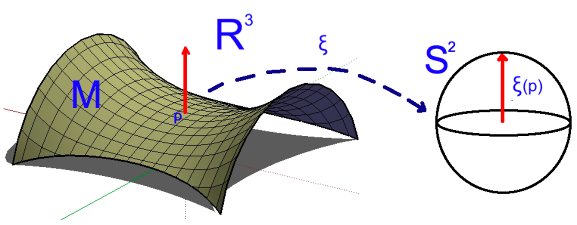

- Let $\mathbf{M}$ be an oriented surface in $\mathbf{R^3}$, let ξ be the unit vector field normal to $\mathbf{M}$:

$$\mathbf{A_p} = -\mathbf{d}\xi_{\mathbf{p}}\colon\, T_{\mathbf{p}}\mathbf{M} \to T_{\xi(\mathbf{p})}\mathbf{S^2} \simeq T_p\mathbf{M}$$

  is the **shape operator** of $\mathbf{M}$.

- The trace of $\mathbf{A}_p$ is twice the mean curvature $\mathbf{H}(p)$ at $p \in \mathbf{M}$.

Introduction to the theory of CMC surfaces.

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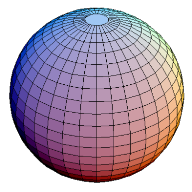
M is an **H-surface** $\iff$ **M** is a critical point for the area functional under compactly supported variations **preserving the volume**.



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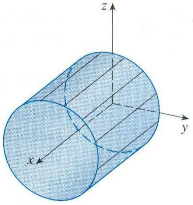
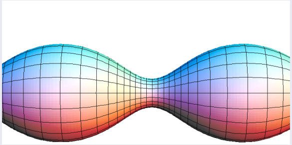


- Cylinder



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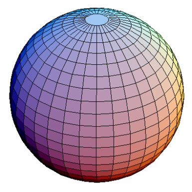

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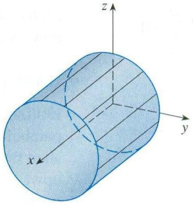

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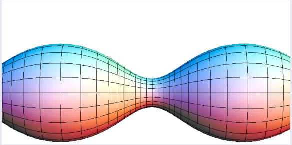

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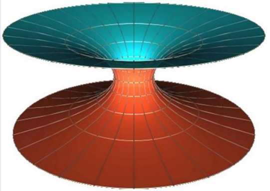
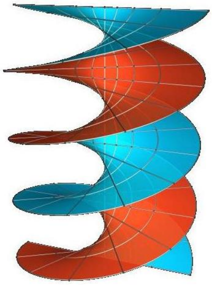
Definition

An **H**-surface **M** is a **minimal surface** $\iff$ **H** $\equiv 0$ $\iff$ **M** is a critical point for the area functional under compactly supported variations.

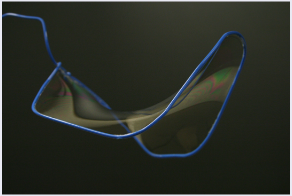


- Catenoid



- Helicoid

Soap films are minimal surfaces.



Soap bubbles are nonzero **H**-surfaces.

Notation and Language

- $\mathrm{Ch}(\mathbf{Y}) = \mathbf{Inf}_{\mathbf{K}\subset\mathbf{Y}\text{ compact}} \dfrac{\mathbf{Area}(\partial\mathbf{K})}{\mathbf{Volume}(\mathbf{K})}$ = Cheeger constant of $\mathbf{Y}$.
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Remark

Proof uses $\mathbf{H}(\mathbf{Y})$-foliations of **Y** to show that if $\mathbf{\Omega}(n) \subset \mathbf{Y}$ is a sequence of isoperimetric domains in **Y** with $\mathbf{Volume}(\mathbf{\Omega}(n)) \to \infty$, then

$$\mathbf{H}_{\partial\mathbf{\Omega}(n)} \geq \mathbf{H}(\mathbf{Y}) \quad \textbf{and} \quad \lim_{n\to\infty} \mathbf{H}_{\partial\mathbf{\Omega}(n)} = \mathbf{H}(\mathbf{Y}).$$

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Simply connected homogeneous **3**-manifolds **X** are either isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$ or to a metric Lie group.

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(2) If **X** is diffeomorphic to $\mathbb{R}^3$, then the values $\mathbf{H} \in \mathbb{R}$ for which there exists a sphere of constant mean curvature **H** in **M** are exactly those with $|\mathbf{H}| > \mathrm{Ch}(\mathbf{X})/2$.

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3. There is a point $p_S \in$ **M** such that every isometry of **M** that fixes p_S also leaves invariant S.

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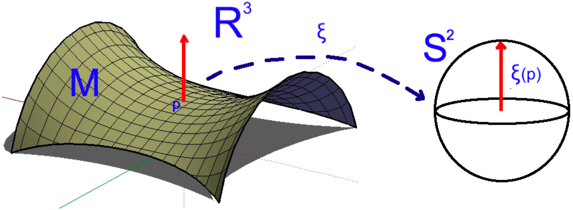

Definition

- Given an oriented immersed surface $f\colon \mathbf{\Sigma} \to \mathbf{X}$ with unit normal vector field $N\colon \mathbf{\Sigma} \to T\mathbf{X}$, the **left invariant Gauss map** of $\mathbf{\Sigma}$ is the map $G\colon \mathbf{\Sigma} \to \mathbb{S}^2 \subset T_e\mathbf{X}$ that assigns to each $p \in \mathbf{\Sigma}$, the unit tangent vector to $\mathbf{X}$ at the identity element e given by left translation:

$$(dl_{f(p)})_e(G(p)) = N_p.$$

Theorem (Representation Theorem, Meeks-Mira-Perez-Ros)

- Suppose $\boldsymbol{\Sigma}$ is a simply connected Riemann surface with conformal parameter z, $\mathbf{X}$ is a simply connected metric Lie group, $\mathbf{H} \in \mathbb{R}$ and $R(q)\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the $\mathbf{H}$-potential.
- Let $g\colon \boldsymbol{\Sigma} \to \overline{\mathbb{C}}$ be a solution of the complex elliptic PDE

$$g_{z\overline{z}} = \frac{R_q}{R}(g)\, g_z g_{\overline{z}} + \left(\frac{R_{\overline{q}}}{R} - \frac{\overline{R_q}}{\overline{R}} \right) (g)\, |g_z|^2, \qquad (1)$$

  such that $g_z \neq 0$ everywhere[a], and such that the $\mathbf{H}$-potential R of $\mathbf{X}$ does not vanish on $g(\boldsymbol{\Sigma})$ (for instance, this happens if $\boldsymbol{\Sigma}$ is closed).
- Then, there exists an immersed $\mathbf{H}$-surface $f\colon \boldsymbol{\Sigma} \looparrowright \mathbf{X}$, unique up to left translations, whose Gauss map is g.
- Conversely, if $g\colon \boldsymbol{\Sigma} \to \overline{\mathbb{C}}$ is the Gauss map of an immersed $\mathbf{H}$-surface $f\colon \boldsymbol{\Sigma} \looparrowright \mathbf{X}$ in a metric Lie group $\mathbf{X}$, and the $\mathbf{H}$-potential R of $\mathbf{X}$ does not vanish on $g(\boldsymbol{\Sigma})$, then g satisfies the equation (1), and moreover $g_z \neq 0$ holds everywhere.

[a] By $g_z \neq 0$ we mean that $g_z(z_0) \neq 0$ if $g(z_0) \in \mathbb{C}$ and that $\lim_{z \to z_0}(g_z/g^2)(z) \neq 0$ if $g(z_0) = \infty$.

Theorem (Classification Theorem for H-spheres, Meeks-Mira-Pérez-Ros)

Suppose **X** is a simply connected **3**-dimensional metric Lie group.

- **X** is diffeomorphic to $\mathbf{R}^3 \implies$ the moduli space of **H**-spheres in **X** is parameterized by the mean curvature values **H** in $(\mathbf{H}(\mathbf{X}), \infty)$.
- **X** is diffeomorphic to $\mathbb{S}^3 \implies$ the moduli space of **H**-spheres in **X** is parameterized by the mean curvature values **H** in $[0, \infty)$.
- **X** diffeomorphic to $\mathbb{S}^3 \implies$ the areas of all **H**-spheres form a half-open interval $(0, \mathbf{A}(\mathbf{X})]$.
- **H**-spheres in **X** are **Alexandrov embedded** with **index 1**, **nullity 3**.

Steps of the proof of the Classification Theorem for **H**-spheres.

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Steps of the proof continued.

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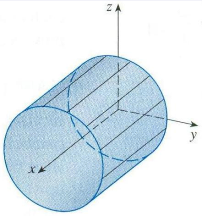
- Jacobi function $\langle \mathbf{K}', N \rangle$ changes sign on $\mathbf{\Sigma}_\infty$, $N =$ unit normal field.
- But on a stable parabolic $\mathbf{H}$-surface, a bounded Jacobi function cannot change sign, a contradiction. $\square$

Question

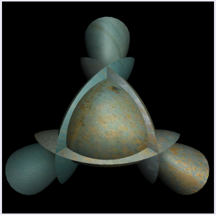
Is the round sphere the only complete simply connected surface **embedded** in $\mathbf{R}^3$ with **non-zero** constant mean curvature?

NOT simply connected



- Cylinder

NOT embedded



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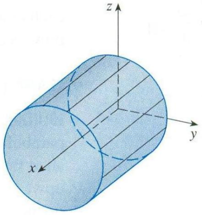

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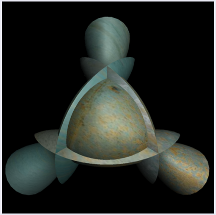

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2007 - Work of **Colding-Minicozzi** and **Meeks-Rosenberg** for $\mathbf{H} = 0$ shows that if $\mathbf{M}$ is a complete, simply connected **0**-surface **embedded** in $\mathbf{R}^3$, then $\mathbf{M}$ is either

a plane or **a helicoid**.

Theorem (Meeks-Tinaglia)

Let $\mathbf{M} \subset \mathbf{R}^3$ be a complete, connected embedded $\mathbf{H}$-surface.

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Item **3** in the above theorem holds for **3**-manifolds which are homogeneously regular; in particular it holds in closed Riemannian **3**-manifolds.

Theorem (Radius Estimates for H-Disks, Meeks-Tinaglia)

$\exists$ $\mathbf{R_0} \geq \pi$ such that every embedded $\mathbf{H}$-disk in $\mathbf{R^3}$ has radius $< \mathbf{R_0}/\mathbf{H}$.

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Theorem (Curvature Estimates for H-Disks, Meeks-Tinaglia)

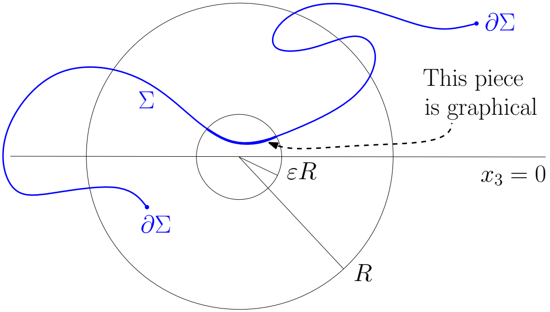
Fix $\varepsilon, \mathbf{H}_0 > 0$ and a complete locally homogenous $\mathbf{3}$-manifold $\mathbf{X}$. $\exists\ \mathbf{C} > 0$ s.t. for all embedded $(\mathbf{H} \geq \mathbf{H_0})$-disks $\mathbf{D}$:

$$|\mathbf{A_D}|(p) \leq \mathbf{C} \quad \text{for all} \quad p \in \mathbf{D} \text{ s.t. } \mathbf{dist_D}(p, \partial\mathbf{D}) \geq \varepsilon.$$

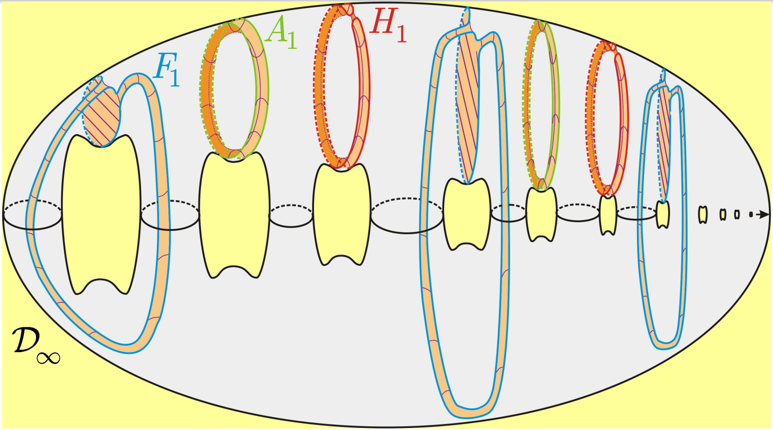
Theorem (One-sided curvature estimate for **H**-disks, Meeks-Tinaglia)

$\exists \mathbf{C}, \varepsilon > 0$ s.t. for any **H**-disk $\mathbf{\Sigma} \subset \mathbf{R}^3$ as in the figure below:

$$|\mathbf{A}_{\mathbf{\Sigma}}| \leq \frac{\mathbf{C}}{R} \text{ in } \mathbf{\Sigma} \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}.$$



This result generalizes the one-sided curvature estimates for minimal disks by Colding-Minicozzi, and uses their work in its proof.



- $\mathcal{D}_\infty$ = the above **bounded domain, smooth except at $\mathbf{p}_\infty$**.
- **Ferrer, Martin** and **Meeks** conjecture: An open surface **properly embeds as a complete minimal surface in $\mathcal{D}_\infty$** $\iff$ every end has **infinite genus** $\iff$ it admits a complete bounded minimal embedding in $\mathbb{R}^3$.

Conjecture (Meeks-Perez-Ros-Tinaglia)

For any complete, connected embedded $\mathbf{H}$-surface $\boldsymbol{\Sigma} \subset \mathbf{R}^{\mathbf{3}}$ of finite genus and compact boundary, there exists a constant $\mathbf{K}_{\boldsymbol{\Sigma}}$ s.t. $\forall R \geq 1$,

$$\mathrm{Area}(\boldsymbol{\Sigma} \cap \mathbb{B}(R)) \leq \mathbf{K}_{\boldsymbol{\Sigma}} \cdot R^3.$$

The embedded Calabi-Yau problem for finite genus

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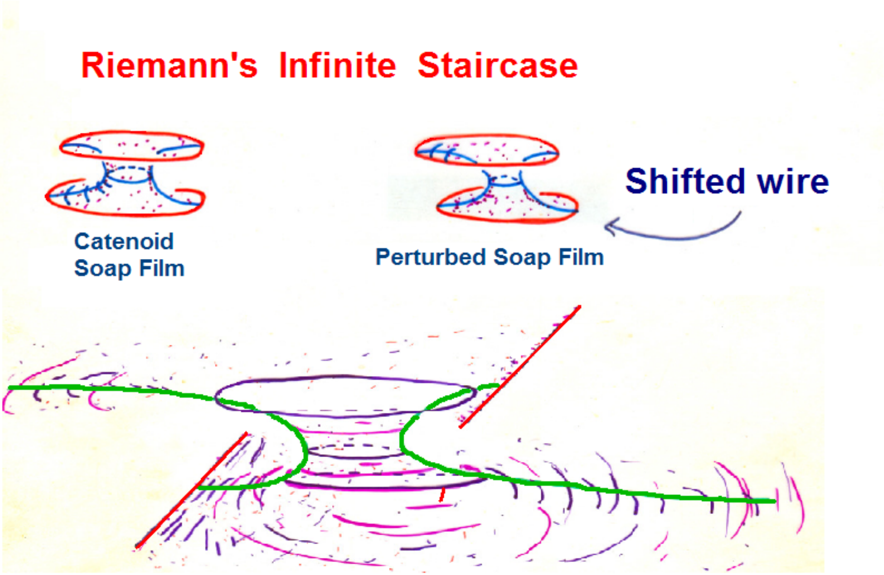
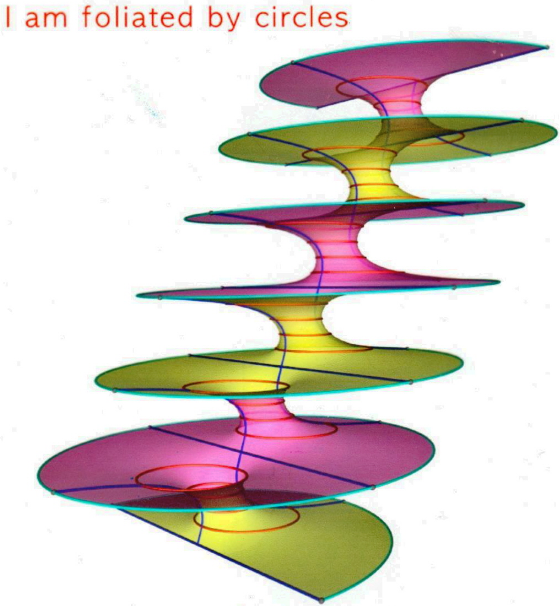
Theorem (Meeks-Perez-Ros)

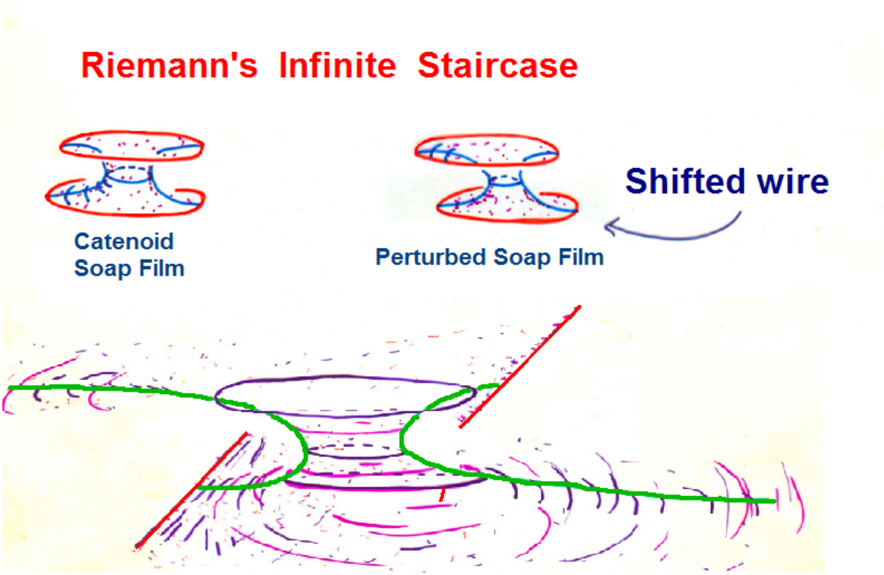
Let $\mathbf{\Sigma} \subset \mathbf{R^3}$ be a complete, connected embedded **0**-surface of finite genus. Then:

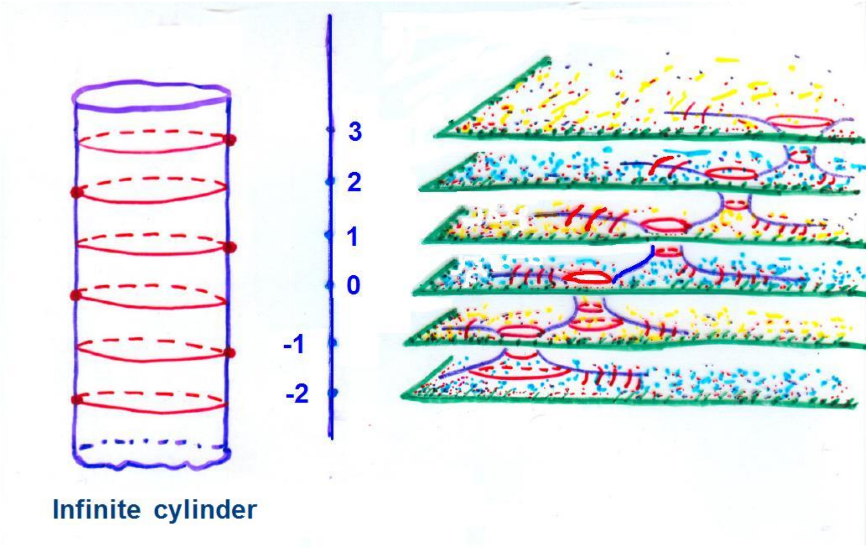
$$\mathbf{\Sigma} \text{ is proper} \iff \mathbf{\Sigma} \text{ has a } \underline{\text{countable}} \; \# \text{ of ends.}$$


Riemann's Infinite Staircase
Catenoid
Soap Film
Perturbed Soap Film
Shifted wire

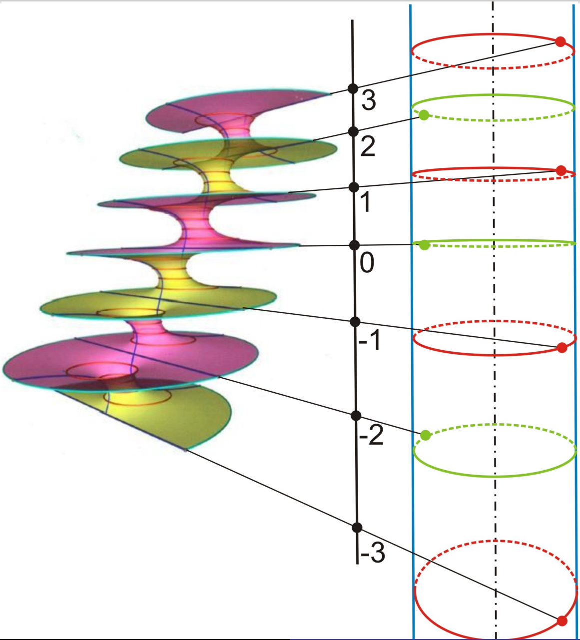

I am foliated by circles

The family $\mathcal{R}_t$ of Riemann minimal examples

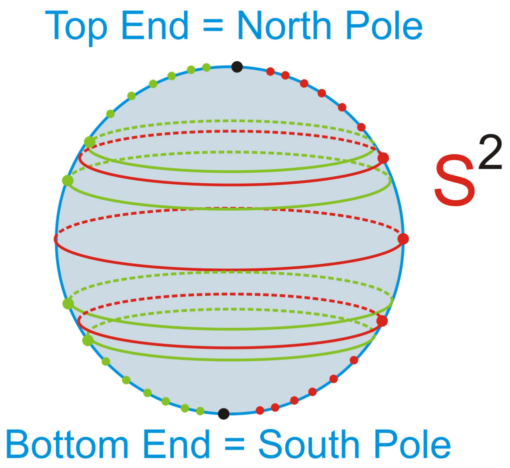



3
2
1
0
-1
-2
Infinite cylinder

Cylindrical parametrization of a Riemann minimal example

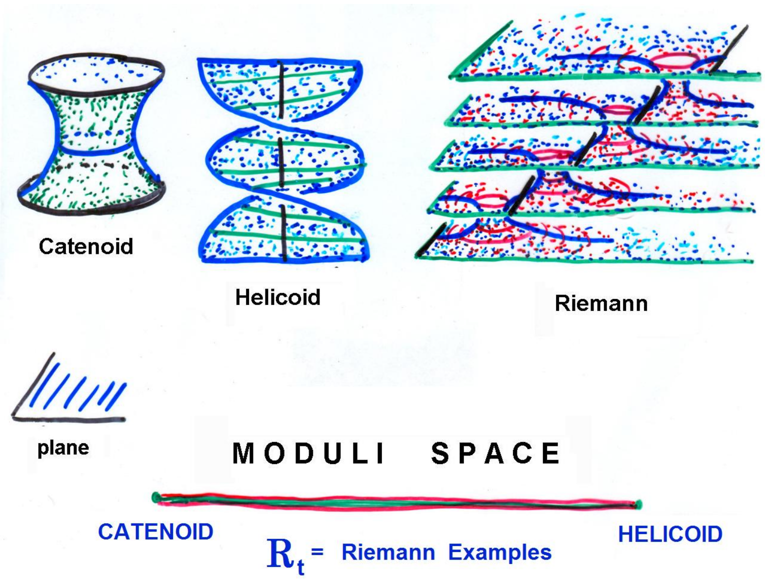


Conformal compactification of a Riemann minimal example



Example

Topologically there is **only one** connected **genus-zero** surface with **two limit ends**. Riemann minimal examples have this property.


Catenoid
Helicoid
Riemann
plane
MODULI SPACE
CATENOID
HELICOID
R_t = Riemann Examples

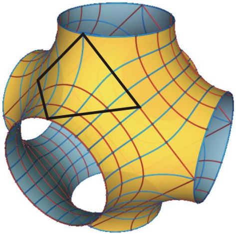
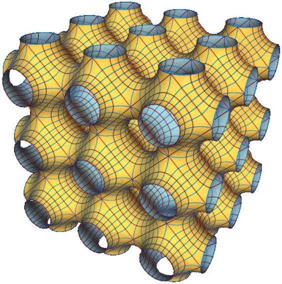
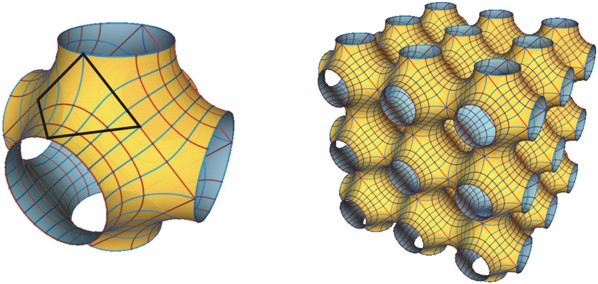
 

Figure: A body-centered cubic interface or Fermi surface in salt crystal.

Next theorem is motivated by the study of **3**-periodic **H**-surfaces that appear as interfaces in material science or as equipotential surfaces in crystals. This result contrasts with the failure of area estimates for compact minimal surfaces of genus $\mathbf{g} > 2$ in any flat **3**-torus (**Traizet**).



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Theorem (Meeks-Tinaglia)

Given a flat **3**-torus $\mathbb{T}^3$ and $\mathbf{H} > 0$, $\exists C_{\mathbf{H}}$ s.t. $\forall \mathbf{g} \in \mathbb{N}$, a closed **H**-surface $\mathbf{\Sigma}$ embedded in $\mathbb{T}^3$ with genus at most **g** satisfies $\mathbf{Area}(\mathbf{\Sigma}) \leq C_{\mathbf{H}}(g+1)$.

Definition

- Suppose $f : \mathbf{\Sigma} \to \mathbf{N}$ is a closed immersed surface positive mean curvature in a Riemannian 3-manifold $\mathbf{N}$.
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Theorem (Meeks-Tinaglia, 2017)

- Let $\mathbf{N}$ be a closed Riemannian 3-manifold.
- Given $\mathbf{H} > 0$ and a non-negative integer g, then the space of strongly Alexandrov embedded closed surfaces in $\mathbf{N}$ of genus at most g and constant mean curvature $\mathbf{H}$ is **compact**.

These studies on the geometry of embedded $\mathbf{H}$-surfaces lead to the following deep results on **CMC** foliations of $\mathbf{3}$-manifolds.

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- Let $\mathcal{F}$ be a *weak* **CMC** foliation of a punctured Riemannian **3**-ball $B(p, r) - \{p\}$.
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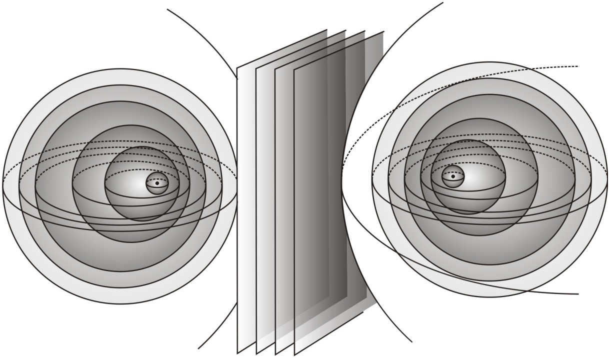
2 key ingredients in the proof.

- Curvature estimates for **CMC** foliations.
- Local removable singularity theorem for weak **H**-laminations. □

CMC foliation of $\mathbf{R}^3$ punctured in two points by spheres and planes



Theorem (Meeks-Perez-Ros)

Suppose $\mathcal{F}$ is a **CMC** foliation of $\mathbf{R}^3 - \mathcal{S}$ where $\mathcal{S}$ is a closed countable set. Then all leaves of $\mathcal{F}$ are contained in planes and round spheres.

Calabi-Yau type problems for embedded **H**-surfaces

Theorem (Meeks-Tinaglia)

For **H** ≥ 1, complete embedded finite topology **H**-surfaces in complete hyperbolic **3**-manifolds are proper.

Theorem (Coskunuzer-Meeks-Tinaglia)

- For every **H** < 1, $\exists$ a complete embedded stable **H**-plane that is **nonproper** in $\mathbb{H}^3$.
- For every **H** $\in (0, 1/2)$, $\exists$ a complete embedded stable **H**-plane that is **nonproper** in $\mathbb{H}^2 \times \mathbb{R}$.

Theorem (Tinaglia-Rodriguez)

$\exists$ a complete embedded stable **0**-plane that is **nonproper** in $\mathbb{H}^2 \times \mathbb{R}$.