

EMMY NOETHER HIGH-SCHOOL MATHEMATICS DAY  
Texas Tech University  
May 8, 2008

SOLUTIONS.

1.) Each of the numbers would have over 1000 decimal places if written in nonexponential form. Thus, it is not practical to try to determine the exact decimal expression for each. Each of the exponents is an approximate multiple of 900. Thus  $2^{3543} < 2^{3600} = (2^4)^{900} = 16^{900}$ ,  $3^{2776} > 3^{2700} = (3^3)^{900} = 27^{900}$  and  $5^{1836} > 5^{1800} = (5^2)^{900} = 25^{900}$ . By this comparison we see that  $2^{3543}$  is the smallest of the three expressions. Looking more closely at the other two terms, we see that  $3^{2776} = 3^{3 \cdot 925 + 1} = 3 \cdot (3^3)^{925} = 3 \cdot 27^{925}$  and  $5^{1836} = (5^2)^{918} = 25^{918}$ . Thus  $3^{2776}$  is the largest of the three expressions.

2.) Venus goes around the Sun more rapidly than the Earth does. Venus will next be on a line between the Earth and the Sun when it has gone around the Sun precisely one more time than the Earth has, i.e. when the Earth has gone around the Sun  $x$  times and Venus has gone around the Sun  $x + 1$  times. It takes the Earth  $365x$  days to go around the Sun  $x$  times and it takes Venus  $225(x + 1)$  days to go around the Sun  $x + 1$  times. Thus  $365x = 225(x + 1)$  or  $x = \frac{225}{140} = \frac{45}{28}$ . Thus, Venus will next be in a line between the Earth

and the Sun after  $365 \cdot \frac{45}{28} \approx 587$  days.

Alternatively, each day the Earth moves  $\frac{360}{365}^\circ$  in its orbit around the sun, while Venus moves  $\frac{360}{225}^\circ$  around the sun each day. Thus Venus, gets  $\frac{360}{225} - \frac{360}{365} = 360 \left( \frac{140}{225 \cdot 365} \right)^\circ$  ahead of the Earth each day. Venus will again be on a line between the Earth and the Sun when it has moved  $360^\circ$  more than the Earth or in  $\frac{360}{360 \cdot \frac{28}{45 \cdot 365}} = \frac{45 \cdot 365}{28}$  days, the same answer as obtained above.

3.) Each of the small circles in the corners of the square is of the same size. The large circle has diameter 2 (the same as the length of each side of the square) and hence radius 1. Consider the line segment going from the center of the large circle to the lower left corner of the square. Since this is at a  $45^\circ$  angle to either the horizontal or vertical side of the square, this line segment has length  $\sqrt{2}$ . If the radius of the smaller circle is  $r$ , then the length of the line segment from the center of the lower left small circle to the lower left corner of the square has length  $\sqrt{2}r$ . The line segment from the center of the large circle to the lower left corner of the square is made up of three parts, a radius of the large circle from the center of the large circle to its point of tangency with the small circle, a radius of the small circle from this point of tangency to the center of the small circle and the line segment from the center of the small circle to the lower left corner of the square. Adding the lengths of these line segments one has  $1 + r + \sqrt{2}r = \sqrt{2}$ . Solving for  $r$ , the radius of

the small circle, gives  $r = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = (\sqrt{2} - 1)^2 \approx 0.1716$ . Students are not assumed to have

access to a calculator during the competition and the exact answer with radicals is preferable to the decimal approximation.

4.) We have to consider a number of cases, since there are three E's and two M's which may be repeated in the five-letter word. For example, in the given sample word RNEME we cannot tell which of the two M's is used nor which two of the three E's is used, or in which order.

Case 1: The word contains three E's and two M's. There are five choices as to where the first E occurs in the five-letter word, four choices as to where the second E occurs and three choices as to where the remaining E occurs. However, the E's are indistinguishable from each other and the same word occurs if the set of three E's occurs in the same three places in the five-letter word but with the E's in different relative order. There are three choices as to which E occurs first, two choices as to which E occurs second, with the remaining E occurring last. There are thus  $(5 \cdot 4 \cdot 3)/(3 \cdot 2 \cdot 1) = 10$  distinct choices for the placement of the E's, with the remaining two spaces being M's.

Case 2: The word contains three E's and at most one M. By the same reasoning as above, there are 10 distinct choices for positioning the three E's in the five-letter word. For the remaining two spaces there are seven choices for the letter in the first position and six choices for the letter in the remaining position. (There are eight distinct letters among the 11 given.) There are thus  $10 \cdot 7 \cdot 6 = 420$  possible five-letter words for this case.

Case 3: The word contains two E's and two M's. There are five choices for the position of the first E and four choices for the position of the second E. However, the same word results independent of the relative order of the two E's, thus there are  $5 \cdot 4/2 = 10$  distinct choices for where the E's are in the five-letter word. There are then three choices for where the first M is in the word and two choices for where the second M is. However, again relative order of the M's is indistinguishable in the word, so there are  $3 \cdot 2/2 = 3$  choices for where the two M's are in the word. There are six choices for the remaining letter of the five-letter word. There is thus a total of  $10 \cdot 3 \cdot 6 = 180$  possible five-letter words for this case.

Case 4. The word contains two E's and at most one M. By the same reasoning as in the previous case, there are 10 choices for where the two E's occur in the word. The remaining three letters are distinct, with seven choices for the first, six choices for the second and five choices for the remaining letter. There is thus a total of  $10 \cdot 7 \cdot 6 \cdot 5 = 2100$  possible five-letter words for this case.

Case 5. The word contains at most one E and contains two M's. There are five choices for where the first M occurs and four choices for where the second M occurs. However the same word occurs independent of the relative order of the two M's, so there are  $5 \cdot 4/2 = 10$  distinct possible choices for where the two M's occur in the word. The remaining three letters are distinct, with seven choices for the first, six choices for the second and five choices for the remaining letter. There is thus a total of  $10 \cdot 7 \cdot 6 \cdot 5 = 2100$  possible five-letter words for this case.

Case 6. No letter occurs more than once in the word. There are eight choices for the first letter, seven choices for the second letter, six choices for the third letter, five choices for the fourth letter and four choices for the last letter. There is thus a total of  $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$  possible five-letter words for this case.

Adding the number of possible words for each of the distinct cases above gives  $10 + 420 + 180 + 2100 + 2100 + 6720 = \boxed{11530}$  distinct five-letter words possible from the given set of letters.

5.) Let  $t$  be the time which it takes Nancy to run back to the entrance of the train tunnel. We are given that Nancy runs at the same constant speed in each direction. Thus, if she runs towards the end of the tunnel, then after time  $t$  she has run two fifths of the length of the tunnel from her starting point. She starts running two fifths of the way through the tunnel from the entrance. Thus, after time  $t$  she is four fifths of the way through the tunnel from the entrance. This is the same time at which she would have been at the entrance if she ran in the other direction, so this is the time when the train reaches the entrance of the tunnel. If Nancy continues running to the end of the tunnel, she reaches it at the same time the train does. Thus the train passes the entire length of the tunnel in the same time it takes Nancy to run the last fifth of the length of the tunnel. Nancy thus runs one fifth as fast as the train, or  $\boxed{10 \text{ miles per hour}}$  and collapses just outside the tunnel as the train whizzes by her.

6.) Let  $n$  be the number of nickels which Debra has,  $d$  the number of dimes which she has and  $q$  the number of quarters which she has. From the given information we have that  $n + d + q = 12$  (from the number of coins) and  $5n + 10d + 25q = 180$  (from the monetary value of the coins). Without any other restrictions on the values of  $n$ ,  $d$  and  $q$  these equations have infinitely many solutions (allowing for non-integer and non-positive values of the variables). However, we are only interested in solutions where each of  $n$ ,  $d$  and  $q$  is a positive integer. Rather than trying to distinguish which of the infinitely many numerical solutions satisfy these additional constraints, we consider cases.

If there is only one quarter, the remaining eleven coins (each worth at most ten cents) have a total value of \$1.55, an impossibility.

Similarly, if there are two quarters, the remaining ten coins (each worth at most ten cents) have a total value of \$1.30, also an impossibility.

If there are three quarters, then the remaining nine coins (each worth at most ten cents) have a total value of \$1.05, still an impossibility.

If there are four quarters, the remaining eight coins have a total value of \$0.80. This is possible if each of the eight remaining coins is a dime, but is not a valid answer since there are no nickels.

If there are five quarters, the remaining seven coins have a total value of \$0.55, which is only possible with 3 nickels and four dimes.

If there are six quarters, the remaining six coins have a total value of \$0.30. As indicated in the statement of the problem, this is possible if each of the remaining coins is a nickel, but it is not a valid answer since there are no dimes.

If there are seven or more quarters, the total value of the twelve coins must exceed the indicated amount of \$1.80.

Thus, the only valid solution, with at least one of each type coin, is  $\boxed{3 \text{ nickels, 4 dimes and 5 quarters.}}$