above averages are possible with a passing grade of 70 and 65% of students doing at or above average.)

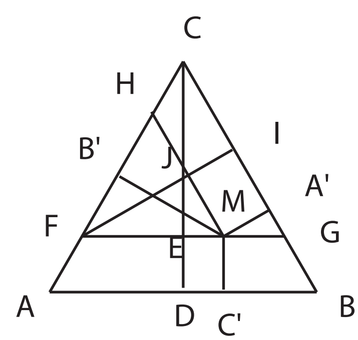
4.) This problem is complicated by the fact that water is continuing to flow into the ocean liner while the pumps work to pump it out. There are three quantities to consider. Let w be the amount of water present in the ship when the pumps begin to operate. Let x be the rate of inflow of water into the ocean liner per hour. Let y be the amount of water pumped out of the ship by a single pump per hour. From the information given we can form two equations, $w + 4x - (10)(4)y = 0$ and $w + 12x - (6)(12)y = 0$, reflecting the fact that with 10 pumps the amount of water in the ship after 4 hours is 0 and with 6 pumps the amount of water in the ship after 12 hours is 0. Subtracting the first equation from the second equation gives $8x - 32y = 0$ or $x = 4y$. Substituting this for x in either of the two equations gives $w - 24y = 0$ or $w = 24y$.

If n is the number of pumps required to pump all of the water out of the ship in 2 hours, then $w + 2x - (n)(2)y = 0$. Substituting $x = 4y$ and $w = 24y$ gives $32y - (n)(2)y = 0$, or $\boxed{n = 16}$ pumps are needed to pump all of the water out of the ship in 2 hours.

(Note that though we had only two equations and three variables, w, x and y, we never solved for the three variables, only for the ratios between them. We do not have enough information to determine the value of each, but since each equation is a linear expression equal to 0, i.e. this is a homogeneous system of equations, with the equations independent, the ratios are uniquely determined.)

5.) Here we need to present an argument rather than do a computation. Notice that if the point M is chosen to coincide with one of the vertices, say A, then MB' and MC' are each equal to 0 and MA' is the length of the altitude of the triangle from the vertex A to the opposite side. For an equilateral triangle all three altitudes from a vertex to the opposite side are equal in length. Thus we show that, for an arbitrary point M in the triangle, the sum $MA' + MB' + MC'$ is equal to the length of an altitude of the triangle. There may be several valid arguments for this. We present one.

We shall denote the line segment joining points X and Y by $\overline{XY}$ and denote the length of this line segment by XY.



Let M be any point in $\triangle ABC$ and construct the perpendiculars from M to each of the sides of the triangle. Construct the altitude $\overline{CD}$ from the vertex C to opposite side AB. Construct the line segment $\overline{FG}$ through the point M parallel to the side $\overline{AB}$. Let E be the point where altitude $\overline{CD}$ intersects line segment $\overline{FG}$. Note that the length ED is equal to length MC', since line segment $\overline{FG}$ is parallel to line segment $\overline{AB}$ and each of $\overline{ED}$ and $\overline{MC'}$ is perpendicular to $\overline{AB}$.

Since $\overline{FG}$ is parallel to $\overline{AB}$, $\triangle CFG$ is also an equilateral triangle with altitude $\overline{CE}$. Construct the altitude $\overline{FI}$ from vertex F to opposite side $\overline{CG}$. Construct the line segment $\overline{HM}$ parallel to side $\overline{CG}$. Let J be the point where altitude $\overline{FI}$ intersects line segment $\overline{HM}$.

Our construction is now complete.

Note that $\triangle FHM$ is also an equilateral triangle. Each of the line segments $\overline{FJ}$ and $\overline{MB'}$ is an altitude of this triangle and hence they have the same length. Thus we have:

$$FJ = MB'$$
$$FJ + JI = FI$$
$$JI = MA'$$
$$FI = MA' + MB'$$
$$FI = CE$$
$$CE = MA' + MB'$$
$$ED = MC'$$

$CD = CE + ED$
$CD = MA' + MB' + MC'$.

Thus, independent of where the point M is chosen in equilateral $\triangle ABC$, $MA' + MB' + MC'$ is equal to the length of the altitude of this triangle.

The above argument can be generalized to the following result for an arbitrary triangle. For an arbitrary triangle the lengths of the three altitudes are not all equal.

Let $\triangle ABC$ be an arbitrary triangle with vertices A, B, and C. Let M be any point in $\triangle ABC$ and let MA', MB' and MC' be the lengths of the perpendiculars from point M to the sides opposite vertices A, B and C respectively. Then $MA' + MB' + MC'$ is at least as large as the length of the shortest altitude of $\triangle ABC$ and is no greater than the length of the longest altitude of $\triangle ABC$.

6.) There are three cases to consider, if all four letters of the word are different, if the word has two identical letters with the other two letters different and if the word has two pairs of identical letters.

If all four letters are different, there are five choices for the first letter, four choices for the second letter, three choices for the third letter and two choices for the last letter, for $5 \cdot 4 \cdot 3 \cdot 2 = 120$ possible words.

If there are two identical letters, there are six choices for which spaces are identical (first and second, first and third, first and fourth, second and third, second and fourth or third and fourth), three choices for what is the repeated letter (C, L or U), four choices for the first non-repeated letter and three choices for the last non-repeated letter, for $6 \cdot 3 \cdot 4 \cdot 3 = 216$ possible words.

If there are two pairs of repeated letters, there are three choices for where the first letter is repeated (first and second, first and third or first and fourth) and three choices for what is the first letter (C, L or U). There are then two choices for what is the repeated letter not in first place, for $3 \cdot 3 \cdot 2 = 18$ possible words.

There is thus a total of $120 + 216 + 18 = \boxed{354}$ possible words.