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Jacobians with prescribed eignvectors.

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Jacobian problem:

Given: $\Omega \underset{open}{\subset} \mathbb{R}^n$, with a fixed coordinate system $u = (u^1, \ldots, u^n)$, a point $\bar{u} \in \Omega$ and vector fields

$$\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}, \quad 1 \leq m \leq n,$$

independent at $\bar{u}$.

Find: all maps $f = [f^1, \ldots, f^n]^T \colon \Omega' \to \mathbb{R}^n$ from some open nbhd. Ω' of $\bar{u}$, such that $\mathcal{R}$ is a (partial) set of eigenvector-fields of the Jacobian matrix

$$[D_u f] = \begin{bmatrix} \mathrm{grad}(f^1) \\ \vdots \\ \mathrm{grad}(f^n) \end{bmatrix}$$

I.e. $\exists$ smooth functions $\lambda^i \colon \Omega' \to \mathbb{R}$, s. t. for $i = 1, \ldots, m$ and $\forall u \in \Omega'$

$$[D_u f]\, \mathbf{r}_i(u) = \lambda^i(u)\, \mathbf{r}_i(u). \qquad \mathcal{F}(\mathcal{R})\text{-system}$$

$\mathcal{F}(\mathcal{R})$-system: $\boxed{[D_u f]\,\mathbf{r}_i(u) = \lambda^i(u)\,\mathbf{r}_i(u)}, \quad i = 1, \ldots, m.$

- $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}, \quad 1 \leq m \leq n$, is called a partial (local) frame.

- f is called a flux.
  - f is called hyperbolic if the Jacobian matrix $[D_u f]$ is diagonalizable over $\mathbb{R}$ at $\bar{u}$, otherwise f is called non-hyperbolic.
  - f is called strictly hyperbolic if the eigenvalues $\lambda^1(u), \ldots, \lambda^n(u)$ of $[D_u f]$ are real and distinct for at $\bar{u}$.

- $\mathcal{F}(\mathcal{R})$ denotes the set of all fluxes corresponding to a partial frame $\mathcal{R}$.

Motivation for the Jacobian problem

- By solving the Jacobian problem, we can construct and study the set of systems conservations laws $\boxed{u_t + f(u)_x = 0}$ with prescribed rarefaction curves and analyze how the geometry of these curves determines behavior of the solutions of conservation laws.

- It is an interesting geometric problem on its own.

- It leads to interesting overdetermined systems of PDE's.

What do we mean by "finding all fluxes"?

- Setting up a system of PDE's for f and then solving it by hand or by computer software?

- What if this fails? Even when we get some solutions, did we find them all?

- Is there any relation between a geometry the partial frame $\mathcal{R}$ and the size of $\mathcal{F}(\mathcal{R})$?

- What types of fluxes $\mathcal{F}(\mathcal{R})$ contains? Hyperbolic? Strictly hyperbolic? Non-hyperbolic?

Goals:

- to determine how the "size" of $\mathcal{F}(\mathcal{R})$ (in terms of the number of arbitrary functions and constants) depends on the geometric properties of $\mathcal{R}$.

- to determine whether or not $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes.

Observations about $\mathcal{F}(\mathcal{R})$-system: $\boxed{[D_u f]\,\mathbf{r}_i = \lambda^i\,\mathbf{r}_i}$.

- $m\,n$ first order PDEs on $m+n$ unknown functions:
  λ^i, $\quad i = 1, \ldots, m$ and n components of f.

- overdetermined for all $n \geq m$, such that $n > 2$ and $m \geq 2$.

- $\mathcal{F}(\mathcal{R})$ is (possibly infinite dimensional) vector space over $\mathbb{R}$.

- for all $\mathcal{R}$, the set $\mathcal{F}(\mathcal{R})$ contains (n+1)-dimensional subspace $\mathcal{F}^{\text{triv}}$ of trivial fluxes:

$$f(u) = \bar{\lambda}\begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \qquad \bar{\lambda}, a_1, \ldots, a_n \in \mathbb{R},$$

  because $[D_u f] = \bar{\lambda} I$.

- scaling invariance: $\mathcal{F}(\mathbf{r}_1, \ldots, \mathbf{r}_m) = \mathcal{F}(\alpha^1\,\mathbf{r}_1, \ldots, \alpha^m \mathbf{r}_m)$
  for any nowhere zero smooth functions α^i on Ω.

Examples of full frames on $\mathbb{R}^3$ ($m = n = 3$, coordinates (u, v, w))

(1) • $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} w \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} u \\ 0 \\ -w \end{bmatrix}$

(integral curves: lines, parabolas, circles)

$$f(u) = \bar{\lambda} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \bar{\lambda}, a_1, a_2, a_3 \in \mathbb{R},$$

• only trivial fluxes: $\mathcal{F}(\mathcal{R}) = \mathcal{F}^{\text{triv}}$.

(2) • $\mathbf{r}_1 = \begin{bmatrix} v \\ u \\ 1 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -v \\ u \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on Ω, where $u\,v \neq 0$.

("hyperbolic spiral":

$$u = \bar{u}\cosh t + \bar{v}\sinh t, v = \bar{u}\sinh t + \bar{v}\cosh t, w = \bar{w} + t,$$

circles, lines)

- $\mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}}$ is a 1-dimensional space

$$f = c\left[v^3,\, u^3,\, \frac{3}{4}(u^2 + v^2)\right]^T + \text{a trivial flux}, \quad c \in \mathbb{R}$$

$$\lambda^1 = 3\,c\,u\,v + \bar{\lambda}, \quad \lambda^2 = -3\,c\,u\,v + \bar{\lambda}, \quad \lambda^3 = \bar{\lambda}.$$

- There are strictly hyperbolic fluxes in a neighborhood of $(\bar{u}, \bar{v}, \bar{w}) \in \Omega$.

(3) (the coordinate frame)

- $$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- $$f = \left[\phi^1(u),\, \phi^2(v),\, \phi^3(w)\right]^T, \quad \phi^i : \mathbb{R} \to \mathbb{R} \text{ arbitrary}$$

  $\mathcal{F}(\mathcal{R})$ is a ∞-dimensional space

- $$\lambda^1 = \left(\phi^1\right)'(u), \quad \lambda^2 = \left(\phi^2\right)'(v), \quad \lambda^3 = \left(\phi^3\right)'(w).$$

  All fluxes are hyperbolic, and almost all are strictly hyperbolic.

What if we prescribe an incomplete eigenframe?

(1) $\mathbf{r}_1 = [0, 1, u]^T$, $\mathbf{r}_2 = [w, 0, 1]^T$, $\mathbf{r}_3 = [u, 0, -w]^T$ only trivial fluxes.

(1a) $\mathbf{r}_1 = [0, 1, u]^T$, $\mathbf{r}_2 = [w, 0, 1]^T$ again only trivial fluxes!

(1b) $\mathbf{r}_1 = [0, 1, u]^T$, $\mathbf{r}_3 = [u, 0, -w]^T$.

$\mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}}$ is 2-dimensional:

$$f = c_1 \begin{bmatrix} \ln(u) \\ 0 \\ \frac{1}{2}\left(\frac{w}{u} - v\right) \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{3}u^3 \\ u\,w \\ w\,u^2 \end{bmatrix} + \text{trivial fluxes}$$

$$\lambda^1 = c_2\,u^2 + \bar{\lambda}, \quad \lambda^3 = c_1 \frac{1}{u} - c_2\,u^2 + \bar{\lambda}$$

(1c) $\mathbf{r}_2 = [w, 0, 1]^T$, $\mathbf{r}_3 = [u, 0, -w]^T$.

$\mathcal{F}(\mathcal{R})$ is ∞-dimensional !

What about the coordinate frame example?

(3) $\mathbf{r}_1 = [1,\, 0,\, 0]^T, \quad \mathbf{r}_2 = [0,\, 1,\, 0]^T, \quad \mathbf{r}_3 = [0,\, 0,\, 1]^T$

$$f = \left[\phi^1(u),\, \phi^2(v),\, \phi^3(w)\right]^T, \quad \phi^i : \mathbb{R} \to \mathbb{R} \text{ arbitrary}$$

$$\lambda^1 = (\phi^1)'(u), \quad \lambda^2 = (\phi^2)'(v), \quad \lambda^3 = (\phi^3)'(w).$$

(3a) $\boxed{\mathbf{r}_1 = [1,\, 0,\, 0]^T, \quad \mathbf{r}_2 = [0,\, 1,\, 0]^T.}$

$$f = \left[\phi^1(u, w),\, \phi^2(v, w),\, \phi^3(w)\right]^T, \quad \phi^1, \phi^2 : \mathbb{R}^2 \to \mathbb{R}; \quad \phi^3 : \mathbb{R} \to \mathbb{R}$$

$$\lambda^1 = \frac{\partial \phi^1}{\partial u}, \quad \lambda^2 = \frac{\partial \phi^2}{\partial v}.$$

(3b) $\boxed{\mathbf{r}_1 = [1,\, 0,\, 0]^T.}$

$$f = \left[\phi^1(u, v, w),\, \phi^2(v, w),\, \phi^3(v, w)\right]^T, \quad \phi^1 : \mathbb{R}^3 \to \mathbb{R}; \quad \phi^2, \phi^3 : \mathbb{R}^2 \to \mathbb{R}$$

$$\lambda^1 = \frac{\partial \phi^1}{\partial u}.$$

Coordinate dependence of the problem formulation.

- Assume $f(u) \in \mathcal{F}(\mathcal{R})$ for $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$, i.e: there exist $\lambda^1(u), \ldots, \lambda^m(u)$, such that

$$[D_u\, f] = \lambda^i(u)\, \mathbf{r}_i.$$

- Let a change of variables be described by a local diffeomorphism

$$u = \Phi(w).$$

- It is not true that $f(\Phi(w))$ belongs to $\mathcal{F}(\Phi^*\mathcal{R})$, where $\Phi^*\mathcal{R} = \{\Phi^*\mathbf{r}_1, \ldots, \Phi^*\mathbf{r}_m\}$, i.e, in general there may not exists functions $\kappa^1(w), \ldots, \kappa^m(w)$, such that

$$[D_w\, f(\Phi(w))] = \kappa^i(u)\, \Phi^*\mathbf{r}_i.$$

Coordinate-free formulation of the Jacobian problem

Given: Given a partial frame

$$\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}, \quad 1 \leq m \leq n$$

on $\Omega \underset{open}{\subset} \mathbb{R}^n$, with a fixed flat, symmetric* connection ∇, and a point $\bar{u} \in \Omega$

Find: all local smooth vector fields $\mathbf{f}$ ("fluxes"), defined on some nbhd Ω' of $\bar{u}$, for which there exist smooth functions $\lambda^i \colon \Omega' \to \mathbb{R}$, such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m. \qquad \text{"new" } \mathcal{F}(\mathcal{R})\text{-system}$$

*Coordinate-free formulation makes sense for non-flat connections, but is not considered here.

Observations:

$$\boxed{\nabla_{\mathbf{r}_i}\mathbf{f} = \lambda^i \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m}.$$

"new" $\mathcal{F}(\mathcal{R})$-system

- Written out in an affine system of coordinates: $(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0, \quad \forall i, j)$ the "new" $\mathcal{F}(\mathcal{R})$-system is the same as the "old" one.

- Integrability conditions for $\mathcal{F}(\mathcal{R})$-system correspond to the flatness conditions

$$\nabla_{\mathbf{r}_i}\nabla_{\mathbf{r}_j}\mathbf{f} - \nabla_{\mathbf{r}_j}\nabla_{\mathbf{r}_i}\mathbf{f} = \nabla_{[\mathbf{r}_i,\mathbf{r}_j]}\mathbf{f}$$

Goals :

- to determine the ”size” of $\mathcal{F}(\mathcal{R})$.

- to determine whether or not $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes.

Methods :

- for the size: C^1 Frobenius and Darboux theorems (and their generalizations), and as the last resort analytic Cartan-Kähler theorem.

- for strict hyperbolicity: a careful examination of integrability conditions.

Involutivity and richness

Definitions: A partial frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ is:

- in involution if $[\mathbf{r}_i, \mathbf{r}_j] \in \mathsf{span}_{C^\infty(\Omega)} \mathcal{R}$ for all $1 \leq i, j \leq m$.
- rich if $[\mathbf{r}_i, \mathbf{r}_j] \in \mathsf{span}_{C^\infty(\Omega)} \{\mathbf{r}_i, \mathbf{r}_j\}$ (pairwise in involution).

Summary of the results:

- Results for all n and all $m \leq n$:
  - Necessary conditions for $\mathcal{F}(\mathcal{R})$ to contain strict. hyp. fluxes.

    $\nabla_{\mathbf{r}_i}\mathbf{r}_j \in \text{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\}$ if and only if $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega')}\{\mathbf{r}_i, \mathbf{r}_j\}$
  - For rich partial frames: we have necessary and sufficient conditions for $\mathcal{F}(\mathcal{R})$ to contain strictly hyperbolic fluxes and for those we know the size of $\mathcal{F}(\mathcal{R})$ (∞-dim.)
- Low dimensional results:
  - $n = 1$ or $n = 2$ or $m = 1$ fall under rich category.
  - non rich, but in involution:
    * $n = 3$ non-rich full frame $(m = 3)$ completely analyzed in:

      K. Jenssen and I.K. (2010)

      1. necessary and sufficient conditions for $\mathcal{F}(\mathcal{R})$ to contain strictly hyperbolic fluxes
      2. under these conditions: $\dim \mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}} = 1$ (unique flux up to scaling)

* for $m = 3$, $n > 3$ we have necessary and sufficient conditions for $\mathcal{F}(\mathcal{R})$ to contain strictly hyperbolic fluxes and for those we know the size of $\mathcal{F}(\mathcal{R})$ (∞-dim.)

– not in involution: for $m = 2$, $n = 3$ we have:

1. (necessary conditions for strict hyperbolicity)
   $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes only if:

   $$\nabla_{\mathbf{r}_1}\mathbf{r}_2 \notin \text{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\} \text{ and } \nabla_{\mathbf{r}_2}\mathbf{r}_1 \notin \text{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\} \qquad (****)$$

2. Under(****), $\mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}}$ contains only strictly hyperbolic and possibly a 1-dimensional subspace of non-hyperbolic fluxes (but no hyperbolic fluxes with repeated eigenfunctions).

3. (size) Under (****) and

   $$\Gamma^3_{22}(\bar{u})\,\Gamma^3_{11}(\bar{u}) - 9\,\Gamma^3_{12}(\bar{u})\,\Gamma^3_{21}(\bar{u}) \neq 0,$$

   $$0 \leq \dim \mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}} \leq 4$$

   (we have examples in all dimensions $0, \dots, 4$).

4. If $\dim \mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}} > 1$, then $\mathcal{F}(\mathcal{R})$ must contain strictly hyperbolic fluxes.

We don't have a sufficient condition for $\mathcal{F}(\mathcal{R})$ to contain non-trivial fluxes,

unless 1) $\mathcal{R}$ is rich or 2) $\mathcal{R}$ is in involution with $m = 3$.

Remark: For all $n \geq m$, such that $n > 2$ and $m \geq 2$, almost all frames admit only trivial fluxes!

Jacobian problem for rich partial frames $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$:

Recall:

- rich means that $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)}\{\mathbf{r}_i, \mathbf{r}_j\}$ $1 \leq i, j \leq m$.
- $\mathcal{F}(\mathcal{R})$ consists of $\mathbf{f}$'s, for which $\exists \lambda^i \colon \Omega \to \mathbb{R}$ such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m.$$

Theorem:

1. (necessary and sufficient conditions for strict hyperbolicity)
   If $\mathcal{R}$ is rich then $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes iff

$$\nabla_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty(\Omega)}\{\mathbf{r}_i, \mathbf{r}_j\} \text{ for all } 1 \leq i, j \leq m. \qquad (*)$$

2. (size) Under (*), $\mathcal{F}(\mathcal{R})$ depends on:
   m arbitrary functions of $n - m + 1$
   (the degree of freedom of prescribing λ's)
   and
   n functions of $n - m$ variables
   (the degree of freedom for prescribing $\mathbf{f}$ for the chosen λ's)

Jacobian problem for involutive partial frames $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$:

Recall:

- involutive means that $[\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)} \mathcal{R}$ for $1 \le i, j \le m$.
- $\mathcal{F}(\mathcal{R})$ consists of $\mathbf{f}$'s, for which $\exists \lambda^i \colon \Omega \to \mathbb{R}$ such that

$$\nabla_{\mathbf{r}_i} \mathbf{f} = \lambda^i \mathbf{r}_i, \quad \text{for } i = 1, \ldots, m.$$

Theorem:

1. (necessary conditions for strict hyperbolicity for arbitrary m)
If $\mathcal{R}$ is involutive then $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes only if for all $1 \le i \ne j \le m$

$$\nabla_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty(\Omega)} \mathcal{R} \qquad (**)$$

$$\nabla_{\mathbf{r}_i} \mathbf{r}_j \in \text{span}_{C^\infty(\Omega)} \{\mathbf{r}_i, \mathbf{r}_j\} \iff [\mathbf{r}_i, \mathbf{r}_j] \in \text{span}_{C^\infty(\Omega)} \{\mathbf{r}_i, \mathbf{r}_j\}$$

2. for $m = 3$ in non-rich case (**) can be completed to necessary and sufficient conditions (***). Under (***), $\mathcal{F}(\mathcal{R})$ depends on $n + 2$ arbitrary functions of $n - 3$ variables.

Jacobian problem for non-involutive partial frames simplest case: $\mathcal{R} = \{\mathbf{r}_1, \mathbf{r}_2\}$ in $\mathbb{R}^3$.

Recall:

- non-involutive means that $[\mathbf{r}_1, \mathbf{r}_2] \notin \text{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\}$.

- $\mathcal{F}(\mathcal{R})$ consists of $\mathbf{f}$'s, for which $\exists \lambda^1, \lambda^2 \colon \Omega \to \mathbb{R}$ such that

$$\nabla_{\mathbf{r}_1} \mathbf{f} = \lambda^1 \mathbf{r}_1 \quad \text{and} \quad \nabla_{\mathbf{r}_2} \mathbf{f} = \lambda^2 \mathbf{r}_2.$$

Theorem:

1. (necessary conditions for strict hyperbolicity)
   $\mathcal{F}(\mathcal{R})$ contains strictly hyperbolic fluxes only if:

   $$\nabla_{\mathbf{r}_1}\mathbf{r}_2 \notin \text{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\} \text{ and } \nabla_{\mathbf{r}_2}\mathbf{r}_1 \notin \text{span}_{C^\infty}\{\mathbf{r}_1, \mathbf{r}_2\} \qquad (****)$$

2. Under(****), $\mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}}$ contains only strictly hyperbolic and possibly a 1-dimensional subspace of non-hyperbolic fluxes (but no hyperbolic fluxes with repeated eigenfunctions).

3. (size) Under (****) and

$$\Gamma^3_{22}(\bar{u})\,\Gamma^3_{11}(\bar{u}) - 9\,\Gamma^3_{12}(\bar{u})\,\Gamma^3_{21}(\bar{u}) \neq 0,$$

$$\boxed{0 \leq \dim \mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}} \leq 4}$$

(we have examples in all dimensions $0, \dots, 4$).

4. If $\dim \mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}} > 1$, then $\mathcal{F}(\mathcal{R})$ must contain strictly hyperbolic fluxes.

Darboux Integrability Theorem [Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. (1910)]

Consider a system of PDE's on $(\phi^1, \dots \phi^p) : \Omega \to \Theta$:

$$\boxed{\frac{\partial \phi^i}{\partial u^j} = h^i_j(u, \phi(u))}, \quad i = 1, \dots, p; \; j \in \alpha(i), \tag{1}$$

where:

1. $\Omega \underset{open}{\subset} \mathbb{R}^n$ (the space of independent variables u's)
2. $\Theta \underset{open}{\subset} \mathbb{R}^p$ (the space of dependent variables ϕ's)
3. $\alpha(i) \subset \{1, \dots, n\}$ for each $i = 1, \dots, p$.
4. $h^i_j(u^1, \dots, u^n, \phi^1, \dots, \phi^p), \quad i = 1, \dots, p, \; j \in \alpha(i)$ are C^1 functions on $\Omega \times \Theta \to \mathbb{R}$, with certain combinatorial restrictions on which ϕ's each of the h^i_j may depend so that (2) become algebraic.

If integrability conditions

$$\frac{\partial}{\partial u^k}\left(\frac{\partial}{\partial u^j}(\phi^i)\right) - \frac{\partial}{\partial u^j}\left(\frac{\partial}{\partial u^k}(\phi^i)\right) = 0 \text{ for all } j, k \in \alpha(i) \tag{2}$$

are identically satisfied on $\Omega \times \Theta$ after substitution of $h^i_j(u, \phi)$ for $\frac{\partial}{\partial u^j}(\phi^i)$ for all $i = 1, \dots, p, \; j \in \alpha(i)$ as prescribed by system (1)

Then $\exists!$ smooth local solution of (1) around $\bar{u}$, for any C^1 initial data for ϕ^i prescribed along submanifold $\Xi_i = \{u^j = \bar{u}^j, \; j \in \alpha_i\} \subset \mathbb{R}^n$ of dimension $n - |\alpha_i|$.

Example of a Darboux system:

- three unknown functions (dependent variables) ϕ, ψ and ξ.
- two independent variables u and v.
- system:

$$\begin{aligned} \phi_u &= F(u, v, \phi, \psi, \xi) \\ \psi_v &= G(u, v, \phi, \psi, \xi) \\ \xi_u &= f(u, v, \psi, \xi) \qquad (f \text{ does not depend on } \phi) \\ \xi_v &= g(u, v, \phi, \xi) \qquad (g \text{ does not depend on } \psi) \end{aligned}$$

- the integrability condition:

$$f_v + f_\psi\, G + f_\xi\, g = g_u + g_\phi\, F + g_\xi\, f.$$

- initial data near $(\bar{u}, \bar{v})$:

$$\begin{aligned} \phi(\bar{u}, v) &= a(v) \\ \psi(u, \bar{v}) &= b(u) \\ \xi(\bar{u}, \bar{v}) &= c \qquad \text{a constant.} \end{aligned}$$

- here F, G f, g, a and b are given C^1 functions of their arguments.

Frobenius Theorem:

PDE version: Is a special case of Darboux Theorem, when each unknown function is differentiated with respect to all variables.

Alternatively, given a full frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$, we can prescribe derivatives with respect to each of the frame directions. The integrability conditions then become:

$$\mathbf{r}_k \left(\mathbf{r}_j(\phi^i)\right) - \mathbf{r}_j \left(\mathbf{r}_k(\phi^i)\right) = \sum_{l=1}^{n} c^l_{kj} \mathbf{r}_l(\phi^i). \tag{3}$$

$$\phi^i(\bar{u}) = c_i.$$

There are equivalent diff. form version and vector field formulation versions about foliating $\mathbb{R}^{n+p}$ by n-dimensional integrable manifolds.

Generalized Frobenius, PDE version [M. Benfield (2016)]:
Consider a system of PDE's on $(\phi^1, \dots \phi^p) \colon \Omega \to \Theta$:

$$\boxed{\mathbf{r}_j(\phi^i(u)) = h^i_j(u, \phi(u))}, \quad i = 1, \dots, p; \; j = 1, \dots, m, \tag{4}$$

where:

1. $\mathcal{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ – a partial frame in involution on $\Omega \underset{open}{\subset} \mathbb{R}^n$.
2. $\Theta \underset{open}{\subset} \mathbb{R}^p$ is the space of dependent variables ϕ's.
3. $h^i_j(u, \phi)$, $\quad i = 1, \dots, p, \; j = 1, \dots, m$ smooth functions on $\Omega \times \Theta \to \mathbb{R}$.

If integrability conditions

$$\mathbf{r}_k\left(\mathbf{r}_j(\phi^i)\right) - \mathbf{r}_j\left(\mathbf{r}_k(\phi^i)\right) = \sum_{l=1}^{m} c^l_{jk} \mathbf{r}_l(\phi^i) \quad i = 1, \dots, p; \; j, k = 1, \dots, m \tag{5}$$

are identically satisfied on $\Omega \times \Theta$ after substitution of $h^i_j(u, \phi)$ for $\mathbf{r}_j(\phi^i)$ for all $i = 1, \dots, p, \; j = 1, \dots, m$ as prescribed by system (4).

Then $\exists!$ smooth local solution of (4), for any smooth initial data prescribed along any embedded submanifold $\Xi \subset \Omega$ of dimension $n - m$ transversal to $\mathcal{R}$.

Generalized Frobenius vector field version (local)
[Benfield, I. K., Jenssen (2016)]:

Given:

1. $\mathrm{s}_1, \ldots, \mathrm{s}_m$ – a partial frame in involution on an open $\mathcal{O} \subset \mathbb{R}^{n+p}$, where $1 \leq m \leq n$ and $p \geq 1$.

2. $\Lambda \subset \mathcal{O}$ be an $(n-m)$-dimensional embedded submanifold, such that

$$\text{span}_{\mathbb{R}}\{\mathrm{s}_1|_z, \ldots, \mathrm{s}_m|_z\} \oplus T_z\Lambda \cong \mathbb{R}^n \qquad \forall z \in \Lambda.$$

Then for $\forall \bar{z} \in \Lambda$, there exists a unique local extension of Λ to an n-dimensional submanifold $\Gamma_{\bar{z}}$ of $\mathbb{R}^{n+p}$, tangent to $\mathrm{s}_1, \ldots, \mathrm{s}_m$

In the classical local Frobenius theorem, $m = n$ and $\Lambda = \{\bar{z}\}$.

Motivation: systems of conservation laws

$$u_t + f(u)_x = 0\,. \tag{1a}$$

- n equations on n unknown functions $u(x,t) \in \Omega \subset \mathbb{R}^n$.
- one space-variable $x \in \mathbb{R}$; one time-variable: $t \in \mathbb{R}$.
- $f(u) : \Omega \to \mathbb{R}^n$ smooth flux.

Equivalently:

$$u_t + [D_u f]\, u_x = 0 \tag{1b}$$

Example: The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} V_t - U_x &= 0 \\ U_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

$V = \frac{1}{\rho}$ is volume per unit mass, U is velocity, S is entropy per unit mass, $\boxed{p(V,S) > 0}$ is pressure as a given function, s.t $\boxed{p_V < 0}$.

- $u_t + f(u)_x = 0$, where $u = [V, U, S]^T$ and $f(u) = [-U, p(V,S), 0]^T$.

- eigenvectors of $[D_u f]$ are:
$\mathbf{r}_1 = [\,1,\ \sqrt{-p_V},\ 0\,]^T$, $\mathbf{r}_2 = [\,-p_S,\ 0,\ p_V\,]^T$, $\mathbf{r}_3 = [\,1,\ -\sqrt{-p_V},\ 0\,]^T$

- eigenvalues of $[D_u f]$ are $\lambda^1 = -\sqrt{-p_V}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_V}$.

Wave curves

are used to construct solution of $u_t + f(u)_x = 0$. A wave curve consists of two parts:

- rarefaction curve - the integral curve of an eigenvector field of $[D_u f]$ - correspond to the smooth part of the self-similar solutions $\boxed{u(x,t) = \zeta\left(\frac{x}{t}\right)}$.

- shock curve – a solution of Rankine-Hugoniot conditions:

$$\{\, u \in \Omega \mid \exists\, s \in \mathbb{R}:\ f(u) - f(\bar{u}) = s \cdot (u - \bar{u}) \,\}.$$

  A shock curve describes the discontinuous part of the solutions.

Through each strictly hyperbolic state $\bar{u} \in \Omega$, there exists n wave curves.

Wave curves are building blocks for the solutions of Cauchy problems:

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x).$$

Lax (1957) under certain condition on f and when u_- and u_+ are close, the solution to the Riemann problem:

$$u_0(x) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0\,. \end{cases}$$

is determined by the wave curves.

Glimm (1965) for u_0 with small total variation, the solutions to the Cauchy problems is determined by solutions of Riemann problems.

Example: The Jacobian problem for the Euler frame.

Given:

- (V, U, S) are coordinate functions in $\mathbb{R}^3$.

- $\boxed{p(V,S) > 0}$, s.t $\boxed{\text{-}p_V < 0}$

- vector fields $\mathbf{r}_1 = \begin{bmatrix} 1 \\ \sqrt{-p_V} \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -p_S \\ 0 \\ p_V \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 1 \\ -\sqrt{-p_V} \\ 0 \end{bmatrix}$

Find: the set $\mathcal{F}(\mathcal{R})$ of <u>all</u> maps $f : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\mathcal{R} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a set of eigenvector-fields of the Jacobian matrix $[D_u f]$.

Answer:

- If $\left(\frac{p_S}{p_V}\right)_V \neq 0$

$$f = c\begin{bmatrix} -U \\ p(v,S) \\ 0 \end{bmatrix} + \bar{\lambda}\begin{bmatrix} V \\ U \\ S \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c\begin{bmatrix} -U \\ p(v,S) \\ 0 \end{bmatrix} + \text{trivial flux.}$$

eigenvalues: $\lambda^1 = -c\sqrt{-p_V} + \bar{\lambda}\,, \qquad \lambda^2 \equiv \bar{\lambda}\,, \qquad \lambda^3 = c\sqrt{-p_V} + \bar{\lambda}$.

- If $\left(\frac{p_S}{p_V}\right)_V \equiv 0$, then $\mathcal{F}(\mathcal{R})$ depends on 3 arbitrary functions of one variable.

How geometry of the eigenframe of $[D_u f]$ affects the properties of hyperbolic conservative systems and their solutions?

- We analyzed relationship between the geometry of the eigenframe and the number of companion conservation laws a system possesses.

  Jenssen, H. K., Kogan, I. A., Extensions for systems of conservation laws *Communications in PDE's*, No. 37, (2012) , pp. 1096 – 1140.

- We would like to better understand relationship between the geometry of the eigenframe and wave interaction patterns, as well as blow-up of the solutions in finite time phenomena.

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Thank you!