DESINGULARIZING COMPACT LIE GROUP ACTIONS

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This talk surveys the well-known structure of G-manifolds and summarizes parts of two papers: [4], joint with J. Brüning and F. W. Kamber, and [5], joint with I. Prokhorenkov.

Suppose that a compact Lie group G acts effectively and isometrically on a connected, closed, Riemannian manifold M. For $x \in M$, the isotropy subgroup $G_x < G$ is defined to be $\{g \in G : gx = x\}$. The orbit $\mathcal{O}_x$ of a point x is defined to be $\{gx : g \in G\}$. We will exhibit simple examples in the lecture.

On any such G-manifold, the conjugacy class of the isotropy subgroups along an orbit is called the *orbit type*. Given subgroups H and K of G, we say that $[H] \leq [K]$ if H is conjugate to a subgroup of K. It is well-known that the union of the principal orbits (those with type $[G_0]$) form an open dense subset M_0 of the manifold M, and the other orbits are called *singular*. Let M_j denote the set of points of M of orbit type $[G_j]$ for each j; the set M_j is called the *stratum* corresponding to $[G_j]$. A stratum M_j is called a *most singular stratum* if there does not exist a stratum M_k such that $[G_j] < [G_k]$. Each stratum is a G-invariant submanifold of M. For each j, the submanifold $M_{\geq j} := \bigcup_{[G_k] \geq [G_j]} M_k$ is a closed, G-invariant submanifold.

1. Desingularization construction and equivariant index

In the lecture, we will describe a new G-manifold N that has a single stratum (of type $[G_0]$) and that is a branched cover of M, branched over the singular strata. A distinguished fundamental domain of M_0 in N is called the desingularization of M and is denoted $\widetilde{M}$. The significance of this construction is that it appears in the equivariant index theorem in [4], and the analysis of transversally elliptic operators on M may be replaced by analysis on $\widetilde{M}$, which is much easier to understand. Further, one may independently desingularize each $M_{\geq j}$, since this submanifold is itself a closed G-manifold.

The desingularizations $\widetilde{M}$ and $\widetilde{M_{\geq j}}$ are the regions of integration present in the following equivariant index formula in [4].

Theorem 1.1. *(Equivariant Index Theorem, in* [4]*)*

$$\begin{aligned} \operatorname{ind}^{\rho}(D) &= \int_{\widetilde{M}\diagup G} A_0^{\rho}(x)\ \widetilde{|dx|} \\ &\quad + \sum_{j,a,b} C_{jab} \left(-\eta\left(D_j^{S+,\sigma_a}\right) + h\left(D_j^{S+,\sigma_a}\right)\right) \int_{\widetilde{M_{\geq j}}\diagup G} A^{\rho_0}_{j,\sigma_b^*}(x)\ \widetilde{|dx|}. \end{aligned}$$

2. Natural equivariant Dirac operators

In this part of the talk, another approach is used to treat difficult transverse analytic problems in a less singular setting. Given a connected, complete G-manifold, the action of $g \in G$ on M induces an action of dg on TM, which in turn induces an action of G on the principal $O(n)$-bundle $F_O \stackrel{p}{\to} M$ of orthonormal frames over M. The G orbits on F_O are diffeomorphic and form a Riemannian fiber bundle, in a natural metric on F_O. The quotient $F_O \stackrel{\pi}{\to} F_O\diagup G = F_O\diagup \mathcal{F}$ is a Riemannian submersion of compact $O(n)$-manifolds.

Let $E \to F_O$ be a Hermitian vector bundle that is $G \times O(n)$-equivariant. Let $\rho : G \to U(V_\rho)$ and $\sigma : O(n) \to U(W_\sigma)$ be irreducible unitary representations. We define the bundle $\mathcal{E}^\sigma \to M$ by $\mathcal{E}^\sigma_x = \Gamma\left(p^{-1}(x), E\right)^\sigma$, where the superscript σ refers to the type σ part of the $O(n)$-representation space $\Gamma\left(p^{-1}(x), E\right)$. Similarly, we define the bundle $\mathcal{T}^\rho \to F_O \diagup G$ by $\mathcal{T}^\rho_y = \Gamma\left(\pi^{-1}(y), E\right)^\rho$.

In the lecture, we will show how to construct equivariant differential operators on M and $F_O \diagup G$, denoted

$$D^\sigma_M : \Gamma(M, \mathcal{E}^\sigma) \to \Gamma(M, \mathcal{E}^\sigma) \text{ and } D^\rho_{F_O\diagup G} : \Gamma(F_O\diagup G, \mathcal{T}^\rho) \to \Gamma(F_O\diagup G, \mathcal{T}^\rho).$$

For an irreducible representation $\alpha : G \to U(V_\alpha)$, let $(D^\sigma_M)^\alpha : \Gamma(M, \mathcal{E}^\sigma)^\alpha \to \Gamma(M, \mathcal{E}^\sigma)^\alpha$ be the restriction of D^σ_M to sections of G-representation type α. The operator $\left(D^\rho_{F_O\diagup G}\right)^\beta$ is defined similarly.

Theorem 2.1. *The operator D^σ_M is transversally elliptic and G-equivariant, and $D^\rho_{F_O\diagup G}$ is elliptic and $O(n)$-equivariant, and the closures of these operators are self-adjoint. The operators $(D^\sigma_M)^\rho$ and $\left(D^\rho_{F_O\diagup G}\right)^\sigma$ have identical discrete spectrum, and the corresponding eigenspaces are conjugate via Hilbert space isomorphisms.*

Thus, questions about the transversally elliptic operator D^σ_M are reduced to questions about the elliptic operators $D^\rho_{F_O\diagup G}$ for each irreducible $\rho : G \to U(V_\rho)$. Further, it turns out that the operators D^σ_M play the same role for equivariant analysis as the standard Dirac operators do in the analysis of elliptic operators on closed manifolds.

3. Further Comments

In both results, a problem of analyzing a transversally elliptic operator (with potentially infinite dimensional eigenspaces) is reduced to an elliptic problem or set of elliptic problems, which are more tractable. For example, the Atiyah-Segal Theorem ([1], 1968) was the first version of an equivariant index theorem. However, the appropriate generalization to transversally elliptic operators appeared only in 1996 (Berline-Vergne, [2],[3]). In a sense, Theorem 1.1 is a Fourier transform version of the Atiyah-Segal and Berline-Vergne results, giving a formula for the Fourier coefficients of the character instead of a function value of the character. Further, Theorem 1.1 gives a method of computing eta invariants of Dirac-type operators on quotients of spheres by compact group actions; this was known previously for finite group actions only.

The new "transversal Dirac operators" on G-manifolds constructed in Section 2 and in [5] should be explored further, and in particular future investigations should lead to generalizations of Dirac operator results to the transversally elliptic setting.

References

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[3] N. Berline and M. Vergne, *L'indice équivariant des opérateurs transversalement elliptiques*, Invent. Math. **124**(1996), no. 1-3, 51-101.

[4] J. Brüning, F. W. Kamber, and K. Richardson, *The eta invariant and equivariant index of transversally elliptic operators*, preprint in preparation.

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