Integrable Surface Classes and Primitive Harmonic Maps from Riemann Surfaces to k-Symmetric Spaces via Loop Groups

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Topics

We will discuss the following topics:

- Surfaces, Gauss maps, Associated Families
- the Loop Group Method for Harmonic Maps
- Some Applications to Surface Theory

The 6 types of surfaces mentioned in this talk will be

- CMC surfaces in $\mathbb{R}^3$
- spacelike CMC surfaces in Minkowski space $\mathbb{R}^{2,1}$
- CMC surfaces in $\mathbb{H}^3$, case $0 < H < 1$
- minimal Lagrangian surfaces in $\mathbb{C}P^2$
- minimal surfaces in Nil_3
- Willmore surfaces in S^{n+2}

I will talk exclusively about surfaces which can be described best in conformal coordinates.

Surface classes and harmonic Gauss maps: 1

Consider the following situation for surfaces in $\mathbb{R}^3$:

$$\begin{array}{ccc} \mathbb{R}^3 & & \\ \uparrow{\scriptstyle f} & & \\ \mathbb{D} & \xrightarrow{\mathcal{G}_f} & S^2 = SU(2)/U(1) \end{array}$$

where

$$\mathcal{G}_f = \frac{\partial_x f \times \partial_y f}{||\partial_x f \times \partial_y f||}$$

is the Gauss map of f.

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is the Gauss map of f.

Theorem (Ruh)

f is CMC $\Leftrightarrow \mathcal{G}_f$ is harmonic

Surface classes and harmonic Gauss maps: 2

Consider the following situation for spacelike surfaces in $\mathbb{R}^{2,1}$:

$$\begin{array}{ccl} \mathbb{R}^{1,2} & & \\ \uparrow {\scriptstyle f} & & \\ \mathbb{D} & \xrightarrow{\ \mathcal{G}_f\ } & \mathbb{H}^2 = Sl(2,\mathbb{R})/SO(2) \end{array}$$

where

$$\mathcal{G}_f = \frac{\partial_x f \times \partial_y f}{||\partial_x f \times \partial_y f||}$$

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where

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is the Gauss map of f.

Theorem (T.K.Milnor)

f is spacelike CMC in $\mathbb{R}^{1,2}$ $\Leftrightarrow$ $\mathcal{G}_f$ is harmonic

Surface classes and harmonic Gauss maps: 3

Consider the following situation for surfaces in $\mathbb{H}^3$:

$$\begin{array}{ccc} \mathbb{H}^3 & & \\ \uparrow f & & \\ \mathbb{D} & \xrightarrow{\mathcal{G}_f} & U\mathbb{H}^3 = Sl(2,\mathbb{C})/U(1) \end{array}$$

where

$$\mathcal{G}_f = (f, n_f)$$

is the "normal Gauss map" of f, where n_f is the normal of f in $\mathbb{H}^3$.

Surface classes and harmonic Gauss maps: 3

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where

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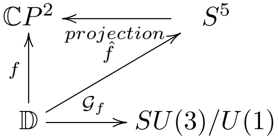
Theorem (T.Ishihara)

f is CMC in $\mathbb{H}^3 \Leftrightarrow \mathcal{G}_f$ is harmonic

Surface classes and harmonic Gauss maps: 4

Consider the following situation for Lagrangian surfaces in $\mathbb{C}P^2$ and their "horizontal lift" $\hat{f}$ into S^5 respectively:



where

$$\mathcal{G}_f \equiv (-ie^{-\frac{u}{2}}\hat{f}_z, -ie^{-\frac{u}{2}}\hat{f}_{\bar{z}}, \hat{f}) \text{ mod } U(1)$$

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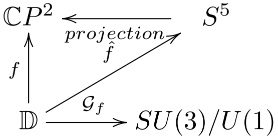

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$$\mathcal{G}_f \equiv (-ie^{-\frac{u}{2}}\hat{f}_z, -ie^{-\frac{u}{2}}\hat{f}_{\bar{z}}, \hat{f}) \text{ mod } U(1)$$

is a "horizontal lift" of f.

Theorem (H.Ma-Y.Ma)

f is a minimal Lagrangian surface in $\mathbb{C}P^2$ $\Leftrightarrow$ $\mathcal{G}_f$ is harmonic

Surface classes and harmonic Gauss maps: 5

Consider the following situation for surfaces in Nil_3:

$$\begin{array}{ccl} \mathrm{Nil}_3 & & \\ \uparrow f & & \\ \mathbb{D} & \xrightarrow{\mathcal{G}_f} & \mathbb{H}^2 = Sl(2,\mathbb{R})/SO(2) \end{array}$$

where

$$\mathcal{G}_f = f^{-1}n$$

is the "normal Gauss map" of f, where n is the normal to f in Nil_3

Note: While in general $\mathcal{G}_f = f^{-1}n$ takes values in the sphere S^2 in the Lie algebra of Nil_3, for non-singular, i.e. non-vertical, minimal surfaces it takes values in $\mathbb{H}^2$.

Surface classes and harmonic Gauss maps: 5

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Theorem (B.Daniel)

f is nowhere vertical and minimal in $\mathrm{Nil}_3 \Leftrightarrow \mathcal{G}_f$ is harmonic

Surface classes and harmonic Gauss maps: 6

Consider the following situation for surfaces in S^{n+2}:

$$\begin{array}{ccl} S^{n+2} & & \\ \uparrow f & & \\ \mathbb{D} & \xrightarrow{\mathcal{G}_f} & Gr_{1,3}(\mathbb{R}^{1,n+3}) = SO^+(1,n+3)/SO^+(1,3) \times SO(n) \end{array}$$

where

$$\mathcal{G}_f = span\langle Y, Y_z, Y_{\bar{z}}, Y_{z\bar{z}}, \rangle$$

is the "conformal Gauss map" of f

and where Y denotes a canonical lift of f into the light cone $S^{n+2} \subset \mathcal{C}^{n+3} \subset R^{1,n+3}$.

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and where Y denotes a canonical lift of f into the light cone $S^{n+2} \subset \mathcal{C}^{n+3} \subset R^{1,n+3}$.

Theorem (Blaschke, Bryant, Eijiri, Rigoli)

f is a Willmore immersion into $S^{n+2} \Leftrightarrow \mathcal{G}_f$ is harmonic

Harmonic maps from Riemann surfaces to k-symmetric spaces: harmonic maps

Let

(M, g) Riemannian manifold   w.l.g. orientable

$(\hat{M}, \hat{g})$ pseudo-Riemannian manifold

$N : M \to \hat{M}$ differentiable map

Then the map N is called harmonic iff for all domains $\mathcal{D} \subset M$ with compact closure and all variations N_t of N with compact support in $\mathcal{D}$ we have

$$\frac{d}{dt} E(N_t, \mathcal{D})|_{t=0} = 0,$$

where

$$E(h, \mathcal{D}) = \int_{\mathcal{D}} ||dh||^2 dvol^M.$$

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Theorem

If $dim M = 2$, then w.l.g. $\mathcal{D}$ an open subset of $\mathbb{C}$ with euclidean metric.

$k-$symmetric spaces

1. G real semi-simple Lie group (finite dimensional)
2. τ automorphism of G of finite order k
3. $Fix_\tau(G)^0 \subset K \subset Fix_\tau(G)$ K closed subgroup

G/K is called a $\mathbf{k}-$symmetric space

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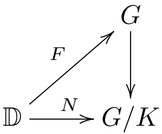
infinitesimally

1. $\mathfrak{g} = Lie(G)$
2. $\mathfrak{k} = Lie(K)$
3. $\mathfrak{g}^{\mathbb{C}} = \sum_{j=0}^{k-1} \mathfrak{g}_j^{\mathbb{C}}$, $\mathfrak{g}_j^{\mathbb{C}}$ an eigenspace of $d\tau \equiv \tau$,
4. $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}}$, $\mathfrak{m}^{\mathbb{C}} = \sum_{j=1}^{k-1} \mathfrak{g}_j^{\mathbb{C}}$,
5. $\mathfrak{g}_0 = \mathfrak{k}$, $\mathfrak{m} = (\sum_{j=1}^{k-1} \mathfrak{g}_j^{\mathbb{C}}) \cap \mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$

Harmonic $N : \mathbb{D} \to G/K$: general case

Consider the following diagram:



Put

$$\alpha = F^{-1}dF = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{m}}$$

and decompose into $(1,0)$-part and $(0,1)$-part

$$\alpha_{\mathfrak{k}} = \alpha'_{\mathfrak{k}} + \alpha''_{\mathfrak{k}} \text{ and } \alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$$

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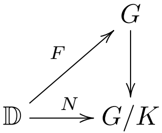

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$$\alpha_{\mathfrak{k}} = \alpha'_{\mathfrak{k}} + \alpha''_{\mathfrak{k}} \quad \text{and} \quad \alpha_{\mathfrak{m}} = \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$$

Theorem

N is harmonic if and only if

1. $d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}] = -\frac{1}{2}[\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]|_{\mathfrak{m}}$
2. $d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]|_{\mathfrak{k}} = 0$

Primitive Harmonic Maps

Theorem

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2. $d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}]|_{\mathfrak{k}} = 0$

Theorem

Assume G/K is symmetric or, more generally, we have $\alpha'_{\mathfrak{m}} \in \mathfrak{g}_{-1}$.

Then $\alpha''_{\mathfrak{m}} \in \mathfrak{g}_{+1}$ and putting $\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$ we obtain:

N is harmonic if and only if $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ for all $\lambda \in S^1$.

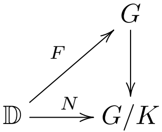
Definition

A harmonic map is called primitive harmonic, iff $\alpha'_{\mathfrak{m}} \in \mathfrak{g}_{-1}$.

Primitive harmonic maps and extended frames

Assume $N : \mathbb{D} \to G/K$ is primitive harmonic.

Recall



Recall $\alpha = F^{-1}dF$ and $\alpha_\lambda = \lambda^{-1}\alpha'_{\mathfrak{m}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{m}}$

Assuming primitive harmonic, i.e. $\alpha'_{\mathfrak{m}} \in \mathfrak{g}_{-1}$.

We have (equivalently) the integrability condition

$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$

Theorem

If $N : \mathbb{D} \to G/K$ is primitive harmonic, then there exists an S^1-family of harmonic maps N_λ with frames F_λ satisfying ${F_\lambda}^{-1}dF_\lambda = \alpha_\lambda$ and $N_\lambda \equiv F_\lambda \mod K$. The "associated family" F_λ is called **extended frame**.

Construction of harmonic maps from extended frames

Theorem

Conversely, consider some family of integrable differential one-forms $\alpha_\lambda = \lambda^{-1}\alpha'_{-1} + \alpha_{\mathfrak{k}} + \lambda\alpha''_1$ *on* $\mathbb{D}$ *and the corresponding frames* F_λ *satisfying* $\alpha_\lambda = {F_\lambda}^{-1} dF_\lambda$.
Then
$N_\lambda \equiv F_\lambda \mod K$ *defines an* S^1*-family of primitive harmonic maps* $N_\lambda : \mathbb{D} \to G/K$.

Construction of harmonic maps from extended frames

Theorem

Conversely, consider some family of integrable differential one-forms $\alpha_\lambda = \lambda^{-1}\alpha'_{-1} + \alpha_{\mathfrak{k}} + \lambda\alpha''_1$ on $\mathbb{D}$ and the corresponding frames F_λ satisfying $\alpha_\lambda = {F_\lambda}^{-1} dF_\lambda$.
Then
$N_\lambda \equiv F_\lambda \mod K$ defines an S^1-family of primitive harmonic maps $N_\lambda : \mathbb{D} \to G/K$.

UPSHOT:

primitive harmonic maps $\mathbb{D} \to G/K$

$\Longleftrightarrow$

integrable differential one forms $\alpha_\lambda = \lambda^{-1}\alpha'_{-1} + \alpha_{\mathfrak{k}} + \lambda\alpha''_1$ on $\mathbb{D}$

with values in $\Lambda\mathfrak{g}_\sigma$. "twisted loops in $\mathfrak{g}$"

Let f be an immersion of one of the six integrable surface classes discussed above.

For each of the surface types under consideration one has a so-called **"Hopf differential"**. This is a differential $r-$form:

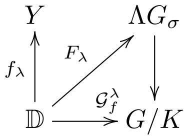
1. CMC surface in $\mathbb{R}^3$: quadratic holomorphic differential
2. spacelike CMC surface in $R^{1,2}$: quadratic holomorphic differential
3. CMC surface in $\mathbb{H}^3$: quadratic holomorphic differential
4. minimal Lagrangian surface in $\mathbb{C}P^2$: cubic holomorphic differential
5. minimal surfaces in Nil_3: quadratic holomorphic differential
6. Willmore surfaces in S^{n+2}: real analytic quadratic differential

For each of the surface types $\rightsquigarrow$ S^1-family of surfaces of the same type **the associated family**. If κ denotes the Hopf differential of f, then the associated family f_λ is essentially determined by the Hopf differential $\kappa_\lambda = \lambda^{-2}\kappa$

The associated family will be denoted by $\mathbf{f}_\lambda$ or $\mathbf{f}(\mathbf{z}, \lambda)$ etc. One has, moreover, $f = f_{\lambda=1}$

Extended frames : Integrable Surface Point of View : 2

Consider the following diagram:



where

- $\mathcal{G}_f^\lambda$ is the "Gauss type map" of f_λ
- ΛG_σ is the **group of loops** in G, i.e. all maps $S^1 \to G$ with some topology and some **twisting**.
- $\Lambda G_\sigma :\to G/K$ is, for fixed λ, the natural projection $g \to g(\lambda) mod K$
- F_λ is the **natural lift** of $\mathcal{G}_f^\lambda$ and is called the **extended frame** of f.

Recall: the homogeneous spaces G/K are $k-$symmetric spaces in general.

Extended Frames: Integrable Surface Point of View: 3 : Sym formulas

We want to complete the surface diagram

where we look for some Sym formula $\mathcal{S}(F_\lambda^{\mathcal{G}})$

This means:

- start from some (λ-dependent) primitive harmonic map $\mathcal{G} = \mathcal{G}_\lambda$ into G/H
- lift to the extended frame $F_\lambda^{\mathcal{G}} : \mathbb{D} \to \Lambda G_\sigma$
- apply the Sym formula
- obtain a surface of the desired type

Sym formulas: a survey

Sym formulas are known in all the six surface types discussed so far.

1. CMC in $\mathbb{R}^3$ — Bobenko

$$f_{\mathcal{G}} = \mathcal{S}(F_\lambda) = i\lambda\partial_\lambda F_\lambda \cdot F_\lambda^{-1} + F_\lambda \frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} F_\lambda^{-1}$$

2. spacelike CMC in $\mathbb{R}^{1,2}$ — Taniguchi

$$f_{\mathcal{G}} = \mathcal{S}(F_\lambda) = i\lambda\partial_\lambda F_\lambda \cdot F_\lambda^{-1} - F_\lambda \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} F_\lambda^{-1}$$

3. CMC in $\mathbb{H}^3$ — Bobenko

$$f_{\mathcal{G}} = \mathcal{S}(F_\lambda^{\mathcal{G}}) = F_\lambda \cdot \begin{pmatrix}e^{-\frac{q}{2}} & 0\\ 0 & e^{\frac{q}{2}}\end{pmatrix} \cdot \bar{F}_\lambda^t$$

4. minimal Lagrangian in $\mathbb{C}P^2$ — Ma

$$f_{\mathcal{G}} = \mathcal{S}(F_\lambda^{\mathcal{G}}) = [F_\lambda e_3]$$

5. **minimal in Nil_3 — Cartier
6. **Willmore surfaces in S^{n+2} — Dorfmeister-Peng Wang

Summing up so far

integrable surface classes	$\Longleftrightarrow$	primitive harmonic maps
	$\Updownarrow$	
associated families integrable surface classes	$\Longleftrightarrow$	associated families of primitive harmonic maps,

$\nwarrow \quad \nearrow$

extended frames

How to construct all extended frames: The loop group method: primitive harmonic $\rightsquigarrow$ potential

η will be called a "holomorphic potential"

$\eta = C^{-1}dC = \lambda^{-1}\eta_{-1} + \lambda^0\eta_0 + \lambda^1\eta_1 + \ldots.. \in \Lambda\mathfrak{g}_\sigma^{\mathbb{C}}$,

$\Uparrow$

$C(z,\lambda)$ holomorphic extended frame

$\Uparrow$

decompose $F_\lambda = CV_+$, with C holomorphic in $z \in \mathbb{D}$ and
V_+ holomorphic in λ for $|\lambda| < 1$

(The matrix V_+ is a global solution to some $\bar{\partial}$-problem on $\mathbb{D}$.)

$\Uparrow$

$F_\lambda : \mathbb{D} \to \Lambda G_\sigma$

Consider the extended frame F_λ of f.

$\Uparrow$

$f : \mathbb{D} \to Y$ Let's start from some surface

$N : \mathbb{D} \to G/K$ Let's start from some primitive harmonic map.

How to construct all extended frames: The loop group method: potential $\rightsquigarrow$ primitive harmonic

Let's start from some "holomorphic potential" ξ

$\xi = \lambda^{-1}\xi_{-1} + \lambda^0\xi_0 + \lambda^1\xi_1 + \ldots.. \in \Lambda\mathfrak{g}_\sigma^{\mathbb{C}}$

$\Downarrow$

$C(z, \lambda)$ " holomorphic extended frame", solution to the ODE $dC = C\xi$.

$\Downarrow$

Decompose $C = F_\lambda \cdot (V_+)^{-1}$. This is a at least locally an "Iwasawa splitting" = infinite dimensional "Gram-Schmidt" procedure

$\Downarrow$

$F_\lambda : \mathbb{D}^\star \rightarrow \Lambda G_\sigma$

Extended frame of the primitive harmonic map
$N_\lambda : \mathbb{D}^* \rightarrow G/K$ given by $N_\lambda \equiv F_\lambda \mod K$.

Combining both directions

Primitive Harmonic $\Rightarrow$ Potential

$\eta = C^{-1}dC =$
$\lambda^{-1}\eta_{-1} + \lambda^0\eta_0 + \lambda^1\eta_1 + ...$

"holomorphic potential"

$\Uparrow$

$C(z,\lambda)$ hol. extended frame

$\Uparrow$

$F_\lambda = CV_+$,
C holomorphic in z
V_+ holomorphic for $|\lambda| < 1$.

$\Uparrow$

$F_\lambda : \mathbb{D} \to \Lambda G_\sigma$ extended frame

$\Uparrow$

Consider $N : \mathbb{D} \to G/K$ primitive harmonic

Potential $\Rightarrow$ Primitive Harmonic

$\xi = \lambda^{-1}\xi_{-1} + \lambda^0\xi_0 + \lambda^1\xi_1 + ...$
"holomorphic potential"

$\Downarrow$

Solve the ODE $dC = C\xi$.

$\Downarrow$

$C = F_\lambda \cdot (V_+)^{-1}$.

Iwasawa splitting $\leftrightarrow$ Gram-Schmidt

$\Downarrow$

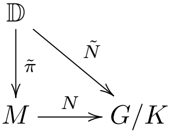
$F_\lambda : \mathbb{D}^* \to \Lambda G_\sigma$ extended frame

$\Downarrow$

$N_\lambda \equiv F_\lambda \mod K$

primitive harmonic

Remarks about general $N : M \to G/K$



Theorem

Let M be a non-compact Riemann surface with universal cover $\mathbb{D}$ and G/K a k-symmetric space.
Let $N : M \to G/K$ be a primitive harmonic map and $\tilde{N} : \mathbb{D} \to G/K$ its lift.
Then $\tilde{N}$ can be derived from some invariant holomorphic potential η on $\mathbb{D}$

$$\gamma^* \eta = \eta \quad \text{for all} \quad \gamma \in \pi_1(M)$$

About the construction of primitive harmonic maps: 1

Step 1: Choose some invariant holomorphic potential η on $\mathbb{D}$,

i.e. we know $\gamma^* \eta = \eta$ for all $\gamma \in \pi_1(M)$.

Step 2: Solve the ODE $dC = C\eta$

Lemma

(1) $\gamma^* C(z, \lambda) = \rho(\gamma, \lambda) C(z, \lambda)$ *for all* $\gamma \in \pi_1(M)$.
(2) $\rho(-, \lambda) : \pi_1(M) \to \Lambda G^{\mathbb{C}}_{\sigma}$ *is a homomorphism of groups*

Step 3: Decompose $C = FV_+$.

Lemma

For all $\gamma \in \pi_1(M)$
$\gamma^* F(z, \bar{z}, \lambda) = \rho(\gamma, \lambda) F(z, \bar{z}, \lambda) \Longleftrightarrow \rho(\gamma, \lambda) \in \Lambda G_{\sigma}$

About the construction of primitive harmonic maps: 2

Theorem

Assume M non-compact and

1. η *holomorphic potential defined on* $\mathbb{D}$
2. $\gamma^*\eta = \eta$ *for all* $\gamma \in \pi_1(M)$.
3. $\gamma^* C(z,\lambda) = \rho(\gamma,\lambda) C(z,\lambda)$ *and* $\rho(\gamma,\lambda) \in \Lambda G_\sigma$ *for all* $\gamma \in \pi_1(M)$.

Decomposing $C = FV_+$ *and putting* $\tilde{N} \equiv F \mod K$ *we obtain the primitive harmonic map* $\tilde{N} : \mathbb{D} \to G/K$ *defined by* η.
It satisfies

$$\gamma^* \tilde{N}(z,\bar{z},\lambda) = \rho(\gamma,\lambda) \tilde{N}(z,\bar{z},\lambda) \text{ for all } \gamma \in \pi_1(M)\ .$$

Moreover,
$\tilde{N}$ *descends (say for* $\lambda = 1$*) to a primitive harmonic map* $N : M \to G/K$
$\Longleftrightarrow$
$\rho(\gamma, \lambda = 1)$ *is in the center of* G.

Some Applications to minimal Lagrangian surfaces in $\mathbb{C}P^2$

- alternative and simplified proof for a result of Costa and Urbano on translationally equivariant minimal Lagrangian immersions , $f(z+t) = g_t f(z)$. (Explicit formulas involving elliptic functions.)

- new examples of generalized equivariant surfaces (metric is a solution to Painleve $PIII(D_7)$)

- construction of all minimal Lagrangian surfaces possessing finite order symmetries around a fixed point

- construction of all minimal immersions with translational symmetry: $f(z+1) = g.f(z)$ for all $z \in \mathbb{D}$

- Any minimal Lagrangian immersion $f : \mathbb{C} \to \mathbb{C}P^2$ which is rotationally equivariant, $f(e^{ict}z) = g_t f(z), z \in \mathbb{C}$, is totally geodesic in $\mathbb{C}P^2$.

- construction of a minimal Lagrangian trinoid with equivariant cylindrical ends

Some Applications to Willmore surfaces in S^{n+2}

- singling out those conformally harmonic maps which occur as Gauss type maps of Willmore immersions
- characterizing those potentials which produce Willmore two-spheres
- constructing explicit, new, Willmore two-spheres

As an example, here is a new, unbranched, non-S-Willmore Willmore two-sphere in S^6.

This is a counterexample to a conjecture of Eijiri, 1988, LMS Proceedings

Explicit example of a Willmore two-sphere in S^6

After projectivization, the family x_λ, $\lambda \in S^1$,

$$x_\lambda = \frac{1}{\left(1 + r^2 + \frac{5r^4}{4} + \frac{4r^6}{9} + \frac{r^8}{36}\right)} \begin{pmatrix} \left(1 - r^2 - \frac{3r^4}{4} + \frac{4r^6}{9} - \frac{r^8}{36}\right) \\ -i\left(z - \bar{z})(1 + \frac{r^6}{9})\right) \\ \left(z + \bar{z})(1 + \frac{r^6}{9})\right) \\ -i\left((\lambda^{-1}z^2 - \lambda\bar{z}^2)(1 - \frac{r^4}{12})\right) \\ \left((\lambda^{-1}z^2 + \lambda\bar{z}^2)(1 - \frac{r^4}{12})\right) \\ -i\frac{r^2}{2}(\lambda^{-1}z - \lambda\bar{z})(1 + \frac{4r^2}{3}) \\ \frac{r^2}{2}(\lambda^{-1}z + \lambda\bar{z})(1 + \frac{4r^2}{3}) \end{pmatrix}$$

with $r = |z|, \ x_1 = x_\lambda|_{\lambda=1}$, yields an

associated family of Willmore two-spheres in S^6 which is

full, non S-Willmore, and totally isotropic.

In particular,

x_λ does not have any branch points.