Singularities of Nijenhuis operators of corank one

D. Akpan

The talk is devoted to the study of Nijenhuis operators of arbitrary dimension n in a neighborhood of a point at which the first $n-1$ coefficients of the characteristic polynomial are functionally independent, and the last coefficient (the determinant of the operator) is an arbitrary function. Also we will see some examples of Nijenhuis operators in small-dimensional cases.

The Nijenhuis geometry is a new section of modern differential geometry, which studies smooth manifolds with a tensor field L of type $(1,1)$, i.e., manifolds with endomorphism field for which the Nijenhuis torsion N_L is identically equal to zero. Fundamental results in this direction have been obtained by A. V. Bolsinov, V. S. Matveev, and A. Yu. Konyaev in the papers [1, 2, 3, 4, 5]. The vanishing of the Nijenhuis tensor for operator fields is a necessary (but not sufficient) condition for their integrability (i.e., the possibility of reduction to a constant form by a change of coordinates).

Definition 1. *Let M^n be a smooth n-dimensional manifold and let L be a tensor field of type $(1,1)$. Then the Nijenhuis torsion or the Nijenhuis tensor N_L is a tensor of type $(1,2)$ which is invariantly defined as follows:*

$$N_L[u,v] = L^2[u,v] + [Lu,Lv] - L[u,Lv] - L[Lu,v],$$

where u,v are arbitrary vector fields and $[u,v]$ is their commutator. In coordinates, this condition is written as follows:

$$(N_L)^i_{jk} = L^l_j \frac{\partial L^i_k}{\partial x^l} - L^l_k \frac{\partial L^i_j}{\partial x^l} - L^i_l \frac{\partial L^l_k}{\partial x^j} + L^i_l \frac{\partial L^l_j}{\partial x^k}.$$

Definition 2. *A tensor field L of type $(1,1)$, i.e., an operator field, is called a Nijenhuis operator if its Nijenhuis torsion is identically zero, i.e., $N_L \equiv 0$.*

Let L be a Nijenhuis operator, and let $\chi(t) = \det(t \cdot \mathrm{Id} - L) = t^n + \sigma_1 t^{n-1} + \ldots + \sigma_n$ be its characteristic polynomial. In [1], a theorem is proved on the form of the matrix of the Nijenhuis operator with functionally independent coefficients of the characteristic polynomial (invariants):

$$L = J^{-1} \tilde{L} J,$$

where

$$\tilde{L} = \begin{bmatrix} -\sigma_1 & 1 & 0 & \ldots & 0 \\ -\sigma_2 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\sigma_n & 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad J = \Big(\frac{\partial \sigma_i(x)}{\partial x^j}\Big).$$

For example, in the two-dimensional case, if the trace $trL = x$ is the first coordinate, and $\det L = f(x,y)$ is a smooth function, then $\sigma_1 = -x$, $\sigma_2 = f(x,y)$ and we obtain the general form of these operators:

$$L = \begin{bmatrix} -1 & 0 \\ \frac{f_x}{f_y} & \frac{1}{f_y} \end{bmatrix} \begin{bmatrix} x & 1 \\ -f & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ f_x & f_y \end{bmatrix} = \begin{bmatrix} x - f_x & -f_y \\ \frac{-xf_x + f_x^2 + f}{f_y} & f_x \end{bmatrix},$$

where f_x, f_y stand for the corresponding partial derivatives of the function f. It is clear that the existence of a smooth two-dimensional Nijenhuis operator in the situation under consideration is equivalent to the smoothness of the fraction $\frac{xf_x - f_x^2 - f}{f_y}$. At the points at which $f_y \neq 0$, this fraction is obviously smooth. A more interesting case occurs when $f_y = 0$ at some point, but $\frac{xf_x - f_x^2 - f}{f_y}$ is smooth in its neighborhood. The Morse and cubic singularities of the determinant (restricted to the level line of the trace) in the two-dimensional case were investigated in [6].

In two-dimensional case we have 4 Nijenhuis operators with non-degenerate trace and with the Morse singularity of the determinant.

$$L_1^{\pm} = \begin{bmatrix} x & \pm 2\tilde{y} \\ \frac{\tilde{y}}{2} & 0 \end{bmatrix} \quad \text{and} \quad L_2^{\pm} = \begin{bmatrix} \frac{x}{2} & \pm 2\tilde{y} \\ \frac{\tilde{y}}{2} & \frac{x}{2} \end{bmatrix}.$$

As we can see, the determinant is equal to $f(x,y) = \pm y^2$ or $f(x,y) = \pm y^2 + \frac{x^2}{4}$.

It's an interesting effect that we got two solutions. In the multidimensional case this effect is not observed.

Theorem 1. *Let L be an n-dimensional Nijenhuis operator for which $\sigma_1 = x_1$, $\sigma_2 = x_2$, $\dots$, $\sigma_{n-1} = x_{n-1}$ in some coordinates $(x_1, x_2, \dots, x_{n-1}, y)$, and $\sigma_n = (-1)^n \det L = f(x_1, \dots, x_{n-1}, y)$ is an arbitrary smooth function such that $f_y \neq 0$. Then, in these coordinates, L has the following form*:

$$L = \begin{bmatrix} -x_1 & 1 & 0 & \dots & 0 & 0 \\ -x_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -x_{n-2} & 0 & 0 & \dots & 1 & 0 \\ -x_{n-1} + f_{x_1} & f_{x_2} & f_{x_3} & \dots & f_{x_{n-1}} & f_y \\ \frac{\sum_{i=1}^{n-1} x_i f_{x_i} - f_{x_1} f_{x_{n-1}} - f}{f_y} & \frac{f_{x_1} + f_{x_2} f_{x_{n-1}}}{-f_y} & \frac{f_{x_2} + f_{x_3} f_{x_{n-1}}}{-f_y} & \dots & \frac{f_{x_{n-2}} + f_{x_{n-1}}^2}{-f_y} & -f_{x_{n-1}} \end{bmatrix}.$$

Remark 1. *It follows from Theorem 1 that the existence of a smooth operator field of the indicated form is equivalent to the smoothness of the following fractions:*

$$\frac{\sum_{i=1}^{n-1} x_i f_{x_i} - f_{x_1} f_{x_{n-1}} - f}{f_y}, \qquad \frac{f_{x_{j-1}} + f_{x_j} f_{x_{n-1}}}{f_y}, \quad j = 2, \dots, n-1.$$

In Theorem **??**, it is actually assumed only that the first $n-1$ coefficients of the characteristic polynomial are functionally independent. Then they can be taken as the first $n-1$ coordinates. If, in addition, the Jacobi matrix $\frac{\partial(\sigma_1, \dots, \sigma_n)}{\partial(x_1, \dots, x_{n-1}, y)}$ is degenerate, then this Nijenhuis operator can be called almost differentially nondegenerate.

Let us now study the case in which such a singularity of the Jacobi matrix is given by a nondegenerate singularity of the last coefficient (the determinant) $\sigma_n = \pm \det L$ with respect to the remaining coordinate y.

Theorem 2. *Let L be an n-dimensional Nijenhuis operator, where $n > 2$, for which $\sigma_1 = x_1$, $\sigma_2 = x_2$, $\dots$, $\sigma_{n-1} = x_{n-1}$ in some coordinates $(x_1, x_2, \dots, x_{n-1}, y)$, and let $\sigma_n = (-1)^n \det L = f(x_1, \dots, x_{n-1}, y)$ be a smooth function with a Morse singularity with respect to the variable y. Then there is a regular change of coordinates $y \to y(x_1, \dots, x_{n-1}, y)$ preserving the remaining coordinates $x_1, \dots, x_{n-1}$, after which $f = \pm y^2$, and the corresponding Nijenhuis operators have the form*

$$L = \begin{bmatrix} -x_1 & 1 & 0 & \dots & 0 & 0 \\ -x_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -x_{n-2} & 0 & 0 & \dots & 1 & 0 \\ -x_{n-1} & 0 & 0 & \dots & 0 & \pm 2y \\ -\frac{y}{2} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Thanks. The work is supported by the interdisciplinary scientific and educational school "Mathematical methods for the analysis of complex systems" of Lomonosov Moscow State University. The author is a scholarship holder of the Basis Foundation.

References

[1] A. Bolsinov, V. Matveev, A. Konyaev, *Nijenhuis Geometry, Advances in Mathematics*, Advances in Mathematics, 394 (2022) 108001.

[2] A. Bolsinov, V. Matveev, A. Konyaev, *Applications of Nijenhuis Geometry II: maximal pencils of multi-Hamiltonian structures of hydrodynamic type*, Nonlinearity, 34 (2021) 5136–5162.

[3] A. Bolsinov, V. Matveev, A. Konyaev, *Nijenhuis Geometry III: gl-regular Nijenhuis operators*, arXiv:2007.09506v1 [math.DG], 18 Jul 2020.

[4] A. Konyaev, *Nijenhuis geometry II: Left-symmetric algebras and linearization problem for Nijenhuis operators*, Differential Geometry and its Applications, 74 (2021) 101706.

[5] A. Konyaev, *Completeness of commutative Sokolov–Odesskii subalgebras and Nijenhuis operators on gl(n)*, Sb. Math., 211:4 (2020), 583–593.

[6] D. Akpan, *Singularities of two-dimensional Nijenhuis operators*, European Journal of Mathematics, 8 (2022) 1328–1340.