

EMMY NOETHER HIGH-SCHOOL MATHEMATICS DAY
Texas Tech University
May 11, 2011

SOLUTIONS.

1.) Since the hour hand is between 10 and 11, the minute hand is directly opposite it between 4 and 5. Note that since the hands move at a uniform rate, the correct answer is not 10:20 as some may be inclined to claim. At 10:20, the hour hand is $\frac{1}{3}$ ($\frac{20}{60}$) of the way between 10 and 11 and the minute hand located at 4 is not directly opposite it. Let x be the number of minutes after 10:20 when the hands are directly opposite. The hour hand has thus moved $(20 + x)/60$ of the way from 10 to 11 and the minute hand has moved $(x/5)$ of the way from 4 to 5. Since the hands are directly opposite each other, these two amounts are equal, or $\frac{20 + x}{60} = \frac{x}{5}$. This gives $x = \frac{20}{11}$ and the time is $\boxed{10:21 \frac{9}{11}}$.

2.) Any one digit number is the sum of its digits. Any number which is 5 times the sum of its digits must be a multiple of 5 and hence end in either 0 or 5. However, any number ending in 0 is at least 10 times the sum of its digits. Any two digit number ending in 5 has sum of its digits at least 6 ($1+5$) and at most 14 ($9+5$) and hence if five times the sum of its digits must be at least 35 and at most 65. Any three digit number ending in 5 has sum of its digits at most 23 ($9+9+5$) and hence if five times the sum of its digits is at most 115. However, 105 and 115 have sum of digits at most 7 and are not five times the sum of their digits. Any number with four or more digits is much more than five times the sum of its digits. Checking the remaining two digit numbers ending in 5 from 35 to 65 shows that $\boxed{45 \text{ is the ONLY number five times the sum of its digits } (45 = 5 \cdot (4 + 5))}$.

3.) There are three cases to consider. If all of the letters of the word are different, there are 8 choices for the first letter, seven choices for the second letter, six choices for the third letter, five choices for the fourth letter and four choices for the fifth letter, for a total $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$ distinct words. If one letter occurs twice, there are $10 = \frac{5 \cdot 4}{2}$ choices for which two places in the word are the same letter, three choices for which letter is repeated, 7 choices for the first nonrepeated letter, six choices for the second nonrepeated letter and 5 choices for the third nonrepeated letter, for a total of $10 \cdot 3 \cdot 7 \cdot 6 \cdot 5 = 6300$ distinct words. If there are two distinct letters each occurring twice, there are three choices for which two letters appear twice, 10 choices for which two places in the word the alphabetically first letter occurs, 3 choices for which two places in the word the other repeated letter occurs and six choices for the nonrepeated letter, for a total of $3 \cdot 10 \cdot 3 \cdot 6 = 540$ distinct words. Combining these three cases gives a total of $6720 + 6300 + 540 = \boxed{13560 \text{ distinct words}}$.

4.) Any positive integer a which is one less than a perfect fourth power is of the form $n^4 - 1$ for some positive integer. If it is a prime, it must be larger than 1 and hence n must be larger than 1. However $n^4 - 1 = (n^2 - 1)(n^2 + 1) = (n - 1)(n + 1)(n^2 + 1)$. However, if n is an integer greater than 1 then each of $n - 1$, $n + 1$ and $n^2 + 1$ is a distinct positive integer dividing a . One of these three must be distinct from 1 and a and hence $\boxed{\text{there is NO prime number which is one less than a perfect fourth power.}}$

5.) Each non-leap year has 365 days, or 52 weeks of 7 days each ($52 \cdot 7 = 364$) plus one extra day. Each leap year has an additional extra day. In the 150 years since 1861 there have been 36 leap years. There have thus been $150 \cdot 1 + 36 \cdot 1 = 186$ extra days, besides the exact number of weeks. However, 186 days is equivalent to 26 weeks ($26 \cdot 7 = 182$) plus 4 extra days. Thus the day of the week this year is four days later than the day of the week in 1861 and the American Civil War began on $\boxed{\text{Friday April 12, 1861}}$.

6.) Each of the triangles to the right or left of the square is a $30^\circ - 60^\circ$ right triangle with height 1. The base opposite the 30° angle thus has length $1/\sqrt{3}$. The equilateral triangle above the square has base of length 1 and thus has height $\sqrt{2}/2$. Each triangle on the side thus has area $\frac{1}{2} \cdot (\text{base}) \cdot (\text{height}) = \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \cdot 1 = \frac{1}{2\sqrt{3}}$. The triangle above the square has area $\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. The sum of the areas of the square and the three triangles

is thus $\boxed{1 + 2 \cdot \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{4} = 1 + \frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{4}}$. Alternatively, the large triangle has base $1 + \frac{2}{\sqrt{3}}$

and height $\frac{\sqrt{3}}{2} + 1$ and hence area $\boxed{\frac{1}{2} \cdot \left(1 + \frac{2}{\sqrt{3}}\right) \cdot \left(\frac{\sqrt{3}}{2} + 1\right) = 1 + \frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{4}}$.