Emma Previato, Boston University, Theta functions and isospectral manifolds

Riemann theta functions, like the Weierstrass $\wp$-function which is their predecessor, can be characterized by partial differential equations (PDEs), which bring out the beautiful and surprising link between their analytic and algebraic nature. Yet more surprisingly, beginning in the 1970s theta functions were used to give exact solutions to "completely integrable hierarchies" of "soliton equations" (references can be found in [P3]).

One overlooked algebraic aspect of this integrability is the link with differential algebra [P1]: Burchnall and Chaundy in the 1920s had characterized the commutative rings $\mathcal{R}$ of ordinary differential operators, up to equivalence, the Jacobi variety of the spectral curve X, a completion of Spec$\mathcal{R}$, and H.F. Baker had written the coefficients of the operators in terms of theta functions on the Jacobian, also deriving the KdV hierarchy in the process of characterizing the generalized $\wp$-functions. In fact, Baker following Klein had generalized the genus-1 σ-function (a modular-scalar multiple of the theta function). The generalized σ-function is associated to a curve (as opposed to a general abelian variety), in fact a "Burchnall-Chaundy" curve, namely one with the property that two functions suffice to generate the ring $H^0(X\backslash\infty, \mathcal{O}_X)$ for a given point ∞, say. Incidentally, these curves have recently been the object of close study in number theory and coding theory, dubbed C_{ab} curves, in analogy to elliptic curves in Weiestrass form. The advantage of Baker's construction is that the dictionary between algebra and analysis is quite explicit: this allows him to generalize the addition rule on the elliptic curve, and derive the PDEs by computing coefficients of the σ-function expanded along the curve.

The object of this talk is an ongoing higher-dimensional version of the theory. On the one hand, almost nothing is known from the differential-algebraic point of view about rings of commuting partial differential operators (PDOs), and the same is true of their spectral varieties. On the other hand, Baker succeeded in constructing the σ-function and its characteristic addition structure and PDEs only for special classes of curves.

After a sketch of the spectral curve theory, the talk will present the following items:

• A non-algebraic example of commuting ring of PDOs, together with a multivariate differential resultant related to the algebraic equations defining the spectral varity of the ring [KP].

• The constructions of Nakayashiki [N] and Mironov [M], respectively, that associate to a theta function in g variables, a hierarchy of PDEs, respectively a commutative ring of $(g \times g)$-matrices for $g = 2$ whose entries are PDOs.

• Barsotti's universal family of PDEs that characterize abelian theta functions. In particular, the following striking generalization of the ring structure of elliptic functions: in the same way in which $\wp$ and $\wp'$ generate the ring of analytic functions that are doubly periodic with respect to a given lattice, on any g-dimensional principally polarized abelian variety there exists a direction with the property that sufficiently many derivatives of the logarithm of the theta function along those directions generate the field of functions of the abelian variety. This theorem was sufficiently surprising for the experts, for me to offer a brute-force proof in $g = 3$, the first 'surprising' case [P2].

Bringing together the two challenges described above, I propose a comparison of Nakayashiki's and Barsotti's equations, both PDEs for non-Jacobians (hence genuine generalizations of the KP hierarchy). In my proof of Barsotti's result on the structure of the abelian functions, I use the generalization of Baker's σ-function, for all trigonal curves of genus 3, constructed in [EEMOP], and describe a stratification of their theta divisor [MP] which closes the circle, by yielding new solutions to integrable hierarchies.

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