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Jacobians with prescribed eignvectors.

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joint work with

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Given: $\Omega \underset{open}{\subset} \mathbb{R}^n$, with a fixed coordinate system $u = (u^1, \ldots, u^n)$, a point $\bar{u} \in \Omega$ and vector fields

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Find: <u>all</u> maps $f = [f^1, \ldots, f^n]^T \colon \Omega' \to \mathbb{R}^n$ from some open nbhd. Ω' of $\bar{u}$, such that $\mathcal{R}$ is a (partial) set of eigenvector-fields of the Jacobian matrix

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$$[D_u f] = \begin{bmatrix} \mathsf{grad}(f^1) \\ \vdots \\ \mathsf{grad}(f^n) \end{bmatrix}$$

I.e. $\exists$ smooth functions $\lambda^i \colon \Omega' \to \mathbb{R}$, s. t. for $i = 1, \ldots, m$ and $\forall u \in \Omega'$

$$[D_u f]\,\mathbf{r}_i(u) = \lambda^i(u)\,\mathbf{r}_i(u). \qquad \mathcal{F}(\mathcal{R})\text{-system}$$

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- $\mathcal{F}(\mathcal{R})$ denotes the set of all fluxes corresponding to a partial frame $\mathcal{R}$.

Motivation for the Jacobian problem

- By solving the Jacobian problem, we can construct and study the set of systems conservations laws $u_t + f(u)_x = 0$ with prescribed rarefaction curves and analyze how the geometry of these curves determines behavior of the solutions of conservation laws.

- It is an interesting geometric problem on its own.

- It leads to interesting overdetermined systems of PDE's.

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Goals:

- to determine how the ”size” of $\mathcal{F}(\mathcal{R})$ (in terms of the number of arbitrary functions and constants) depends on the geometric properties of $\mathcal{R}$.

- to determine whether or not $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes.

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- for all $\mathcal{R}$, the set $\mathcal{F}(\mathcal{R})$ contains (n+1)-dimensional subspace $\mathcal{F}^{\text{triv}}$ of trivial fluxes:

$$f(u) = \bar{\lambda}\begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \qquad \bar{\lambda}, a_1, \ldots, a_n \in \mathbb{R},$$

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- scaling invariance: $\mathcal{F}(\mathbf{r}_1, \ldots, \mathbf{r}_m) = \mathcal{F}(\alpha^1\,\mathbf{r}_1, \ldots, \alpha^m\mathbf{r}_m)$
  for any nowhere zero smooth functions α^i on Ω.

Examples of full frames on $\mathbb{R}^3$ ($m = n = 3$, coordinates (u, v, w))

(1) • $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} w \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} u \\ 0 \\ -w \end{bmatrix}$

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• only trivial fluxes: $\mathcal{F}(\mathcal{R}) = \mathcal{F}^{\text{triv}}$.

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("hyperbolic spiral":

$$u = \bar{u} \cosh t + \bar{v} \sinh t, v = \bar{u} \sinh t + \bar{v} \cosh t, w = \bar{w} + t,$$

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$$f = c \left[v^3,\, u^3,\, \frac{3}{4}(u^2 + v^2) \right]^T + \text{a trivial flux}, \quad c \in \mathbb{R}$$

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• There are strictly hyperbolic fluxes in a neighborhood of $(\bar{u}, \bar{v}, \bar{w}) \in \Omega$.

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- $$f = \left[\phi^1(u),\, \phi^2(v),\, \phi^3(w)\right]^T, \quad \phi^i : \mathbb{R} \to \mathbb{R} \text{ arbitrary}$$

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- $$\lambda^1 = \left(\phi^1\right)'(u), \quad \lambda^2 = \left(\phi^2\right)'(v), \quad \lambda^3 = \left(\phi^3\right)'(w).$$

  All fluxes are hyperbolic, and almost all are strictly hyperbolic.

What if we prescribe an <u>incomplete</u> eigenframe?

(1) $\mathbf{r}_1 = [0, 1, u]^T$, $\mathbf{r}_2 = [w, 0, 1]^T$, $\mathbf{r}_3 = [u, 0, -w]^T$ only trivial fluxes.

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$\mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}}$ is 2-dimensional:

$$f = c_1 \begin{bmatrix} \ln(u) \\ 0 \\ \frac{1}{2}\left(\frac{w}{u} - v\right) \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{3}\, u^3 \\ u\, w \\ w\, u^2 \end{bmatrix} + \text{trivial fluxes}$$

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What about the coordinate frame example?

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- It is not true that $f(\Phi(w))$ belongs to $\mathcal{F}(\Phi^*\mathcal{R})$, where $\Phi^*\mathcal{R} = \{\Phi^*\mathbf{r}_1, \ldots, \Phi^*\mathbf{r}_m\}$, i.e, in general there may not exists functions $\kappa^1(w), \ldots, \kappa^m(w)$, such that

$$[D_w\, f(\Phi(w))] = \kappa^i(u)\, \Phi^*\mathbf{r}_i.$$

Coordinate-free formulation of the Jacobian problem

Given: Given a partial frame

$$\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}, \quad 1 \leq m \leq n$$

on $\Omega \underset{open}{\subset} \mathbb{R}^n$, with a fixed flat, symmetric* connection ∇, and a point $\bar{u} \in \Omega$

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Find: all local smooth vector fields $\mathbf{f}$ ("fluxes"), defined on some nbhd Ω' of $\bar{u}$, for which there exist smooth functions $\lambda^i \colon \Omega' \to \mathbb{R}$, such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \, \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m. \qquad \text{"new" } \mathcal{F}(\mathcal{R})\text{-system}$$

*Coordinate-free formulation makes sense for non-flat connections, but is not considered here.

Observations:

$$\boxed{\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m}.$$

"new" $\mathcal{F}(\mathcal{R})$-system

- Written out in an affine system of coordinates: $(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0, \quad \forall i, j)$ the "new" $\mathcal{F}(\mathcal{R})$-system is the same as the "old" one.

- Integrability conditions for $\mathcal{F}(\mathcal{R})$-system correspond to the flatness conditions

$$\nabla_{\mathbf{r}_i} \nabla_{\mathbf{r}_j} \mathbf{f} - \nabla_{\mathbf{r}_j} \nabla_{\mathbf{r}_i} \mathbf{f} = \nabla_{[\mathbf{r}_i, \mathbf{r}_j]} \mathbf{f}$$

Goals :

- to determine the ”size” of $\mathcal{F}(\mathcal{R})$.
- to determine whether or not $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes.

Methods :

- for the size: C^1 Frobenius and Darboux theorems (and their generalizations), and as the last resort analytic Cartan-Kähler theorem.
- for strict hyperbolicity: a careful examination of integrability conditions.

Involutivity and richness

Definitions: A partial frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ is:

- in involution if $[\mathbf{r}_i, \mathbf{r}_j] \in \mathsf{span}_{C^\infty(\Omega)} \mathcal{R}$ for all $1 \leq i, j \leq m$.

- rich if $[\mathbf{r}_i, \mathbf{r}_j] \in \mathsf{span}_{C^\infty(\Omega)} \{\mathbf{r}_i, \mathbf{r}_j\}$ (pairwise in involution).

Summary of the results:

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  - For rich partial frames: we have necessary and sufficient conditions for $\mathcal{F}(\mathcal{R})$ to contain strictly hyperbolic fluxes and for those we know the size of $\mathcal{F}(\mathcal{R})$ (∞-dim.)

- Low dimensional results:
  - $n = 1$ or $n = 2$ or $m = 1$ fall under rich category.
  - non rich, but in involution:
    - * $n = 3$ non-rich full frame ($m = 3$) completely analyzed in:

      K. Jenssen and I.K. (2010)

      1. necessary and sufficient conditions for $\mathcal{F}(\mathcal{R})$ to contain strictly hyperbolic fluxes
      2. under these conditions: $\dim \mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}} = 1$ (unique flux up to scaling)

- $*$ for $m = 3$, $n > 3$ we have necessary and sufficient conditions for $\mathcal{F}(\mathcal{R})$ to contain strictly hyperbolic fluxes and for those we know the size of $\mathcal{F}(\mathcal{R})$ (∞-dim.)

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– not in involution: for $m = 2$, $n = 3$ we have:

1. (necessary conditions for strict hyperbolicity)
   $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes only if:

   $$\nabla_{\mathbf{r}_1}\mathbf{r}_2 \notin \mathrm{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\} \text{ and } \nabla_{\mathbf{r}_2}\mathbf{r}_1 \notin \mathrm{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\} \qquad (****)$$

2. Under(****), $\mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}}$ contains only strictly hyperbolic and possibly a 1-dimensional subspace of non-hyperbolic fluxes (but no hyperbolic fluxes with repeated eigenfunctions).

3. (size) Under (****) and

   $$\Gamma^3_{22}(\bar{u})\,\Gamma^3_{11}(\bar{u}) - 9\,\Gamma^3_{12}(\bar{u})\,\Gamma^3_{21}(\bar{u}) \neq 0,$$

   $4 \leq \dim(\mathcal{F}(\mathcal{R})) \leq 8$ (we have examples in all dimensions $4, \ldots, 8$).

4. If $\dim \mathcal{F}(\mathcal{R}) > 5$, then $\mathcal{F}(\mathcal{R})$ must contain strictly hyperbolic fluxes.

We don't have a sufficient condition for $\mathcal{F}(\mathcal{R})$ to contain non-trivial fluxes,

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unless 1) $\mathcal{R}$ is rich or 2) $\mathcal{R}$ is in involution with $m = 3$.

Remark: For all $n \geq m$, such that $n > 2$ and $m \geq 2$, almost all frames admit only trivial fluxes!

Jacobian problem for rich partial frames $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$:

Recall:

- rich means that $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)}\{\mathbf{r}_i, \mathbf{r}_j\}$ $1 \leq i, j \leq m$.
- $\mathcal{F}(\mathcal{R})$ consists of $\mathbf{f}$'s, for which $\exists \lambda^i \colon \Omega \to \mathbb{R}$ such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m.$$

Theorem:

1. (necessary and sufficient conditions for strict hyperbolicity)
   If $\mathcal{R}$ is rich then $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes iff

$$\nabla_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty(\Omega)}\{\mathbf{r}_i, \mathbf{r}_j\} \text{ for all } 1 \leq i, j \leq m. \qquad (*)$$

2. (size) Under (*), $\mathcal{F}(\mathcal{R})$ depends on:
   m arbitrary functions of $n - m + 1$
   (the degree of freedom of prescribing λ's)
   and
   n functions of $n - m$ variables
   (the degree of freedom for prescribing $\mathbf{f}$ for the chosen λ's)

Jacobian problem for involutive partial frames $\mathcal{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_m\}$:

Recall:

- involutive means that $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)} \mathcal{R}$ for $1 \le i, j \le m$.
- $\mathcal{F}(\mathcal{R})$ consists of $\mathbf{f}$'s, for which $\exists \lambda^i \colon \Omega \to \mathbb{R}$ such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \qquad \text{for } i = 1, \dots, m.$$

Theorem:

1. (necessary conditions for strict hyperbolicity for arbitrary m)
   If $\mathcal{R}$ is involutive then $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes only if for all $1 \le i \ne j \le m$

$$\nabla_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty(\Omega)} \mathcal{R} \qquad (**)$$
$$\nabla_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty(\Omega)} \{\mathbf{r}_i, \mathbf{r}_j\} \iff [\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)} \{\mathbf{r}_i, \mathbf{r}_j\}$$

2. for $m = 3$ in non-rich case (**) can be completed to necessary and sufficient conditions (***). Under (***), $\mathcal{F}(\mathcal{R})$ depends on $n + 2$ arbitrary functions of $n - 3$ variables.

Jacobian problem for non-involutive partial frames simplest case: $\mathcal{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ in $\mathbb{R}^3$.

Recall:

- non-involutive means that $[\mathbf{r}_1, \mathbf{r}_2] \notin \text{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\}$.

- $\mathcal{F}(\mathcal{R})$ consists of $\mathbf{f}$'s, for which $\exists \lambda^1, \lambda^2 \colon \Omega \to \mathbb{R}$ such that

$$\nabla_{\mathbf{r}_1} \mathbf{f} = \lambda^1 \mathbf{r}_1 \quad \text{and} \quad \nabla_{\mathbf{r}_2} \mathbf{f} = \lambda^2 \mathbf{r}_2.$$

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   $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes only if:

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$\boxed{4 \leq \dim(\mathcal{F}(\mathcal{R})) \leq 8}$ (we have examples in all dimensions $4, \ldots, 8$).

4. If $\dim \mathcal{F}(\mathcal{R}) > 5$, then $\mathcal{F}(\mathcal{R})$ must contain strictly hyperbolic fluxes.

Darboux Integrability Theorem [Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. (1910)]

Consider a system of PDE's on $(\phi^1, \dots \phi^p) : \Omega \to \Theta$:

$$\boxed{\frac{\partial \phi^i}{\partial u^j} = h^i_j(u, \phi(u))}, \quad i = 1, \dots, p;\ j \in \alpha(i), \tag{1}$$

where:

1. $\Omega \underset{open}{\subset} \mathbb{R}^n$ (the space of independent variables u's)
2. $\Theta \underset{open}{\subset} \mathbb{R}^p$ (the space of dependent variables ϕ's)
3. $\alpha(i) \subset \{1, \dots, n\}$ for each $i = 1, \dots, p$.
4. $h^i_j(u^1, \dots, u^n, \phi^1, \dots, \phi^p), \quad i = 1, \dots, p,\ j \in \alpha(i)$ are C^1 functions on $\Omega \times \Theta \to \mathbb{R}$, with certain combinatorial restrictions on which ϕ's each of the h^i_j may depend so that (2) become algebraic.

If integrability conditions

$$\frac{\partial}{\partial u^k}\left(\frac{\partial}{\partial u^j}(\phi^i)\right) - \frac{\partial}{\partial u^j}\left(\frac{\partial}{\partial u^k}(\phi^i)\right) = 0 \text{ for all } j, k \in \alpha(i) \tag{2}$$

are identically satisfied on $\Omega \times \Theta$ after substitution of $h^i_j(u, \phi)$ for $\frac{\partial}{\partial u^j}(\phi^i)$ for all $i = 1, \dots, p,\ j \in \alpha(i)$ as prescribed by system (1)

Then $\exists!$ smooth local solution of (1) around $\bar{u}$, for any C^1 initial data for ϕ^i prescribed along submanifold $\Xi_i = \{u^j = \bar{u}^j,\ j \in \alpha_i\} \subset \mathbb{R}^n$ of dimension $n - |\alpha_i|$.

Example of a Darboux system:

- three unknown functions (dependent variables) ϕ, ψ and ξ.
- two independent variables u and v.
- system:

$$\begin{aligned} \phi_u &= F(u, v, \phi, \psi, \xi) \\ \psi_v &= G(u, v, \phi, \psi, \xi) \\ \xi_u &= f(u, v, \psi, \xi) && \text{(no } \phi\text{)} \\ \xi_v &= g(u, v, \phi, \xi) && \text{(no } \psi\text{)} \end{aligned}$$

- the integrability condition:

$$f_v + f_\psi \, G + f_\xi \, g = g_u + g_\phi \, F + g_\xi \, f.$$

- initial data near $(\bar{u}, \bar{v})$:

$$\begin{aligned} \phi(\bar{u}, v) &= a(v) \\ \psi(u, \bar{v}) &= b(u) \\ \xi(\bar{u}, \bar{v}) &= c && \text{a constant.} \end{aligned}$$

- here F, G f, g, a and b are given C^1 functions of their arguments.

Frobenius Theorem:

PDE version: Is a special case of Darboux Theorem, when each unknown function is differentiated with respect to all variables.

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Alternatively, given a full frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$, we can prescribe derivatives with respect to each of the frame directions. The integrability conditions then become:

$$\mathbf{r}_k\left(\mathbf{r}_j(\phi^i)\right) - \mathbf{r}_j\left(\mathbf{r}_k(\phi^i)\right) = \sum_{l=1}^{n} c^l_{kj}\mathbf{r}_k(\phi^i). \qquad (3)$$

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If (3) are satisfied then there is a unique solution with any initial data

$$\phi^i(\bar{u}) = c_i.$$

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There are equivalent diff. form version and vector field formulation versions about foliating $\mathbb{R}^{n+p}$ by n-dimensional integrable manifolds.

Generalized Frobenius, PDE version [M. Benfield (2016)]:
Consider a system of PDE's on $(\phi^1, \dots \phi^p) : \Omega \to \Theta$:

$$\boxed{\mathbf{r}_j(\phi^i(u)) = h^i_j(u, \phi(u))}, \quad i = 1, \dots, p; \ j = 1, \dots, m, \tag{4}$$

where:

1. $\mathcal{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_\mathrm{m}\}$ – a partial frame in involution on $\Omega \underset{open}{\subset} \mathbb{R}^\mathrm{n}$.
2. $\Theta \underset{open}{\subset} \mathbb{R}^p$ is the space of dependent variables ϕ's.
3. $h^i_j(u, \phi), \quad i = 1, \dots, p, \ j = 1, \dots, m$ smooth functions on $\Omega \times \Theta \to \mathbb{R}$.

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$$\mathbf{r}_k\left(\mathbf{r}_j(\phi^i)\right) - \mathbf{r}_j\left(\mathbf{r}_k(\phi^i)\right) = \sum_{l=1}^{m} c^l_{jk}\mathbf{r}_l(\phi) \quad i = 1, \dots, p; \ j, k = 1, \dots, m \tag{5}$$

are identically satisfied on $\Omega \times \Theta$ after substitution of $h^i_j(u, \phi)$ for $\mathbf{r}_j(\phi^i)$ for all $i = 1, \dots, p, \ j = 1, \dots, m$ as prescribed by system (4).

Then $\exists!$ smooth local solution of (4), for any smooth initial data prescribed along any embedded submanifold $\Xi \subset \Omega$ of dimension $n - m$ transversal to $\mathcal{R}$.

Generalized Frobenius vector field version (local)
[Benfield, I. K., Jenssen (2016)]:

Given:

1. $\mathrm{s}_1, \ldots, \mathrm{s}_m$ – a partial frame in involution on an open $\mathcal{O} \subset \mathbb{R}^{n+p}$, where $1 \leq m \leq n$ and $p \geq 1$.

2. $\Lambda \subset \mathcal{O}$ be an $(n-m)$-dimensional embedded submanifold, such that

$$\text{span}_{\mathbb{R}}\{\mathrm{s}_1|_z, \ldots, \mathrm{s}_m|_z\} \oplus T_z\Lambda \cong \mathbb{R}^n \qquad \forall z \in \Lambda.$$

Then for $\forall \bar{z} \in \Lambda$, there exists a unique local extension of Λ to an n-dimensional submanifold $\Gamma_{\bar{z}}$ of $\mathbb{R}^{n+p}$, tangent to $\mathrm{s}_1, \ldots, \mathrm{s}_m$

In the classical local Frobenius theorem, $m = n$ and $\Lambda = \{\bar{z}\}$.

Motivation: systems of conservation laws

$$u_t + f(u)_x = 0\,. \tag{1a}$$

- n equations on n unknown functions $u(x,t) \in \Omega \subset \mathbb{R}^n$.
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Equivalently:

$$u_t + [D_u f]\, u_x = 0 \qquad (1b)$$

Example: The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} V_t - U_x &= 0 \\ U_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

$V = \frac{1}{\rho}$ is volume per unit mass, U is velocity, S is entropy per unit mass, $\boxed{p(V,S) > 0}$ is pressure as a given function, s.t $\boxed{p_V < 0}$.

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- $u_t + f(u)_x = 0$, where $u = [V, U, S]^T$ and $f(u) = [-U, p(V,S), 0]^T$.

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- eigenvectors of $[D_u f]$ are:
$\mathbf{r}_1 = [\,1,\ \sqrt{-p_V},\ 0\,]^T,\ \mathbf{r}_2 = [\,-p_S,\ 0,\ p_V\,]^T,\ \mathbf{r}_3 = [\,1,\ -\sqrt{-p_V},\ 0\,]^T$

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- eigenvalues of $[D_u f]$ are $\lambda^1 = -\sqrt{-p_V}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_V}$.

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- shock curve – a solution of Rankine-Hugoniot conditions:

$$\{\, u \in \Omega \mid \exists\, s \in \mathbb{R} :\ f(u) - f(\bar{u}) = s \cdot (u - \bar{u})\,\}.$$

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  A shock curve describes the discontinuous part of the solutions.

Through each strictly hyperbolic state $\bar{u} \in \Omega$, there exists n wave curves.

Wave curves are building blocks for the solutions of Cauchy problems:

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Lax (1957) under certain condition on f and when u_- and u_+ are close, the solution to the Riemann problem:

$$u_0(x) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0\,. \end{cases}$$

is determined by the wave curves.

Glimm (1965) for u_0 with small total variation, the solutions to the Cauchy problems is determined by solutions of Riemann problems.

Example: The Jacobian problem for the Euler frame.

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Given:

- (V, U, S) are coordinate functions in $\mathbb{R}^3$.
- $\boxed{p(V, S) > 0}$, s.t $\boxed{\text{-}p_V < 0}$
- vector fields $\mathbf{r}_1 = \begin{bmatrix} 1 \\ \sqrt{-p_V} \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -p_S \\ 0 \\ p_V \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 1 \\ -\sqrt{-p_V} \\ 0 \end{bmatrix}$

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Find: the set $\mathcal{F}(\mathcal{R})$ of <u>all</u> maps $f : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\mathcal{R} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a set of eigenvector-fields of the Jacobian matrix $[D_u f]$.

Answer:

Answer:

- If $\left(\frac{p_S}{p_V}\right)_V \neq 0$

$$f = c \begin{bmatrix} -U \\ p(v,S) \\ 0 \end{bmatrix} + \bar{\lambda} \begin{bmatrix} V \\ U \\ S \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c \begin{bmatrix} -U \\ p(v,S) \\ 0 \end{bmatrix} + \text{trivial flux.}$$

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eigenvalues: $\lambda^1 = -c\sqrt{-p_V} + \bar{\lambda}\,, \qquad \lambda^2 \equiv \bar{\lambda}\,, \qquad \lambda^3 = c\sqrt{-p_V} + \bar{\lambda}.$

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eigenvalues: $\lambda^1 = -c\sqrt{-p_V} + \bar{\lambda}\,, \qquad \lambda^2 \equiv \bar{\lambda}\,, \qquad \lambda^3 = c\sqrt{-p_V} + \bar{\lambda}.$

- If $\left(\frac{p_S}{p_V}\right)_V \equiv 0$, then $\mathcal{F}(\mathcal{R})$ depends on 3 arbitrary functions of one variable.

How geometry of the eigenframe of $[D_u f]$ affects the properties of hyperbolic conservative systems and their solutions?

- We analyzed relationship between the geometry of the eigenframe and the number of companion conservation laws a system possesses.

  Jenssen, H. K., Kogan, I. A., Extensions for systems of conservation laws *Communications in PDE's*, No. 37, (2012) , pp. 1096 – 1140.

- We would like to better understand relationship between the geometry of the eigenframe and wave interaction patterns, as well as blow-up of the solutions in finite time phenomena.

1. Jenssen, H. K., Kogan, I. A., Conservation laws with prescribed eigencurves. *J. of Hyperbolic Differential Equations (JHDE)* Vol. 7, No. 2., (2010) pp. 211– 254.

2. Jenssen, H. K., Kogan, I. A., Extensions for systems of conservation laws *Communications in PDE's*, No. 37, (2012) , pp. 1096 – 1140.

3. Benfield, M., Some Geometric Aspects of Hyperbolic Conservation Laws Ph.D. thesis, NCSU, (2016)

4. Benfield, M., Jenssen, H. K., Kogan, I. A., Jacobians with prescribed eignvectors, in preparation.

Thank you!

Additional slides

Rich partial frames: $[\mathbf{r}_i, \mathbf{r}_j] \in \mathrm{span}_{C^\infty(\Omega)}\{\mathbf{r}_i, \mathbf{r}_j\}$

Properties

- $\exists$ smooth functions $\alpha^i \colon \Omega \to \mathbb{R}$, $i = 1, \ldots, m$ such that

$$\tilde{\mathbf{r}}_1 := \alpha^1(u)\,\mathbf{r}_1, \quad \ldots, \quad \tilde{\mathbf{r}}_m := \alpha^n\,\mathbf{r}_m$$

  commute.

$$\Downarrow$$

- $\exists$ coordinates $w^1, \ldots, w^n = \rho(u)$ (called Riemann invariants)

$$\tilde{r}_i = \frac{\partial}{\partial w^i}, \quad i = 1, \ldots, m.$$

Necessary condition for strict hyperbolicity

For $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$, the exists strictly hyperbolic flux $\mathbf{f} \in \mathcal{F}(\mathcal{R})$ only if for each pair of indices $i \neq j \in \{1, \ldots, m\}$ the following equivalence holds:

$$\nabla_{\mathbf{r}_i}\mathbf{r}_j \in \mathsf{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\} \iff [\mathbf{r}_i, \mathbf{r}_j] \in \mathsf{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\}$$

Differential-Algebraic system (the $\lambda(\mathcal{R})$-system) for full frames *

$n(n-1)$ linear, 1st order PDEs and $\frac{n(n-1)(n-2)}{2}$ algebraic equations:

$$\begin{cases} r_i(\lambda^j) = \Gamma^j_{ji}(\lambda^i - \lambda^j) & i \neq j, \qquad (\lambda(\mathcal{R})\text{-diff}) \\ \Gamma^k_{ji}(\lambda^i - \lambda^k) = \Gamma^k_{ij}(\lambda^j - \lambda^k) & i < j,\ i \neq k,\ j \neq k \quad (\lambda(\mathcal{R})\text{-alg}), \end{cases}$$

---

where $\boxed{\Gamma^k_{ij} := L^k(DR_j)R_i}$ are the Christoffel symbols of the connection $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0$ computed relative to the frame $\mathcal{R}$ i.e. $\nabla_{r_i} r_j = \sum_{k=1}^{n} \Gamma^k_{ij} r_k$.

$$c^i_{km} = \Gamma^i_{km} - \Gamma^i_{mk} \qquad \text{(Symmetry)}$$

$$r_m\left(\Gamma^j_{ki}\right) - r_k\left(\Gamma^j_{mi}\right) = \sum_{s=1}^{n} \left(\Gamma^j_{ks}\Gamma^s_{mi} - \Gamma^j_{ms}\Gamma^s_{ki} - c^s_{km}\Gamma^j_{si}\right) \text{ (Flatness).}$$

---

*In different contexts the λ-system appeared in Sévannec (1994), Tsarëv (1985).

Rich system with non-trivial algebraic constraints

$$\partial_i \lambda^j = \Gamma^j_{ji}(\lambda^i - \lambda^j) \quad \text{for} \quad 1 \leq i \neq j \leq n, \quad \partial_i := \frac{\partial}{\partial w_i}.$$

$$\Gamma^k_{ij}(\lambda^j - \lambda^i) = 0 \qquad \text{for} \quad 1 \leq k \neq i < j \neq k \leq n.$$

- $\exists$ distinct i, j, k s.t. $\Gamma^k_{ij} \neq 0$

- multiplicity conditions on eigenvalues are implied by the algebro-differential system (no strictly hyperbolic conservation laws in this case).

- Darboux theorem $\Rightarrow$ general solution depends on s_0 constants and s_1 functions of one variable, where
  - s_0 is the number of distinct eigenvalues of multiplicity > 1,
  - s_1 is the number of eigenvalues of multiplicity 1.

$\lambda(\mathcal{R})$-system for $n = 3$

I. $\mathsf{rank}(\lambda(\mathcal{R})\text{-alg}) = 0 \Rightarrow \mathcal{R}$ is rich; a general solution of $\lambda(\mathfrak{R})$ depends on 3 functions of 1 variable; $\exists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$.

II. $\mathsf{rank}(\lambda(\mathcal{R})\text{-alg}) = 1$ (a single algebraic constraint):

IIa. All three λ^i appear in the algebraic constraint $\Rightarrow \lambda(\mathcal{R})$ can be analyzed by Fronebious theorem; the solution of the λ-system is either trivial or depends on 2 arbitrary constants; In the latter case, $\exists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$; $\nexists$ rich systems in class IIa.

IIb. Exactly two λ^i appear in the algebraic constraint $\Rightarrow$ two λ^i coincide; $\lambda(\mathcal{R})$ can be analyzed by Cartan-Kähler theorem; the general solution is either trivial or depends on 1 arbitrary function of 1 variables and 1 constant; $\nexists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$; but $\exists$ rich systems, in class IIb.

III. $\mathsf{rank}(\lambda(\mathcal{R})\text{-alg}) = 2 \Rightarrow$ only trivial solutions $\lambda^1(u) = \lambda^2(u) = \lambda^3(u) = \bar{\lambda} \in \mathbb{R}$.

The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

  $v = \frac{1}{\rho}$ is volume per unit mass, u is velocity, S is entropy per unit mass, $\boxed{p(v,S) > 0}$ is pressure as a given function of v and S, s.t $\boxed{p_v < 0}$.

- $U_t + f(U)_x = 0$, where $U = (v,u,S)$ and $f(U) = (-u, p(v,S), 0)^T$.

- eigenvalues of Df are $\lambda^1 = -\sqrt{-p_v}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_v}$.

- eigenvectors of D_f are $R_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T$, $R_2 = [\,-p_S,\ 0,\ p_v\,]^T$, $R_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$

Inverse problem: Coordinates $U = (v, u, S)$

- For a given pressure function $p = p(v, S) > 0$, with $p_v < 0$ define a frame $\mathcal{R}$:

  $$R_1 = [\,1,\, \sqrt{-p_v},\, 0\,]^T,\; R_2 = [\,-p_S,\, 0,\, p_v\,]^T,\; R_3 = [\,1,\, -\sqrt{-p_v},\, 0\,]^T$$

- determine the class of conservative systems with eigenfields $\mathcal{R}$ by solving the λ-system for λ^1, λ^2, λ^3.

- Observation: frame is rich $\Leftrightarrow$ $\left(\frac{p_S}{p_v}\right)_v \equiv 0 \Leftrightarrow p(v, S) = \Pi(v + F(S))$.

λ-system:

- differential equations

$$
\begin{aligned}
r_1(\lambda^2) &= 0 \\
r_1(\lambda^3) &= \frac{p_{vv}}{4p_v}(\lambda^3 - \lambda^1) \\
r_2(\lambda^1) &= \frac{p_v}{2}\left(\frac{p_S}{p_v}\right)_v(\lambda^1 - \lambda^2) \\
r_2(\lambda^3) &= \frac{p_v}{2}\left(\frac{p_S}{p_v}\right)_v(\lambda^3 - \lambda^2) \\
r_3(\lambda^1) &= \frac{p_{vv}}{4p_v}(\lambda^1 - \lambda^3) \\
r_3(\lambda^2) &= 0\,.
\end{aligned}
$$

- one independent algebraic equation:

$$\frac{p_v}{4}\left(\frac{p_S}{p_v}\right)_v(\lambda^1 + \lambda^3 - 2\lambda^2) = 0.$$

- Rich frame $\Leftrightarrow$ $\left(\frac{p_S}{p_v}\right)_v \equiv 0$ $\Leftrightarrow$ no algebraic constraints.

Solution of the $\lambda(\mathcal{R})$-system:

in the non-rich case:

- $\lambda(\mathcal{R})$-alg consists of:

$$\frac{p_v}{4}\left(\frac{p_S}{p_v}\right)_v(\lambda^1+\lambda^3-2\lambda^2)=0 \Leftrightarrow \lambda^2=\frac{1}{2}(\lambda^1+\lambda^3)$$

  that involves all three λ's (case IIa) $\Rightarrow$ the general solution depends on two constants.

- from the differential part of λ-system we obtain:

$$\lambda^1=-C_1\sqrt{-p_v}+C_2\,,\quad \lambda^2=C_2\,,\quad \lambda^3=C_1\sqrt{-p_v}+C_2\,.$$

$$f(U)=C_1\begin{pmatrix}-u\\ p(v,S)\\ 0\end{pmatrix}+C_2\begin{pmatrix}v\\ u\\ S\end{pmatrix}+\bar{v}.$$

Solution of the $\lambda(\mathcal{R})$-system:

in the rich case:, i.e. $\left(\frac{p_S}{p_v}\right)_v \equiv 0$

- this is rich case with no algebraic constraints $\Rightarrow$ solution depends on 3 arbitrary functions in one variable.

- $\left(\frac{p_S}{p_v}\right)_v \equiv 0 \quad \Leftrightarrow \quad p(v,S) = \Pi(\xi)$, where $\xi = v + F(S)$.

- from the differential part of λ-system we obtain:

$$\lambda^2 = \lambda^2(S), \quad \lambda^1 = A(\xi, u), \quad \lambda^3 = B(\xi, u),$$

  where

$$A_\xi - \sqrt{-\Pi'(\xi)} A_u = a\,(B - A)\,,\; B_\xi + \sqrt{-\Pi'(\xi)} B_u = a\,(A - B)$$

  and $a = -\frac{p_{vv}}{4p_v}$.

Example: rich orthogonal frame (cylindrical coordinates)

$$R_1 = [u^1,\, u^2,\, 0]^T, \quad R_2 = [-u^2,\, u^1,\, 0]^T, \quad R_3 = [0,\, 0,\, 1]^T.$$

Riemann coordinates: (in the first octant):

$$w^1 = \tfrac{1}{2}\ln\left[(u^1)^2 + (u^2)^2\right], \quad w^2 = \arctan\left(\frac{u^2}{u^1}\right), \quad w^3 = u^3.$$

$$\begin{aligned}
\lambda^1 &= \psi_1(w^1),\\
\lambda^2 &= e^{-w^1}\int_*^{e^{w^1}} \psi_1(\ln(\tau^2))\,d\tau + e^{-w^1}\psi_2(w^2),\\
\lambda^3 &= \psi_3(w^3).
\end{aligned}$$

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\lambda^3 &= \psi_3(w^3).
\end{aligned}$$

Necessary condition for strict hyperbolicity

For $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$, the exists strictly hyperbolic flux $\mathbf{f} \in \mathcal{F}(\mathcal{R})$ only if for each pair of indices $i \neq j \in \{1, \ldots, m\}$ the following equivalence holds:

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Coordinate-free definition of the Jacobian map:

Definition: The Jacobian of a vector field $\mathbf{f}$ on open $\Omega \subset \mathbb{R}^n$, relative to a flat, symmetric connection on Ω connection ∇ is a map

$$J\mathbf{f} : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega) \text{ defined by} J\mathbf{f}(\mathbf{r}) = \nabla_{\mathbf{r}}\mathbf{f}$$

If $\mathbf{f} = F^1 \frac{\partial}{\partial u^1} + \cdots + F^n \frac{\partial}{\partial u^n}$ and $\mathbf{r} = R^1 \frac{\partial}{\partial u^1} + \cdots + R^n \frac{\partial}{\partial u^n}$, where $u^1, \ldots, u^n$ are affine coordinates $\left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0 \right)$ then

$$J\mathbf{f}(\mathbf{r}) = [D_u F] R,$$

where $F = [F^1, \ldots, F^n]^T$ and $R = [R_1, \ldots, R^n]^T$.

Definition: $\mathbf{f}$ is called hyperbolic on Ω if eigenvector-fields of $J\mathbf{f}$ form a frame on Ω. (This implies that all eignefunctions of $J\mathbf{f}$ are real)

$\mathbf{f}$ is called strictly hyperbolic if, in addition, at every point of Ω all n eignefunctions of $J\mathbf{f}$ have distinct values.

Jacobian problem:

Given a partial frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ on open $\Omega \subset \mathbb{R}^n$ ($n \geq m$), and a fixed point $\bar{u} \in \Omega$, describe the set of smooth vector fields

$$\mathcal{F}(\mathcal{R}) = \{\mathbf{f} \in \mathcal{X}(\Omega') \,|\, \bar{u} \in \Omega' \subset \Omega\}$$

s. t. there $\exists$ smooth functions $\lambda^i \colon \Omega' \to \mathbb{R}$ for which

$$J\mathbf{f}(\mathbf{r}_i) := \nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \, \mathbf{r}_i, \qquad \text{for } i = 1, \ldots, m,$$

where ∇ is a flat, symmetric connection on Ω.

Elements of $\mathcal{F}(\mathcal{R})$ will be called fluxes.

- $\mathcal{F}(\mathcal{R})$ is, possibly ∞-dimensional, $\mathbb{R}$-vector space.

- scaling invariance: if $\tilde{\mathcal{R}} = \{\phi^1 \, \mathbf{r}_1, \ldots, \phi^m \, \mathbf{r}_m\}$, where $\phi^i \colon \Omega \to \mathbb{R}$ are nowhere zero, then $\mathcal{F}(\mathcal{R}) = \mathcal{F}(\tilde{\mathcal{R}})$.

- $\forall \mathcal{R}$, the set $\mathcal{F}(\mathcal{R})$ contains a trivial fluxes:

$$(a\, u^1 + b^1)\frac{\partial}{\partial u^1} + \cdots + (a\, u^n + b^n)\frac{\partial}{\partial u^n}, \text{ for all } a, b^1, \ldots, b^n \in \mathbb{R}.$$

Explicit form of integrability conditions (5). For $i = 1, \ldots, p; \; j, k = 1, \ldots, m$

$$\mathbf{r}_j\left(\mathbf{r}_k(\phi^i)\right) - \mathbf{r}_k\left(\mathbf{r}_j(\phi^i)\right) = \sum_{l=1}^{m} c_{jk}^l \mathbf{r}_l(\phi^i),$$

The 1st substitution of the derivatives of ϕ's prescribed by (4) into (5):

$$\mathbf{r}_j\left(h_k^i(u, \phi(u)\right) - \mathbf{r}_k\left(h_j^i(u, \phi(u)\right) = \sum_{l=1}^{m} c_{jk}^l \, h_l^i\,(u, \phi(u))$$

The chain rule and the 2nd substitution for the derivatives of ϕ's :

$$\sum_{l=1}^{n}\left(\frac{\partial h_k^i\,(u,\phi)}{\partial u^l}\mathbf{r}_j(u^l) - \frac{\partial h_j^i\,(u,\phi)}{\partial u^l}\mathbf{r}_k(u^l)\right) + \sum_{s=1}^{p}\left(\frac{\partial h_k^i\,(u,\phi)}{\partial \phi^s}\,h_j^s\,(u,\phi) - \right.$$

$$= \sum_{l=1}^{m} c_{jk}^l(u)\, h_l^i\,(u,\phi)\,.$$

Extensions and entropies:

Assume that $\exists$ functions $q\colon \Omega \to \mathbb{R}$ and $\eta\colon \Omega \to \mathbb{R}$, s.t.
$\boxed{\operatorname{grad} q = \operatorname{grad} \eta (D_u f)}$, then multiplication of $\boxed{u_t + f(u)_x = 0}$ by $\boxed{\operatorname{grad} \eta}$ from the left (assuming that u is smooth) leads to a companion conservation law:

$$\eta(u)_t + q(u)_x = 0$$

η is called an extension of conservative system.

Proposition: η is an extension iff:

for each pair $1 \leq i \neq j \leq n$: $\boxed{\lambda^j = \lambda^i \quad \text{or} \quad R_i^T \left(D_u^2 \eta\right) R_j = 0}$.

An extension η is called an entropy if $D_u^2 \eta$ is positive semidefinite and is called strict entropy if $D_u^2 \eta$ is positive definite.

Admissibility criterion:

A weak solution of $u_t + f(u)_x = 0$ is admissible if it is a limit of smooth solutions

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \,, \qquad \text{as } \varepsilon \downarrow 0.$$

If η is an entropy with flux q, then:

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \leq \varepsilon \eta(u^\varepsilon)_{xx} \qquad (\varepsilon > 0)$$

A weak solution of $u_t + f(u)_x = 0$ is admissible if it satisfies the entropy inequality

$$\eta(u)_t + q(u)_x \leq 0 \qquad \text{(distributional sense)}$$