

EMMY NOETHER HIGH-SCHOOL MATHEMATICS DAY
Texas Tech University
May 6, 2009

SOLUTIONS.

1.) We present two slightly different arguments. First consider the case of two consecutive odd numbers. These can be expressed as n and $n + 2$. The difference of their squares is $(n + 2)^2 - n^2 = (n^2 + 4n + 4) - n^2 = 4n + 4 = 4(n + 1)$. Since n is odd, $n + 1$ is even (a multiple of 2) and $4(n + 1)$ is thus a multiple of 8. The difference between the squares of any two odd numbers is a sum of differences between the squares of consecutive odd numbers (e.g. $(n + 4)^2 - n^2 = \{(n + 4)^2 - (n + 2)^2\} + \{(n + 2)^2 - (n)^2\}$) and thus a sum of multiples of 8 (and hence itself a multiple of 8).

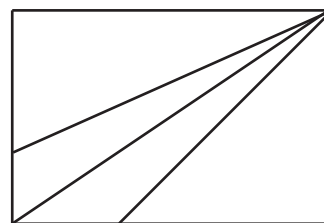
Without considering consecutive odd numbers, any two odd numbers differ by an even number and so can be expressed as n and $n + 2k$ for some number k . The difference between their squares is then $(n + 2k)^2 - n^2 = (n^2 + 4nk + 4k^2) - n^2 = 4nk + 4k^2 = 4k(n + k)$. If k is odd, then $n + k$ is even (since n is also odd) and thus $4(n + k)$ is a multiple of 8 and so is $4k(n + k)$. If k is even then $4k$ is a multiple of 8 and so is $4k(n + k)$.

2.) Consider each of the integers 0, 1, 2, 3 and 4 and express each in the form $5 \cdot x + 7 \cdot y$ with minimal usage of negative values for x or y . We have $0 = 5 \cdot 0 + 7 \cdot 0$, $1 = 5 \cdot 3 + 7 \cdot (-2)$, $2 = 5 \cdot (-1) + 7 \cdot 1$, $3 = 5 \cdot 2 + 7 \cdot (-1)$ and $4 = 5 \cdot (-2) + 7 \cdot 2$. Adding $24 = 5 \cdot 2 + 7 \cdot 2$ to each of these expressions gives each of the numbers 24, 25, 26, 27 and 28 expressed in the form $5 \cdot x + 7 \cdot y$ with each of x and y nonnegative. Every integer greater than 28 can be expressed as one of the numbers 24, 25, 26, 27 or 28 plus a positive integer multiple of 5, and thus can also be expressed in the form $5 \cdot x + 7 \cdot y$ with each of x and y nonnegative. A check shows that the number 23 cannot be expressed in the form $5 \cdot x + 7 \cdot y$ with each of x and y nonnegative and so it is the largest integer with this property.

3.) The points of intersection of the lines drawn from the upper right corner with the bottom and left sides of the rectangle divide each of these sides into line segments in the ratio of 2:1. This ratio is independent of the lengths of the sides of the rectangle or their relative proportions. We give two arguments for this.

Suppose that the rectangle has height h and width w . It then has area $h \cdot w$. Since the line segments drawn from the upper right corner divide the rectangle into three regions of equal area, each of these regions has area $\frac{h \cdot w}{3}$. Consider the triangle containing the lower right corner of the rectangle. The area A of a triangle is given by $A = \frac{1}{2}(\text{base}) \cdot (\text{height})$. The height is h and the area is $\frac{h \cdot w}{3}$, so the base is $\frac{2}{3}w$. The part of the bottom of the rectangle in the quadrilateral formed in the subdivision thus has length $\frac{1}{3}w$ and the bottom of the rectangle is subdivided into parts with lengths in the ratio 2:1. A similar argument holds for the triangle containing the upper left corner of the rectangle.

Alternatively, consider a diagonal line segment joining the upper right and lower left corners of the rectangle. This divides the quadrilateral into regions of equal area. (To observe that these areas are equal, note that this line segment divides the rectangle into regions of equal area, each containing triangles of equal areas.) The area of the quadrilateral is given as equal to the area of the triangle containing the lower right corner of the rectangle. The area of the triangle formed by the newly introduced diagonal, the original line segment from upper right corner to bottom side of the rectangle and the line segment consisting of the left piece of the subdivision of the bottom side is thus equal to half the area of the triangle containing the bottom right corner. Both triangles have the same height. Therefore the base of the smaller triangle must be $\frac{1}{2}$ the length of the base of the larger triangle and the bottom side is divided into line segments in the ratio 2:1. A similar argument holds for the left side of the rectangle.



4.) One can quickly determine that the time is between 4:54 and 4:55. (Since the hour hand is between 4 and 5, the minute hand is between 10 and 11. More than four fifths of the hour has thus passed and the hour hand must be more than four fifths of the way from 4 to 5, with the minute hand more than four fifths of the way from 10 to 11.) However, we wish to determine the **exact** time when the hour and minute hands are **exactly** 180° apart. Let x represent the number of minutes after 4:00 when this occurs. At this time the minute hand has gone $\frac{x}{60}$ of the way around from the straight up 12 position it was in at 4:00. Since the hour hand is exactly opposite it, the hour hand has gone $\frac{x}{60} - \frac{1}{2}$ of the way around from the straight up 12 position which it was in at 12:00. This occurred 4 hours and x minutes or $240 + x$ minutes ago. It takes the hour hand 12 hours or 720 minutes to make a complete circle on the clock, so at this time the hour hand has gone $\frac{240 + x}{720}$ of the way around from the straight up 12 position it was in at 12:00. Thus

$$\frac{x}{60} - \frac{1}{2} = \frac{240 + x}{720}. \text{ Solving this equation gives } x = 54\frac{6}{11} \text{ and thus the exact time is } \boxed{4 : 54\frac{6}{11}}.$$

5.) Each non-leap year has 365 days, or 52 weeks of 7 days each ($52 \cdot 7 = 364$) plus one extra day. Each leap year has 366 days, or 52 weeks of 7 days each plus 2 extra days. In the 200 years since the birth of Abraham Lincoln and Charles Darwin there have been 151 non-leap years and 49 leap years (except for the fact that 1900 was not a leap year there would normally be 50 leap years in 200 years), so there have been $151 \cdot 1 + 49 \cdot 2 = 249$ extra days, besides the exact number of weeks. However, 249 days is equivalent 35 weeks ($7 \cdot 35 = 245$) plus four extra days. Thus the 200th birthday of Abraham Lincoln and Charles Darwin on a Thursday occurred four week days (and many weeks) after the day they were born on

Sunday, February 12, 1809.

6.) We are given that Alice intended to withdraw x dollars and y cents, but instead withdrew y dollars and x cents. After spending exactly one dollar she has $(y - 1)$ dollars and x cents, which is exactly half the amount she intended to withdraw. If we separately equate the dollars and cents she has after spending one dollar with half of the original dollar and cents she intended to withdraw, we have $y - 1 = \frac{x}{2}$ for the dollars and $x = \frac{y}{2}$ for the cents. Solving this pair of equations gives $x = \frac{2}{3}$ and $y = \frac{4}{3}$. However, a meaningful answer should have each of x and y a nonnegative integer.

The original amount which she intended to withdraw, x dollars and y cents is equivalent to $(x - 1)$ dollars and $(100 + y)$ cents, and would be so interpreted when considering half of the amount if x were odd. In this case, half of the amount which she intended to withdraw is now $\frac{x - 1}{2}$ dollars and $\frac{100 + y}{2}$ cents. Now separately equating the dollars and cents amounts she has after spending one dollar with half of the amount she intended to withdraw gives $y - 1 = \frac{x - 1}{2}$ and $x = \frac{100 + y}{2}$. Solving this set of equations gives $x = 67$

and $y = 34$, a meaningful answer. Thus she intended to withdraw \$67.34. Instead, she withdrew \$34.67 and after spending \$1.00 had exactly \$33.67, precisely one half the amount she intended to withdraw.

(Note that, if appropriately interpreted, the solution of $x = \frac{2}{3}$ and $y = \frac{4}{3}$ can be considered a meaningful answer even though one does not normally talk in terms of thirds of dollars or cents. With these values, x dollars and y cents is equivalent to $\$0.66\frac{2}{3} + \$0.01\frac{1}{3} = \$0.68$, an exact monetary value which can be realized in current coinage. The amount of y dollars and x cents is equivalent to $\$1.33\frac{1}{3} + \$0.00\frac{2}{3} = \$1.34$, also an exact monetary amount which can be realized in current coinage. After spending \$1.00 from this amount of y dollars and x cents, the person has \$0.34, which is exactly half of the intended amount of x dollars and y cents. We leave it to the student to determine if there are other solutions which can be interpreted as meaningful, e.g. by interpreting x dollars and y cents as $(x - 2)$ dollars and $(200 + y)$ cents, or by equating total monetary values without separately equating amounts for dollars and for cents.)