EMMY NOETHER MIDDLE SCHOOL MATHEMATICS DAY
Texas Tech University
May 11, 2011

SOLUTIONS.

1.) At any time after 12:00 noon, the minute hand of an analog clock will have travelled farther since noon than the hour hand. The two hands will point in the same direction when the difference between the angle moved by the minute hand and the angle moved by the hour hand is a multiple of $360°$, a complete circle. Between 12:00 noon and 12:00 midnight, the minute hand moves $12 \times 360° = 4320°$ and the hour hand moves $1 \times 360° = 360°$. Thus, the minute hand moves $3960°$ more than the hour hand and will point in the same direction as the hour hand when it has moved $360°, 720°, \ldots, 3600°$. When it has moved $3960°$ more than the hour hand, it is midnight and this time when the hands point in the same direction is not counted. There are thus $\boxed{10 \text{ times}}$ between noon and midnight when the hour hand minute hand point in the same direction.

2.) Any one digit number is equal to the sum of its digits. Any number between 10 and 54 has the sum of its digits at most 13 $(4 + 9)$ and hence can be no larger than 39 if it is three times the sum of its digits. Any number from 55 to 99 has the sum of its digits at most 18 and thus three times the sum of its digits is at most 54. Any number with three or more digits is more than three times the sum of its digits. Thus, the only positive integers to check are the multiples of three from 12 through 54. (Additional argument can eliminate many of these.) A quick check shows that $\boxed{27 = 3(2 + 7)}$ is the $\boxed{\text{only}}$ positive integer which is equal to three times the sum of its digits.

3.) There are two cases to consider. If all three letters of the word are distinct, there are 8 possibilities for the first letter, 7 possibilities for the second letter and 6 possibilities for the third letter, hence $8 \cdot 7 \cdot 6 = 336$ possible words. If two letters of the word are the same, there are 3 possibilities for which positions of the letters are the same, e.g. first and second letters, 3 possibilities for the repeated letter and 7 possibilities for the other letter, hence $3 \cdot 3 \cdot 7 = 63$ possible words. There are thus a total of $336 + 63 = \boxed{399 \text{ possible three letter words.}}$

4.) If $a^2 + b^2 = 2011$, then one of a^2 or b^2 must be greater than 1010, hence a or b must be greater than or equal to 32 since each is an integer. Each of a^2 and b^2 must be less than 2011, hence each of a and b must be less than or equal to 44, since each is an integer. A check of the small number of case shows that there are $\boxed{\text{NO } a \text{ and } b \text{ with } a^2 + b^2 = 2011.}$ However, one can also give a more general argument without checking specific cases. If $a^2 + b^2 = 2011$, then one term, say a^2, is even and the other, b^2, is odd. If a^2 is even, then a is even, i.e. $a = 2m$ for some integer m. Hence $a^2 = (2m)^2 = 4m^2$ and a^2 is a multiple of 4. If b^2 is odd, then b is odd, i.e. $b = 2n + 1$ for some integer n. Hence $b^2 = (2n + 1)^2 = 4n^2 + 4n + 1$ and b^2 is one more than a multiple of 4. If a^2 is a multiple of 4 and b^2 is one more than a multiple of 4, then $a^2 + b^2$ is one more than a multiple of 4. However, 2011 is three more than a multiple of 4 and hence $\boxed{2011 \text{ cannot be expressed in the form } a^2 + b^2.}$

5.) Lines from the center of the circle to each of the vertices of the hexagon divide the hexagon into six equilateral triangles, i.e. triangles with each side equal. Each such equilateral triangle has height 1 and thus base $2/\sqrt{3}$. The area of each triangle is thus $\frac{1}{2} \cdot (\text{base}) \cdot (\text{height}) = \left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{3}}\right)(1) = \frac{1}{\sqrt{3}}$. Since the hexagon is composed of six such triangles, its area is $\boxed{6 \cdot \dfrac{1}{\sqrt{3}} = 2\sqrt{3}.}$ Note that this is consistent with the area of the hexagon, $2\sqrt{3} \approx 3.4641$, being slightly larger than the area of the circle, $\pi \cdot (1)^2 = \pi \approx 3.1416$.

6.) Each non-leap year has 365 days, or 52 weeks of 7 days each $(52 \cdot 7 = 364)$ plus one extra day. Each leap year has 366 days, or 52 weeks of 7 days plus two extra days. In the 50 years since the flight of Yuri Gagarin there have been 38 non-leap years and 12 leap years, so there have been $38 \cdot 1 + 12 \cdot 2 = 62$ extra days, besides the exact number of weeks. However, 62 days is equivalent to 8 weeks $(8 \cdot 7 = 56)$ plus 6 extra das. Thus the day of the week on the 50th anniversary this year is six week days later than the initial day in 1961 and the orbit of Yuri Gagarin occurred on $\boxed{\text{Wednesday April 12, 1961.}}$