The Gopakumar-Vafa conjecture for symplectic manifolds

Eleny Ionel

based on joint work with Tom Parker/Penka Georgieva

TGTC conference, February 2017

Background: GW invariants

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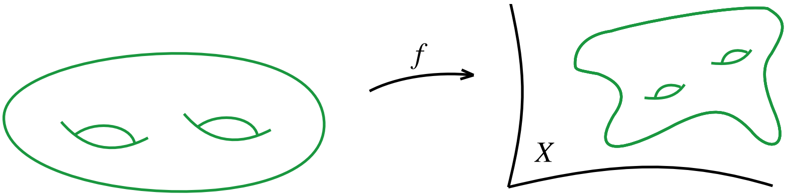

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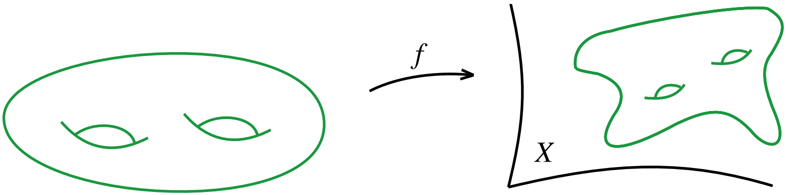

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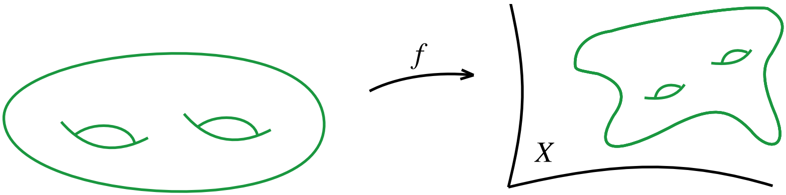

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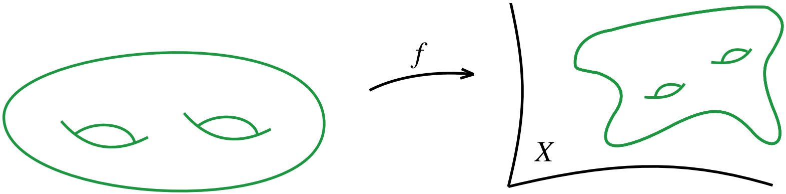

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Work in progress (w/ Penka Georgieva) on a related structure theorem for Real GW invariants.

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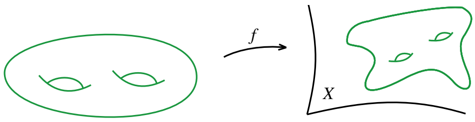

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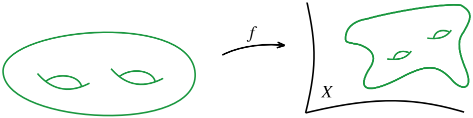

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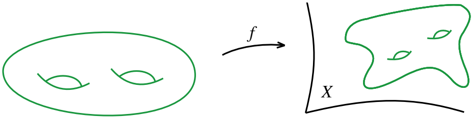

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(Need to fix extra data to orient these moduli spaces.)

$\implies$ Real GW invariants $GW^{\mathbb{R}}(X) \in \mathbb{Q}$.

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Work in progress (w/ P. Georgieva): understand R_g for $g \geq 2$.

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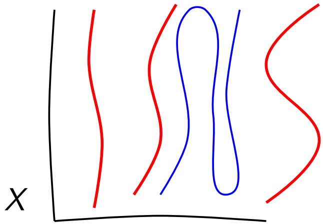
4. conclude that each local contribution is a linear combination with *integer* coef of elementary contributions.

Structure of the moduli space

Thm. For generic J, all J-holomorphic maps are (branched) covers of disjoint embeddings.



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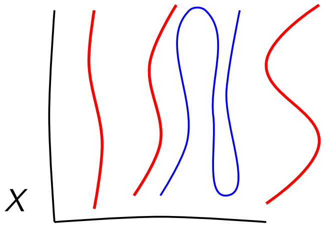

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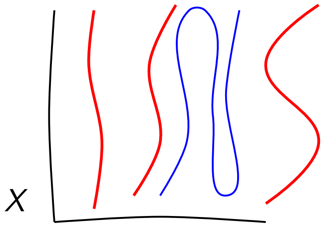

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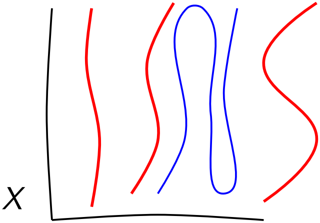

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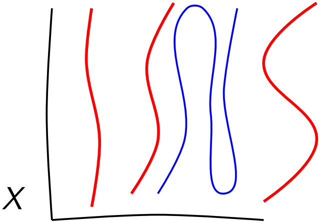

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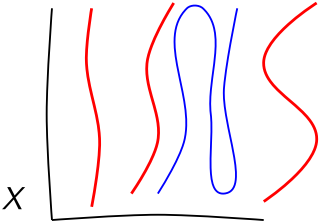

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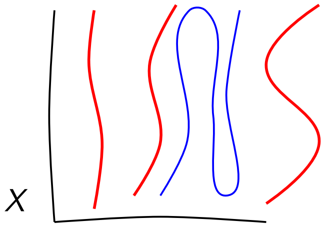

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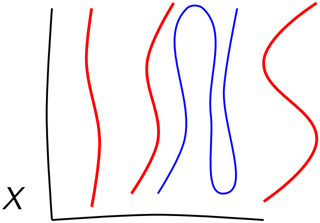

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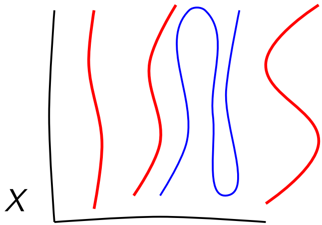

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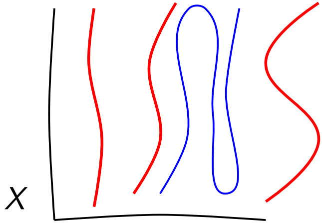

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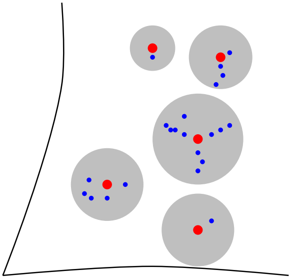

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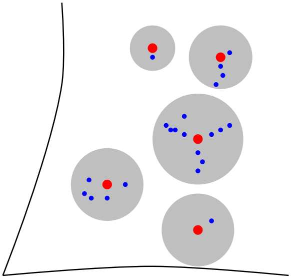

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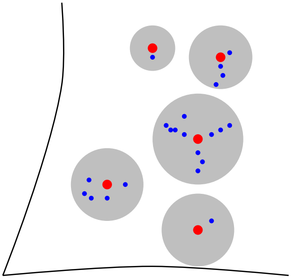

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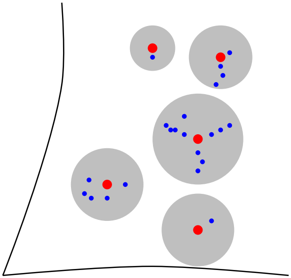

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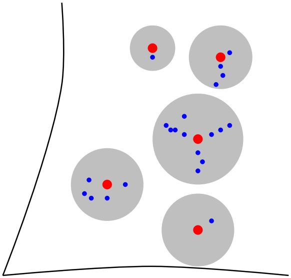

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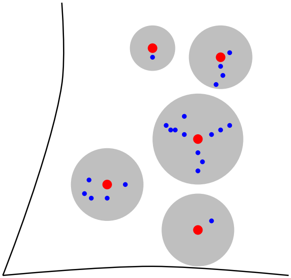

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$\Longrightarrow$ reduce to proving local GV Conj (for every cluster).

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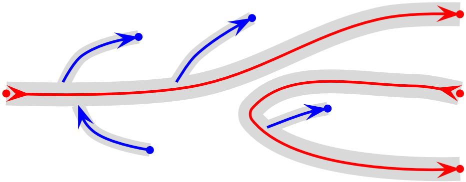

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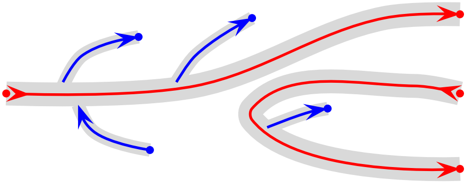

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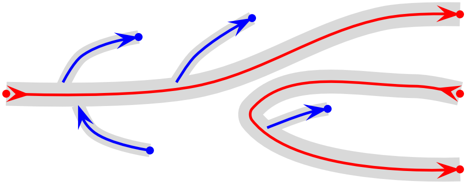

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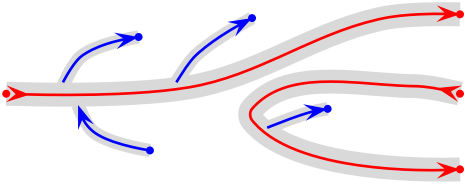

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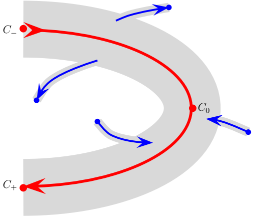

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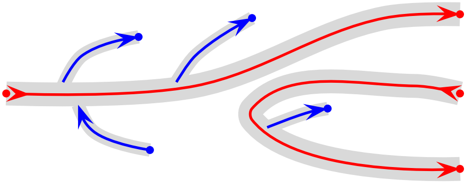

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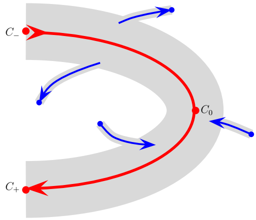

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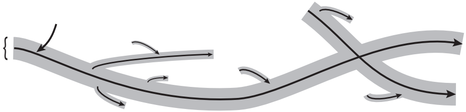

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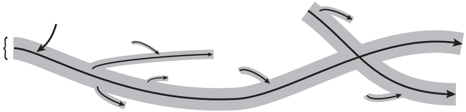

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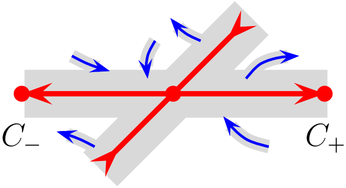

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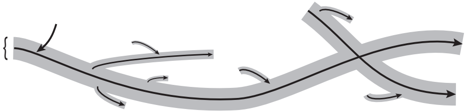

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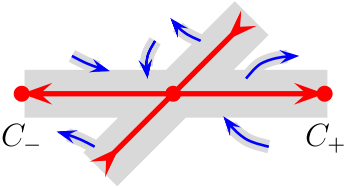

Bifurcation analysis: Kuranishi local model (quadratic) $\Longrightarrow$

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via degenerating the core curve to a nodal one.