EMMY NOETHER HIGH-SCHOOL MATHEMATICS DAY
Texas Tech University
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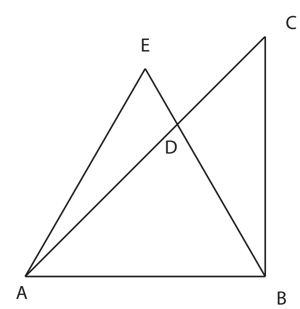
SOLUTIONS.

1.) All of the number involved in the equation are powers of 3. Expressing them all in exponential notation and using that $\frac{1}{3^\alpha} = 3^{-\alpha}$, the equation become $3^{-3} \cdot 3^{100} \cdot 3^{-4} \cdot (3^2)^x = 3^{-1} \cdot 3^x$. When multiplying powers of 3, one adds exponents and $(3^2)^x = 3^{2x}$. The equation thus becomes $3^{-3+100-4+2x} = 3^{-1+x}$ or, equating exponents, $93 + 2x = -1 + x$, giving the solution $\boxed{x = -94.}$

2.) The minute hand of a clock moves at 6° per minute (a complete circuit of 360° in one hour, or 60 minutes). The hour hand of a clock moves at 0.5° degrees per minute (one-twelfth of a complete circuit, i.e. 30°, in one hour, or 60 minutes). At 12:00 noon the hour hand and minute hand are together, both pointing straight upward. Each minute the minute hand moves 5.5° more than the hour hand. It has thus moved 120° farther than the hour hand after $\frac{120}{5.5} \approx 21.81818$ minutes, or at $\boxed{12:00 + \frac{120}{5.5} \approx 12:21:49.1 \text{ pm.}}$

The second hand moves at 360° per minute, a complete circuit every minute. It gains 354° on the minute hand each minute, or 120° in $\frac{120}{354} \approx 0.338983$ minutes and gains 240° on the minute hand in twice this time. During a 12 hour (720 minute) period, between when the hands are all straight up and when they are again all together and straight up, there are $\frac{720}{\frac{120}{5.5}} = 33$ times when the hour hand and the minute hand are an exact multiple of 120° apart. During the same 12 hour period, there are $\frac{720}{\frac{120}{354}} = 2108$ times when the minute hand and the second hand are an exact multiple of 120° apart. The numbers 33 and 2108 have no common integer divisors other than one, i.e. are relatively prime. There is thus no time shorter than 12 hours when all three hands are exact multiples of 120° apart and hence $\boxed{\text{NO TIME}}$ when the three hands divide the face of the clock into equal thirds.

3.) Rather than directly trying to compute the area of triangle ADE it is probably easier to compute the areas of triangles ABE and ABD and take the difference. The area of a triangle is given by area $= \frac{1}{2}$ (base) (height). Each of triangles ABE and ABD has base AB of length 1. Triangle ABE is an equilateral triangle of base 1 and so has height of $\frac{\sqrt{3}}{2}$. (This is a standard fact for equilateral triangles or can be derived from properties of $30° - 60°$ right triangles and the Pythagorean theorem.) The area of triangle ABE is thus $\frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$. .



To determine the area of triangle ADE we need to determine its height and hence the location of the point D. The point D is at the intersection of line segment AC with line segment EB. Consider a coordinate system with origin at the point A, $x-$axis horizontal and $y-$axis vertical. Line segment AC has equation $y = x$. Line segment EB has equation $y = \sqrt{3} - \sqrt{3}x$. (Consider its slope and $y-$intercept.) Solving these two equations for the point of intersection D, one has $y = \frac{\sqrt{3}}{1+\sqrt{3}}$, giving the area of triangle ABD as $\frac{1}{2}(1)\left(\frac{\sqrt{3}}{1+\sqrt{3}}\right)$. The area of triangle ADE is thus $\boxed{\frac{\sqrt{3}}{4} - \frac{1}{2}\frac{\sqrt{3}}{1+\sqrt{3}} = \frac{2\sqrt{3}-3}{4}.}$

4.) We do not know how many students are in the class, but this information is not needed. Let x be the percentage of students who passed the exam and $100 - x$ be the percentage of students who failed the exam. From the given information $\frac{(x)(65) + (100-x)(35)}{100} = 53$. The solution of this equation is $x = 60$ and hence

$\boxed{60\%}$ of the students passed the exam. (Note that though we know that over half of the students passed the exam and the average grade of all students was 53, we cannot conclude that 53 was a passing grade. The above averages are possible with a passing grade of 60 and at least 60% of students doing above average.)

5.) This problem is complicated by the fact that water is continuing to flow into the ocean liner while the pumps work to pump it out. There are three quantities to consider. Let w be the amount of water present in the ship when the pumps begin to operate. Let x be the rate of inflow of water into the ocean liner per hour. Let y be the amount of water pumped out of the ship by a single pump per hour. From the information given we can form two equations, $w + 3x - (12)(3)y = 0$ and $w + 10x - (5)(10)y = 0$, reflecting the fact that with 12 pumps the amount of water in the ship after 3 hours is 0 and with 5 pumps the amount of water in the ship after 10 hours is 0. Subtracting the first equation from the second equation gives $7x - 14y = 0$ or $x = 2y$. Substituting this for x in either of the two equations gives $w - 30y = 0$ or $w = 30y$.

If n is the number of pumps required to pump all of the water out of the ship in 2 hours, then $w + 2x - (n)(2)y = 0$. Substituting $x = 2y$ and $w = 30y$ gives $34y - (n)(2)y = 0$, or $\boxed{n = 17}$ pumps are needed to pump all of the water out of the ship in 2 hours.

(Note that though we had only two equations and three variables, w, x and y, we never solved for the three variables, only for the ratios between them. We do not have enough information to determine the value of each, but since each equation is a linear expression equal to 0, i.e. this is a homogeneous system of equations, with the equations independent, the ratios are uniquely determined.)

6.) We give a much more extensive analysis of this problem than was expected of students. Students were only instructed to determine one possible set of 15 actual ballots giving the indicated results and hence only had to consider one of each possible case or sub-case. We consider all possibilities here. (It is assumed that all ballots did contain a ranking for all three soft drinks with no one making only a first choice and leaving the others blank.)

Since Dr. Pepper won when plurality was used, it received more first place votes than either Coca Cola or Pepsi Cola. Thus, it had to receive at least six first place votes. If had received at least eight first place votes, it would also have won when instant runoff was used, which it did not. Thus, Dr. Pepper received either six or seven first place votes..

Case 1. Dr. Pepper received six first place votes.

Case 2. Dr. Pepper received seven first place votes.

Pepsi Cola won when instant runoff was used and so must have received the second highest number of first place votes, with Coca Cola being eliminated in the instant runoff..

Suppose that Dr. Pepper received six first place votes (Case 1). Then Pepsi Cola received five first place votes and Coca Cola received four first place votes. (There were nine first place votes which did not go to Dr. Pepper and Pepsi Cola received the majority of these nine. If Pepsi Cola received six or more of these votes, it would have tied or exceeded Dr. Pepper's number of first place votes, which it did not.) Since Pepsi Cola won when instant runoff was used, it received at least three second place rankings on ballots which placed Coca Cola first. Thus the four ballots which placed Coca Cola first either all placed Pepsi Cola second and Dr. Pepper last or three of these placed Pepsi Cola second with Dr. Pepper last and one placed Dr. Pepper second with Pepsi Cola last.

Case 1 can be divided into two sub-cases.

Case 1a. Pepsi Cola was second on three of the ballots placing Coca Cola first.

Case 1b. Pepsi Cola was second on all four of the ballots placing Coca Cola first.

Coca Cola won when the point system was used. Thus, it had to receive at least 16 of the possible 45 points. (Total points for a set of 15 ballots was 2×15 for first place plus 1×15 for second place.) Coca Cola received at most 19 points, eight points for its four first place votes plus possibly eleven points if everyone voting for either Dr. Pepper or Pepsi Cola as first place chose Coca Cola second.

Each of Cases 1a and 1b can be divided into four sub-cases.

Case 1a6 or Case 1b6. Coca Cola receives 16 points.

Case 1a7 or Case 1b7. Coca Cola receives 17 points.

Case 1a8 or Case 1b8. Coca Cola receives 18 points.

Case 1a9 or Case 1b9. Coca Cola receives 19 points.

Suppose that Dr. Pepper received seven first place votes (Case 2). Pepsi Cola must have received the majority of the remaining eight votes.

Case 2 can be divided into two sub-cases.

Case 2a. Pepsi Cola receives five first place votes and Coca Cola receives three first place votes.

Case 2b. Pepsi Cola receives six first place votes and Coca Cola receives two first place votes.

In each case, for Pepsi Cola to win when instant runoff was used, it must have been placed second on every ballot placing Coca Cola first.

Again, Coca Cola must receive at least 16 points to win when the point system was used. In Case 2a it could have received a maximum of 18 points, six from the three ballots placing it first and one on each of the remaining 12 ballots if everyone placing either Dr. Pepper or Pepsi Cola first placed Coca Cola second. In Case 2b, Coca Cola could have received a maximum of 17 points, four from its two first place votes and one on each of the remaining 13 ballots if everyone placing either Dr. Pepper or Pepsi Cola first placed Coca Cola second.

Thus Case 2a and Case 2b can be divided into the following sub-cases.

Case 2a6 or Case 2b6. Coca Cola receives 16 points.

Case 2a7 or Case 2b7. Coca Cola receives 17 points.

Case 2a8. Coca Cola receives 18 points.

Listed below are various possible votes giving these results in each case. For convenience below, notation such as 6 DCP means that six ballots each ranked Dr. Pepper (D) first, Coca Cola (C) second and Pepsi Cola (P) third. Each line indicates a different possible set of 15 ballots.

Case 1a6. 3 DCP, 3 DPC, 5 PCD, 3 CPD, 1 CDP
4 DCP, 2 DPC, 4 PCD, 1 PDC, 3 CPD, 1 CDP
5 DCP, 1 DPC, 3 PCD, 2 PDC, 3 CPD, 1 CDP
6 DCP, 2 PCD, 3 PDC, 3 CPD, 1 CDP

Case 1a7. 4 DCP, 2 DPC, 5 PCD, 3 CPD, 1 CDP
5 DCP, 1 DPC, 4 PCD, 1 PDC, 3 CPD, 1 CDP
6 DCP, 3 PCD, 3 PDC, 3 CPD, 1 CDP

Case 1a8. 5 DCP, 1 DPC, 5 PCD, 3 CPD, 1 CDP
6 DCP, 4 PCD, 1 PDC, 3 CPD, 1 CDP

Case 1a9. 6 DCP, 5 PCD, 3 CPD, 1 CDP

Cases 1b6 through 1b9 correspond to Cases 1a6 through 1a9 with 4 CPD replacing 3 CPD, 1 CDP in each case.

Case 2a6. 5 DCP, 2 DPC, 5 PCD, 3 CPD
6 DCP, 1 DPC, 4 PCD, 1 PDC. 3 CPD
7 DCP, 3 PCD, 2 PDC, 3 CPD

Case 2a7. 6 DCP, 1 DPC, 5 PCD, 3 CPD
7 DCP, 4 PCD, 1 PDC, 3 CPD

Case 2a8. 7 DCP, 5 PCD, 3 CPD

Case 2b6. 6 DCP,1 DPC, 6 PCD, 2 CPD
7 DCP, 5 PCD, 1 PDC, 2 CPD

Case 2a7. 7 DCP, 6 PCD, 2 CPD

While there are many possible sets of ballots giving the results indicated in the problem, again the student was only instructed to determine one of these, not all of them as we have done here.

As this example illustrates, when there are more than two choices on a ballot, there are several possible and reasonable ways to determine a winner. The three listed here are only some of the possible methods of deciding. Different methods can give quite different results.

In all of the sets of ballots listed here, Dr. Pepper received the most first place votes. However, in none of these did it receive a majority of the first place votes. In some of the cases, while Dr. Pepper received the most (though less than half) first place votes there was also a majority ranking Dr. Pepper as third place. One might interpret this situation as indicating that most persons prefer cola, but with the preference for cola divided between Pepsi Cola and Coca Cola, while a substantial minority preferred Dr. Pepper.

It is also worth observing that in all of the cases described here, Coca Cola received the fewest first place votes, never more than four out of fifteen. However, in several of these cases a majority of persons ranked Coca Cola above Pepsi Cola.