

EMMY NOETHER HIGH-SCHOOL MATHEMATICS DAY  
Texas Tech University  
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SOLUTIONS.

1.) The first 5 digits represent the odd numbers from 1 through 9. The next 90 digits ( $2 \times 45$ ) represent the odd numbers from 11 through 99. The next 150 digits ( $3 \times 50$ ) represent the odd numbers from 101 through 199. At this point, the first 245 digits ( $5 + 90 + 150$ ) have been written. The next several digits are 201203205, with the 250th digit being the 0 in 203.

2.) One starts with a single triangle. At the first stage 3 triangles are added. At the second stage 6 triangles are added. At the third stage 9 triangles are added. This pattern continues, so when triangles are added at the tenth stage one has  $1 + 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30 =$  166 triangles. (If this process is continued, then by the  $n$ th stage there have been  $3(1 + 2 + 3 + \cdots + n) = 3 \cdot \frac{n(n+1)}{2}$  triangles added, plus the one that one started with.)

3.) During one revolution of Mercury about the Sun, the direction from Mercury to the Sun has changed by one complete circuit or  $360^\circ$ . During this same time, Mercury has rotated 1.5 times on its axis, so the direction overhead for a given spot on the surface of Mercury has changed by 1.5 circuits or by  $1.5 \times 360^\circ$ . Since both motions are in the same direction, the direction of the Sun relative to the overhead direction for a given point on the surface of Mercury has changed by  $540^\circ - 360^\circ = 180^\circ$ . The solar day on Mercury is twice this long, or the length of time for the angle between the Sun and the direction overhead for a given spot on the surface of Mercury to change by  $360^\circ$ . Thus the solar day on Mercury is  $2 \times 87.9 =$  175.8 days. (Mercury has the longest solar day of any planet in the solar system, though Venus takes longer to rotate on its axis. The solar "day" on Mercury is twice as long as the "year" on Mercury.)

4.) The number 3 is a prime which is one less than the perfect square 4. Suppose that the positive integer  $n$  is a prime number which is one less than a perfect square. Then there exists a positive integer  $x$  such that  $n = x^2 - 1 = (x - 1)(x + 1)$ . However,  $x - 1$  must be nonzero, so  $x$  must be at least 2. Since  $n$  is a prime number whose only positive integer factors are 1 and  $n$ , then  $x - 1 = 1$  and  $x + 1 = n$ . This implies that  $n = 3$  is the only prime number which is one less than a perfect square.

5.) One can form 8 cents postage with one 3 cent stamp and one 5 cent stamp. One can form 9 cents postage with three 3 cent stamps. One can form 10 cents postage with two 5 cent stamps. Every positive integer  $n$  greater than 10 can be formed as one of 8, 9 or 10 plus a positive integer multiple of 3. (If  $n$  is a multiple of 3, then it is also 9 plus a multiple of 3. If  $n$  has a remainder of 1 when divided by 3, then it is also 10 plus a multiple of 3. If  $n$  has a remainder of 2 when divided by 3, then it is also 8 plus a multiple of 3.) Thus every exact postage amount of eight cents or more can be formed from exact postage for 8, 9 or 10 cents using only 3 cent and 5 cent stamps, plus a quantity of 3 cent stamps. (There is nothing special about the numbers 3 and 5 in this problem other than the fact that they are relatively prime, i.e. their only common divisor is one. For any relatively prime positive integers  $a$  and  $b$  there is a positive integer  $k$  such that every integer greater than  $k$  can be formed as the sum of nonnegative multiples of  $a$  and  $b$ .)

6.) Not that by the Pythagorean theorem each side of the large square has length  $\sqrt{2}$ . Consider the triangle with one vertex at the center of the circle, one vertex at the upper right corner of the rightmost small square and one vertex at the midpoint of the right side of the rightmost small square. This is a right triangle with hypotenuse a radius of the circle, hence of length 1. If each side of the smaller square (All of the smaller squares are of the same size.) has length  $x$  then the right side of the triangle described has length  $\frac{x}{2}$ . The distance from the center of the circle to the right side of the large square is half the length of a side of the square and so is  $\frac{\sqrt{2}}{2}$ . The length of the base of the triangle describe is then this length plus the width of the smaller

square or  $\frac{\sqrt{2}}{2} + x$ . By the Pythagorean theorem, 
$$x = \frac{-\sqrt{2} + \sqrt{4.5}}{2.5} = \frac{\sqrt{2}}{5} \approx 0.2828427..$$