

Risk-Based and Factor Investing

Quantitative Finance Set

coordinated by
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Risk-Based and Factor Investing

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Preface

This book contains a collection of 20 exclusive new Chapters written by leading academics and practitioners in the area of risk-based and factor investing (RBF), a term that encompasses both alternative non-return based portfolio construction techniques and investing style risk premia strategies. The Chapters are intended to introduce readers to some of the latest, cutting edge research encountered by academics and professionals dealing with RBF solutions. The articles deal with new methods of building strategic and tactical risk-based portfolios, constructing and combining systematic factor strategies and assessing the related rules-based investment performances. Although numerous articles are technical in nature, this book can assist portfolio managers, asset owners, consultants, academics, as well as students wanting to further understand the science and the art of risk-based and factor investing.

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Advances in Portfolio Risk Control

Spurred by the increased interest in applying “risk control” techniques in an asset allocation context, we offer a practitioner’s review of techniques that have been newly proposed or revived from academic history. We discuss minimum variance, “1/N” or equal-weighting, maximum diversification, volatility weighting and volatility targeting – and especially equal risk contribution or “risk parity”, a concept that has become a real buzz word. We start from a taxonomy of risk control techniques. We discuss their main characteristics and their positives and negatives and we compare them against each other and against the maximum Sharpe ratio (SR) criterion. We illustrate their implications by means of an empirical example. We also highlight some key papers from the vast and still growing literature in this field. All in all, we aim to provide a practical and critical guide to risk control strategies. It may help to demystify risk control techniques, to appreciate both the “forest” and “trees” and to judge these techniques on their potential merits in practical investment applications.

1.1. Introduction

Recently, there has been increased interest in applying “risk control” techniques in an asset allocation context. Some examples of techniques that have been newly proposed or revived from academic history are “1/N” or equal-weighting, minimum variance, maximum diversification, volatility weighting and volatility targeting – and especially equal risk contribution or “risk parity”, a concept that has become a real buzz word. In this chapter, we start from a taxonomy of risk control techniques. We discuss their main characteristics and their positives and negatives, we compare

them against each other and against the maximum Sharpe ratio (SR) criterion and we illustrate their implications by means of an empirical example. We also highlight some key papers from the vast and still growing literature in this field. All in all, we aim to provide a practical and critical guide to risk control strategies that may help to appreciate both the “forest” and “trees” and to judge these techniques on their actual potential merits in practical investment applications. For an in-depth exposition, comparison and evaluation of these strategies, we recommend [RON 14].

The main question in risk control is: “does it work?” Do risk control techniques achieve the *ex ante* targeted risk balance or risk profile? Can we avoid hot spots (pockets of risk concentration in a portfolio) and can we achieve diversification against losses? Apart from this risk-budgeting perspective, a large part of the literature has promoted risk control as a full-fledged investment criterion – suggesting that controlling the risk dimension is sufficient to build a portfolio or an opportunity to reap risk-adjusted outperformance. But, why would ignoring the return dimension *ex ante* produce portfolios that are superior in terms of *ex post* risk-adjusted performance?

Several studies indicate that the historical outperformance of risk control strategies can be linked to overweighting asset classes that in the rear-view mirror have paired high historical risk premia with low-risk levels (as is the case for bonds, e.g.), or to implicit exposures to factor premia [JUR 15]. However, focusing directly on factor exposures, as is done in factor investing (see [ANG 14]), provides a much more efficient and effective way to capture factor premia. Still, focusing only on risk aspects when forming a portfolio can be a perfectly sensible heuristic (see [FIS 15]) or a starting point if we have only low confidence in *ex ante* risk premia estimates. From the perspective of estimation risk, mis-estimation of risk premia has the greatest impact on portfolio composition and especially risk premia are notoriously hard to estimate *ex ante*. For example, suppose that *ex ante* we cannot meaningfully differentiate between all assets’ SRs (so assuming that all SRs are equal), then constructing a maximum diversification portfolio (MDP) gives the maximum Sharpe ratio portfolio (MSRP). If, in addition to equal SRs, we cannot meaningfully differentiate between asset correlations (so also assuming that all correlations are uniform), then applying risk parity gives the MSRP. So, besides the risk-budgeting dimension, also the potential relevance of risk control techniques in full-fledged risk-return optimization is not to be underestimated.

This chapter is divided as follows. Section 1.2 introduces our empirical example and provides some preliminaries. Section 1.3 outlines the MSRP within the familiar mean-variance framework. Next, we discuss the risk control strategies. The main skeleton of the taxonomy of risk control strategies has a cross-section and a time-series branch. The objective of risk control in the cross-section is to control a portfolio’s risk profile at a given point in time. The focus is across assets: reweighting the portfolio constituents so as to obtain a desired risk profile. The main

cross-sectional risk control strategies are: $1/N$, or the equally weighted portfolio (section 1.4), the minimum variance portfolio (MVP) (section 1.5) and the MDP (section 1.6). Next, we have risk parity, which comes in two flavors, the equal risk contribution portfolio (ERCP) or “full” risk parity (section 1.7) and the inverse volatility portfolio (IVP) or “naive” risk parity, implying volatility weighting in cross-section (section 1.8). The objective of time-series risk control is to control the portfolio risk level over time. There are two closely related time-series techniques: volatility weighting over time, or adjusting the exposure to risky assets according to the level of forecasted volatility, and volatility targeting, or volatility weighting with the specific goal to achieve a prespecified level of portfolio volatility (section 1.9). Each of these sections is organized according to a fixed format, starting with main references, followed by the recipe to calculate the particular portfolio, its characteristics and evaluation. Section 1.10 concludes with an overall evaluation. Section 1.11 contains technical details.

1.2. The empirical example and preliminaries

We consider monthly returns in excess of the risk-free rate over the decade January 2005 through December 2014 for a selection of US asset classes: equities, treasuries (Tsies), investment grade (IG) corporates and high yield (HY) corporates. The risk-free return comes from the Ibbotson “Stocks, Bills, Bonds and Inflation” database. Equity is the market factor from Kenneth French’s database¹. The fixed income series are taken from Barclays Capital Live². All returns are in USD. The composition of the market capitalization weighted portfolio “Mkt Cap” is estimated as per 2014Q4³. “EqWtd” is the equally weighted portfolio.

Table 1.1 shows the descriptive statistics. Over the past decade, fixed income assets were the real winners in terms of risk-adjusted performance. This is not surprising given the substantial tail wind from decreasing interest rates. Especially Tsies paired a 3% average return with a relatively low level of risk. Equities showed the highest volatility, but viewing the SR this was not matched by a proportionally higher risk premium. Equities and Tsies were negatively correlated, providing hedge opportunities (see the small negative correlation between Tsies and the market cap

1 The Ibbotson risk free rate and the equity market factor can be downloaded from http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

2 Download from <https://live.barcap.com/>.

3 Sources are (1) Securities Industry and Financial Markets Association (SIFMA), US Bond Market Outstanding, download from <http://www.sifma.org/research/statistics.aspx>, (2) World Bank, year-end market capitalization of listed companies by country, download from <http://data.worldbank.org/indicator/CM.MKT.LCAP.CD> and (3) Barclays Live (for the relative IG and HY capitalizations).

portfolio). The highest correlation is between equities and HY, pointing at a strong link between equity risk and credit risk. Credit risk is dominant in HY and the negative correlation between interest rates and credit spreads manifests itself in the negative correlation between Tsies and HY.

	Equities	Tsies	IG	HY	Mkt Cap	EqWtd
<i>Market cap weight:</i>	53%	29%	14%	4%	100%	
<i>Return statistics:</i>						
avge p.a. %	7.54	2.97	4.17	6.62	5.71	5.33
stdev p.a. %	15.11	4.16	6.04	10.54	8.46	6.78
Sharpe ratio	0.50	0.71	0.69	0.63	0.67	0.78
<i>Correlations:</i>						
Equities		-0.30	0.35	0.74	0.98	0.88
Tsies			0.44	-0.24	-0.11	-0.01
IG				0.63	0.53	0.73
HY					0.78	0.90
Mkt Cap						0.95

Table 1.1. *Statistics of US excess returns (p.a.) over the risk-free rate (January 2005 – December 2014)*

1.2.1. Money allocation versus risk allocation

The money allocation in the market cap portfolio is given in Table 1.1. For the risk allocation within the market cap portfolio, we compute the Ordinary Least Squares (OLS) regression slope or beta of the assets against the market cap portfolio. This beta represents the relative marginal contribution of the corresponding asset to the overall portfolio risk (for details, see section 1.11). The component risk contribution is given by the product of the portfolio weight and beta. Hence, the betas can be interpreted as the adjustment factors to transform money allocation into risk allocation (note that the weighted average value of beta is unity). The risk allocation within the market cap portfolio is given in Table 1.2. From this

table, we see a nasty surprise: at first sight, the market cap portfolio appears to be a properly diversified portfolio but in reality more than 90% of the portfolio risk is due to equities (this was already forewarned by the high correlation between equities and the market cap portfolio as shown in Table 1.1). The same finding is widely reported for conventional 60/40 equity-bond portfolios in general, and for typical “Yale-type” portfolios (where alternatives and/or commodities are added to main holdings of equities and bonds).

	Eq	Tsies	IG	HY	sum
<i>Market cap weight</i>	53%	29%	14%	4%	100%
<i>Beta</i>	1.74	-0.06	0.38	0.97	
<i>% Risk contribution</i>	92%	-2%	5%	4%	100%

Table 1.2. Risk attribution with respect to Mkt Cap portfolio

Although we focus on volatility as the risk measure, most of the results in this chapter carry over to downside risk measures such as portfolio loss or value-at-risk (VaR). Table 1.3 shows the average of the six largest monthly losses against the risk-free rate on the market cap portfolio: equities contributed by far the most to the realized losses. The excessive contribution of equities to (downside) risk within portfolios that seem only moderately geared toward equities provided the impetus to the research into risk control strategies (see also [QIA 05]). In the remainder of this chapter, we use this empirical example to illustrate various risk control strategies.

Mkt Cap index	Eq	Tsies	IG	HY
-6.02	-5.39	-0.04	-0.40	-0.19
100%	90%	1%	7%	3%

Table 1.3. Absolute (in %) and relative contribution of assets to the average of the six largest losses on the market cap portfolio (in terms of excess returns), 2005–2014

1.2.2. Implied risk premia and the implied Sharpe ratios

There is one additional perspective that we want to highlight – a perspective that is helpful in evaluating risk control strategies *vis à vis* the MSRP. For each of the portfolios that we discuss, we present the implied risk premia and the implied SRs of the individual assets. Instead of using actual risk premia and the variance-covariance matrix to calculate the MSRP, we reverse the process and assume that the reference portfolio at hand actually *is* the MSRP. Together with the variance-covariance matrix of excess returns, this allows us to derive the “imputed” risk premia

(pioneered by Sharpe [SHA 74]); together with the actual (historical) asset standard deviations, we can then compute the implied SRs. Hence, given a particular portfolio, these implied risk premia (or implied SRs) would make this reference portfolio the MSRP. For more details, we refer the reader to section 1.11.

This reverse portfolio optimization is relevant when there is an uncertainty about *ex ante* risk premia. After all, since the MSRP is the tangency portfolio to the mean-variance efficient frontier without including risk-free borrowing and lending, this portfolio is very sensitive to the input risk premia. Slight differences in these inputs can result in very different (and sometimes “unrealistic” or extreme and hence unacceptable) portfolios. At the same time, estimating *ex ante* risk premia is a very difficult task. Reverse optimization can help since the assets’ implied risk premia, derived from a reference portfolio such as the market capitalization weighted portfolio or a risk control portfolio, serve as a sensible starting point. Depending on the confidence placed in one’s *ex ante* views, next we can adjust the implied risk premia accordingly. After reoptimization, the resulting portfolio is closer to the original portfolio and is less extreme. This two-stage portfolio optimization process is originally proposed by Black and Litterman [BLA 92] and extended by Haesen *et al.* [HAE 14] who take the risk parity portfolio as the reference portfolio.

Table 1.4 presents the implied risk premia and the implied SRs of the market cap portfolio. For equities, the implied risk premium is about 25% larger than the historical risk premium. For IG, the implied risk premium is about half the historical risk premium. So, if the market cap portfolio were the MSRP, equities would have to offer a risk premium of 10% and IG of 2%. Conversely, if we felt confident in extending the historical risk premia to the future, this implies that we should increase the weight of IG and lower the weight of equities in order to increase the SR of the market cap portfolio. For Tsies, the implied risk premium (and hence the implied SR) is even slightly negative, reflecting the role of Tsies as a hedge in the market cap portfolio. Because of the negative correlation of Tsies with equities (and HY), their inclusion in the market cap portfolio would be justified even when their risk premium were negative.

	Eq	Tsies	IG	HY	Mkt Cap
<i>Implied risk premium</i>	9.96	-0.32	2.14	5.55	5.71
<i>Implied Sharpe ratio</i>	0.66	-0.08	0.35	0.53	0.67

Table 1.4. *Implied risk premia (%) and implied SRs within the market cap portfolio*

In order to derive the implied risk premia from a portfolio different from the market cap portfolio, we first calculate the relative risk aversion coefficient λ^* as

implied by the market cap portfolio (see [SHA 74]): dividing the historical risk premium on the market cap portfolio by the historical variance of the market cap portfolio yields $\lambda^* = 8.0$. The implied risk premium on an alternative portfolio, when assuming that this portfolio is mean-variance optimal, is then given by the product of λ^* and its historical variance. This portfolio risk premium is next attributed to the assets comprised in the portfolio according to their relative risk contributions (or betas) with respect to this portfolio (see section 1.11).

NOTE.— We use fairly conventional notation. The weight of asset i in portfolio p is w_i . Individual and portfolio risk premia or “rewards” (average returns in excess of the risk-free rate) are denoted by \bar{r}_i and \bar{r}_p . Individual asset standard deviations or volatilities are denoted by σ_i and the portfolio volatility by σ_p . The beta of asset i with respect to portfolio p is β_{ip} (reflecting its relative marginal contribution to portfolio volatility) and its correlation with the portfolio is denoted as ρ_{ip} . Where deemed necessary, technical details are mentioned in the main text; section 1.11 contains a general background and additional derivations.

1.3. Maximum Sharpe ratio portfolio (MSRP)

For a discussion of the SR, see [SHA 94]. For mean-variance portfolio theory and for finding the composition of the MSRP, we refer to standard investment texts.

Recipe: Choose the portfolio weights to maximize the SR:

$$\max_{\{w\}} SR_p = \frac{\bar{r}_p}{\sigma_p} \quad [1.1]$$

Characteristics: (1) For the MSRP, it is not possible to further increase its risk premium without increasing its risk. This implies that for all assets the ratios of marginal contributions to risk and reward are constant. An asset’s marginal contribution to portfolio risk equals $\partial\sigma_p / \partial w_i$, whereas an asset’s marginal contribution to the portfolio risk premium simply equals its risk premium, $\partial\bar{r}_p / \partial w_i = \bar{r}_i$. Hence, for all assets within the MSRP, we must have:

$$\frac{1}{\bar{r}_i} \frac{\partial\sigma_p}{\partial w_i} = \frac{1}{\bar{r}_j} \frac{\partial\sigma_p}{\partial w_j} \quad [1.2]$$

Substituting the definition of beta from equation [1.23] in section 1.11, this translates into $\bar{r}_i / \beta_{ip} = \bar{r}_j / \beta_{jp} = \bar{r}_p$ (where the last equality follows from the fact that the portfolio beta equals unity). Note that \bar{r}_i / β_{ip} is the Treynor's [TRE 66] risk-adjusted performance ratio. Hence, for each asset within the MSRP, its risk premium should be equal to the product of (i) its beta with respect to the MSRP and (ii) the risk premium of the MSRP:

$$\bar{r}_i = \beta_{ip} \bar{r}_p \quad [1.3]$$

This is the first-order condition of mean-variance optimality. (2) Since we can rewrite the beta as the product of (i) the correlation with the portfolio and (ii) the quotient of the asset and portfolio volatility, $\beta_{ip} = \rho_{ip} \sigma_i / \sigma_p$, it follows that in the MSRP the stand-alone asset SRs and the portfolio's SR are related by $SR_i = \rho_{ip} \cdot SR_p$. If $SR_i > \rho_{ip} SR_p$, we can increase the SR of the portfolio by increasing the weight of (or adding) asset i to the portfolio p . (3) Without any additional constraints, the long-only MSRP can be a very concentrated portfolio. (4) When all volatilities, correlations and risk premia are the same, then the MSRP is the $1/N$ portfolio (which then also coincides with the *ERCP* and the *MVP*). After all, diversification lowers risk but in the portfolio context all assets are perfect substitutes. It is not possible to lower portfolio risk or increase the portfolio risk premium by changing the weights. Hence, we end up with the equally weighted portfolio.

Evaluation: Table 1.5 shows the historical statistics of the portfolios we consider. Historically, the MSRP has the maximum SR. This is so by construction, since we optimized the SR over the full historical sample period (in-sample). In practical applications, we would use trailing historical windows (avoiding a look-ahead bias) to periodically recalculate the weights. In this way, the out-of-sample properties of the MSRP can be evaluated. The same argument applies to the other strategies that we discuss below. Whether the MSRP indeed delivers the maximum SR *ex post* depends on the quality of the inputs, especially the risk premia.

	Cap Wtd	MSRP	1/N	MVP	MDP	ERCP	IVP
<i>avge p.a.</i>	5.71	3.97	5.33	3.68	4.05	4.24	4.46
<i>stdev p.a.</i>	8.46	3.57	6.78	3.43	3.70	4.19	4.74
<i>Sharpe ratio</i>	0.67	1.11	0.78	1.07	1.10	1.01	0.94

Table 1.5. Comparative historical excess return portfolio statistics, 2005–2014

In our example, the MSRP is indeed a concentrated portfolio, containing mostly Tsies supplemented with HY and only 7% equities, see Table 1.6. Tsies dominate because of their low volatility and negative correlation with HY. The smaller than unity beta of Tsies reveals that Tsies are included as a diversifier; the larger than unity betas of HY and equities show that these assets are included because of their high average return. Slight changes in risk premia will change the composition of the MSRP markedly. Note that the implied risk premia are different from the historical risk premia as shown in Table 1.1. After all, the implied risk premia are calculated from the derived risk aversion parameter $\lambda^* = 8.0$ and the historical portfolio variance (and hence are proportional to betas).

	Eq	Tsies	IG	HY	sum
<i>Weight</i>	7%	74%	0%	19%	100%
<i>Beta</i>	1.90	0.75	1.41	1.67	
<i>% Risk contribution</i>	13%	55%	0%	31%	100%
<i>Implied risk premium</i>	1.93	0.76	1.43	1.69	1.01
<i>Implied Sharpe ratio</i>	0.13	0.18	0.24	0.16	0.28

Table 1.6. Risk attribution with respect to MSRP, and implied risk premia (%) and Sharpe ratios

1.4. 1/N or equal-weighting

Recipe: In equally weighted portfolios, each of the N assets is assigned a weight of $1/N$. In our example, each asset class gets a weight of 25% in the portfolio, with monthly rebalancing [DEM 09].

Characteristics: (1) $1/N$ avoids concentrated positions – in terms of money allocation. (2) Within equities, $1/N$ implies an exposure to the small-cap anomaly. The market cap portfolio is tilted toward large cap stocks. The $1/N$ portfolio is tilted toward small cap stocks and hence will capture a size premium. (3) Furthermore, $1/N$ implies a disciplined and periodical rebalancing of positions. By definition, the market cap portfolio is a buy-and-hold portfolio. The $1/N$ portfolio, in contrast, implies a “volatility pumping” effect: in order to maintain the $1/N$ allocation, we have to buy (sell) out- (under-) performing assets. This is effectively a “buy low, sell high” strategy, which profits from reversals⁴. Depending on the frequency, the rebalancing process implies portfolio turnover with the associated transaction cost and exposure to potential illiquidity (since even the smallest market cap assets get a weight of $1/N$). (4) In Bayesian terms, the $1/N$ portfolio is the “uninformed prior”:

⁴ For the effects of rebalancing on portfolio returns, see [HAL 14].

the naively diversified portfolio that is optimal when we have no information to discriminate between the attractiveness of assets. (5) It can be shown that when all assets have the same volatility and when all pairwise correlations are the same, then the 1/N portfolio is the *MVP*. In this case, the MVP also coincides with the *ERCP*, see below. (6) From Tables 1.1 and 1.5, we see that the 1/N portfolio has a lower risk than the market cap portfolio and a higher historical SR. This arises from underweighting equities (with a lower SR) and overweighting IG and HY (with a higher SR).

Evaluation: (1) Table 1.7 shows the 1/N portfolio statistics. It clearly shows that equal money weights are very different from equal risk weights. Notably, Tsies act as a strong diversifier (negatively correlated with equities and HY) and show (virtually) zero risk contribution. Still, equity risk dominates in the 1/N portfolio, accounting for about half of the portfolio volatility. (2) For equities, the implied risk premium is 7.17% p.a. (which given historical volatility implies a SR of 0.47). When we believe that the *ex ante* equity risk premium is below 7.17%, the weight of equities should be lowered in order to improve the risk-adjusted portfolio performance. Likewise, when we believe that the *ex ante* bond risk premium is above -3 bps p.a., the weight of Tsies should be increased. Equivalent reasoning applies to IG and HY.

	Eq	Tsies	IG	HY	EqWtd
<i>Weight</i>	25%	25%	25%	25%	100%
<i>Beta</i>	1.95	-0.01	0.65	1.41	
<i>% Risk contribution</i>	49%	0%	16%	35%	100%
<i>Implied risk premium</i>	7.17	-0.03	2.39	5.16	3.67
<i>Implied Sharpe ratio</i>	0.47	-0.01	0.40	0.49	0.54

Table 1.7. Risk attribution with respect to 1/N portfolio, and implied risk premia (%) and Sharpe ratios

1.5. Minimum variance portfolio (MVP)

Haugen and Baker [HAU 91] show that market cap weighted portfolios are inefficient (suboptimal) when there are market frictions and highlight the high-relative performance of low volatility portfolios. Clarke *et al.* [CLA 06] extend Haugen and Baker's empirical research. Blitz and van Vliet [BLI 07] revive the interest in the low volatility anomaly and provide possible explanations (behavioral biases, leverage restrictions, and delegated portfolio management and benchmarking).

Recipe: Choose the portfolio weights to minimize the portfolio variance:

$$\max_{\{w_i\}} \sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij} \quad [1.4]$$

The optimal portfolio is characterized by equal marginal contributions to portfolio risk:

$$\frac{\partial \sigma_p}{\partial w_i} = \frac{\partial \sigma_p}{\partial w_j} \quad [1.5]$$

Characteristics: (1) Note that marginal risk contributions are given by $\partial \sigma_p / \partial w_i = \sigma_{ip} / \sigma_p = \beta_{ip} \cdot \sigma_p$, so all asset betas with respect to the MVP are identical (and hence equal to unity). (2) Since an asset's risk contribution equals $w_i \cdot \partial \sigma_p / \partial w_i \propto w_i$, the risk contribution is proportional to the portfolio weight, so risk allocation equals money allocation. (3) When all assets have the same volatility and when all pairwise (imperfect) correlations are the same, then the MVP is the $1/N$ portfolio: it pays to diversify over the assets but in the portfolio context, all assets are perfect substitutes. (4) The MVP is the *MSRP* when all assets have the same risk premium, $\bar{r}_i = \bar{r}$ (implying that all SRs SR_i are proportional to $1/\sigma_i$). After all, in this case, we have (see equation [1.2]):

$$\frac{1}{\bar{r}} \frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\bar{r}} \frac{\partial \sigma_p}{\partial w_j} \quad [1.6]$$

Evaluation: (1) The MVP favors low volatility assets and low beta assets and hence benefits from the low volatility anomaly. The MSCI Minimum Variance Index and the S&P Low Volatility Index are examples of low-risk portfolios that are designed to benefit from this anomaly. For more information on the low volatility anomaly, see [BLI 07]. (2) Several studies have documented that MVPs also pick up other priced anomalies. Clarke *et al.* [CLA 06] find that, in general, the MVP has a substantially higher value (B/P) exposure than the market (since value stocks tend to have low volatilities), which explains at least part of its higher average realized return. Scherer [SCH 11] shows that the MVP not only loads significantly on the Fama-French factors (large size and high value) but also finds that MVPs have a negative beta bias (favor low beta assets) and favor assets with low residual volatility. The latter effects crowd out the Fama-French factors in the sense that low beta and low residual volatility alone can explain more of the variation in the MVP's excess returns than the Fama-French factors. This leads Scherer to conclude that low beta and low residual volatility is a more efficient and effective way to capture the low volatility anomaly than minimum variance. (3) When time passes and the MVP

is reoptimized, we will need to apply constraints on turn-over in order to mitigate transaction costs. However, turnover constraints make the MVP a path-dependent strategy. (4) The MVP is a concentrated portfolio. Assets with low volatility and/or low correlations with other assets will carry a large weight. Conversely, assets with high volatility and/or high correlations with other assets will carry a small or even negative weight; when excluding short positions, these assets will not appear in the MVP. This is illustrated in Table 1.8: IG is not included in the long-only MVP. Table 1.8 confirms that when the assets comprised in the MVP have identical risk premia, then the MVP is the *MSRP*. Note again that this only applies to assets that are comprised in the MVP in the first place.

	Eq	Tsies	IG	HY	MVP
<i>Weight</i>	5%	82%	0%	13%	100%
<i>Beta</i>	1.00	1.00	1.35	1.00	
<i>% Risk contribution</i>	5%	82%	0%	13%	100%
<i>Implied risk premium</i>	0.94	0.94	1.27	0.94	0.94
<i>Implied Sharpe ratio</i>	0.06	0.23	0.21	0.09	0.27

Table 1.8. Risk attribution with respect to MVP, and implied risk premia (%) and Sharpe ratios

(5) Table 1.8 also confirms that marginal risk contributions of MVP constituents are identical (all betas equal unity) so the money allocation equals the risk allocation in an MVP. (6) Table 1.5 shows that the MVP has the lowest historical risk premium as well as the lowest volatility (by construction), yielding a SR slightly lower than the MSRP and MDP. This lower volatility is achieved by overweighting Tsies at 82%, supplemented by positions in equities and HY which are negatively correlated with Tsies. (7) Last but not least, the quadratic optimization underlying the MVP has the property of being “error maximizing”, see [MIC 89]. This means that the composition of the MVP is very sensitive to slight differences in variances and covariances. When (part of) these differences are not real but due to sampling error, this will propagate into portfolio composition. Again, note that we use the full historical sample to calculate the weights of the MVP.

1.6. Maximum diversification portfolio (MDP)

The MDP is proposed by Choueifaty and Coignard [CHO 08].

Recipe: The weights of the MDP are obtained by maximizing the “diversification ratio”, which is defined as the ratio of weighted volatilities and portfolio volatility:

$$\max_{\{w\}} \frac{\sum_i w_i \sigma_i}{\sigma_p} \quad [1.7]$$

For obtaining an insight into this ratio, note that the portfolio volatility can be written as the weighted sum of the product of each asset’s individual volatility and its correlation with the portfolio (see equation [1.18] in section 1.11). Hence, we can rewrite the diversification ratio as:

$$\max_{\{w\}} \frac{\sum_i w_i \sigma_i}{\sum_i w_i \sigma_i \rho_{ip}} \quad [1.8]$$

This expression reveals that the diversification ratio compares (1) the portfolio volatility when ignoring correlations in the numerator, with (2) the actual portfolio volatility when taking into account correlations (and hence diversification) in the denominator. Imperfect (<1) correlations increase the diversification ratio above unity.

Characteristics: (1) It can be shown that for the MDP it holds that (see [CHO 08]):

$$\frac{1}{\sigma_i} \frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\sigma_j} \frac{\partial \sigma_p}{\partial w_j} \quad [1.9]$$

By definition, within the global MVP, all assets’ marginal risk contributions $\partial \sigma_p / \partial w_i$ are equal, see equation [1.5]. It follows that for equal volatilities, $\sigma_i = \sigma_j$, the MPD coincides with the global MVP. (2) From equation [1.9], it also follows that when risk premia $\{\bar{r}_i\}$ are proportional to volatilities $\{\sigma_i\}$, thus implying that all assets have the same SR, then the MDP is the MSRP. After all, in this case, equation [1.9] is equivalent to equation [1.2]. (3) Finally, Choueifaty and Coignard [CHO 08] also show that all constituent assets have the same correlation with the MDP.

Evaluation: (1) Why should we want to maximize this specific diversification ratio? After all, there are many definitions of “diversified”. (2) The diversification ratio is a differential diversification measure. It applies with respect to the specific portfolio at hand. It is not an absolute diversification measure from which we can read the degree of diversification; we cannot compare the diversification ratios of two different portfolios to infer which portfolio is more diversified than the other.

(3) The MDP is not unique and may be very concentrated in weights (money allocation) or in risk and loss contributions (risk allocations). Indeed, in our example, IG carries zero weight in the MDP, see Table 1.9. Tsies have the highest weight in the MDP; the money allocation of 74% here implies that Tsies account for almost 50% of the portfolio risk. This can hardly be termed a “diversified portfolio”.

	Eq	Tsies	IG	HY	MDP
<i>Weight</i>	14%	74%	0%	13%	100%
<i>Beta</i>	2.34	0.64	1.29	1.63	
<i>% Risk contribution</i>	32%	47%	0%	21%	100%
<i>Implied risk premium</i>	2.55	0.70	1.40	1.78	1.09
<i>Implied Sharpe ratio</i>	0.17	0.17	0.23	0.17	0.30

Table 1.9. Risk attribution with respect to MDP, and implied risk premia (%) and Sharpe ratios

(4) Table 1.9 also shows that the implied SRs of the three portfolio components equal 0.17. This confirms that when SRs of the portfolio constituents are the same, then the MDP is the *MSRP*. Note that this only applies to assets comprised in the MDP; by construction, the composition of the MDP does not depend on risk premia or SRs.

Table 1.5 shows that, for the historical inputs, the MDP beats all portfolios except the *MSRP* in risk-adjusted performance. This is due to the large overweight of Tsies which over the past decade showed the highest SR. Since we use the full historical sample to calculate the weights of the MDP (and other portfolios), these are in-sample results. In practice, we would sequentially derive the *ex ante* MDP from trailing data windows.

1.7. Equal risk contribution portfolio (ERCP): full risk parity

[QIA 05] is the seminal paper on risk parity. Qian [QIA 06] discusses the linear decomposition of volatility. Hallerbach [HAL 03] extends risk decomposition to VaR and shows how to decompose risk in parametric and non-parametric (simulation) settings. Maillard *et al.* [MAI 10] discuss the theoretical properties of risk parity portfolios and provide a comparison with other risk control techniques. Roncalli [RON 14] provides a good discussion of risk control techniques and Lee [LEE 11] critically evaluates risk parity (see also section 1.10). Asness *et al.* [ASN 12] document the empirical outperformance of a risk parity strategy over a

market cap weighted portfolio and refer to the leverage aversion effect to explain this outperformance. Anderson *et al.* [AND 12] critically review and refute the empirical evidence provided by Asness *et al.* [ASN 12].

Recipe: The ERCP rests on the premise that no asset should dominate the portfolio risk profile. Consequently, all assets' contributions to portfolio risk are equalized. The contribution of an asset to portfolio risk equals its investment weight multiplied with its marginal contribution to portfolio risk. An asset's marginal contribution to portfolio risk is proportional to its beta with respect to the portfolio. Hence, the weights of the ERCP satisfy:

$$w_i \frac{\partial \sigma_p}{\partial w_i} = \frac{\partial \sigma_p}{\partial w_j} w_j \Leftrightarrow w_i \beta_{ip} = w_j \beta_{jp} \quad [1.10]$$

Consequently, the weights in the ERCP are proportional to the inverse of the corresponding betas:

$$w_i^{ERC} \propto 1 / \beta_{ip} \quad [1.11]$$

For algorithms to calculate the composition of the ERCP, we refer to [RON 14].

Characteristics: (1) The ERCP is the $1/N$ portfolio when all assets have the same volatility σ and when all pairwise correlations are uniform at ρ . After all, in this case, equation [1.10] implies that $w_i \sigma \rho = w_j \sigma \rho$, which is satisfied for $w_i = w_j = 1/N$. (2) The ERCP is the *MDP* when all correlations are uniform: $\rho_{ip} = \rho_{jp}$. (3) The ERCP is the *MVP* when correlations are uniform (pairwise equal) and at their theoretically lowest level of $\rho = -1/(N-1)$, see [MAI 10]. (4) The ERCP is the *MSRP* when all correlations are uniform and all assets have the same SR, see [MAI 10]. (5) With only two assets, their correlation is not relevant and the ERCP equals the *IVP* (see section 1.8).

Evaluation: (1) "Risk" is usually equated with standard deviation of return (volatility), but in principle any other risk measure can be chosen as long as the risk measure is linearly homogeneous in the portfolio weights. This means that when multiplying all portfolio weights with a constant c , the risk measure is also multiplied by the same constant c . Portfolio loss, VaR and conditional VaR (or expected tail loss) satisfy this property (see, for example, [HAL 03]). (2) Since we can rewrite beta as the product of (i) the correlation with the portfolio and (ii) the quotient of the asset and portfolio volatility, so $\beta_{ip} = \rho_{ip} \sigma_i / \sigma_p$, equation [1.11]

implies that ERCPs favor assets with low levels of volatility and low correlations with other assets (hence, “portfolio diversifiers”).

(3) The ERCP is perfectly diversified in terms of risk (loss) contributions. (4) The ERCP is less concentrated than the MVP and the MDP, and it contains all N assets. (5) The ERCP is more robust, i.e. less error maximizing, than the MVP. The intuitive reason is that the MVP is found by means of optimization, i.e. by equating marginal risk contributions, whereas the ERCP is found by a restriction on the product of weights and marginal risk contributions. (6) Maillard *et al.* [MAI 10] show that $\sigma_{MVP} \leq \sigma_{ERC} \leq \sigma_{1/N}$, where the MVP is error maximizing and the 1/N portfolio focuses on money allocation, and not on risk allocation. Hence, the *ex ante* volatility of the ERCP is between the lowest level (from the MVP) and the volatility of the naively diversified 1/N portfolio. (7) Calculating the ERCP is a daunting task when the number of assets is very large. A solution would be to resort to a hierarchical procedure in which risk parity is first applied within groups (e.g. sectors and countries) and next across groups. However, pregrouping directly influences the ERCP, see below under point (9).

Panel A: ERC (4)	Eq	Tsies	IG	HY	ERCP
<i>Weight</i>	11%	55%	20%	14%	100%
<i>Beta</i>	2.19	0.45	1.28	1.77	
<i>% Risk contribution</i>	25%	25%	25%	25%	100%
<i>Implied risk premium</i>	3.07	0.64	1.79	2.48	1.40
<i>Implied Sharpe ratio</i>	0.20	0.15	0.30	0.24	0.33
Panel B: ERC (3)	Eq	Tsies	IG+HY		
<i>Weight</i>	15%	60%	25%		100%
<i>Beta</i>	2.21	0.56	1.34		
<i>% Risk contribution</i>	33%	33%	33%		100%
<i>Implied risk premium</i>	2.86	0.72	1.74		1.30
<i>Implied Sharpe ratio</i>	0.19	0.17	0.27		0.32

Table 1.10. Risk attribution with respect to ERCP, and implied risk premia (%) and Sharpe ratios

(8) Table 1.10, Panel A, shows the composition of the ERCP. Note the large 55% weight of Tsies, this is due to both their low volatility and negative correlation with equities and HY. The high volatility of equities implies a lower than 25% weight. The implied risk premia and SRs can be interpreted as before. (9) The composition of the ERCP depends on choosing the number of assets N and hence on

any pregrouping of assets (see [LEE 11]). ERCP(4) is on the basis of the four original assets. When aggregating IG and HY into a single credits subportfolio, ERCP(3), the risk allocations shift from 25 to 33%; see Table 1.10, Panel B. (10) Leverage is needed to boost the low risk and return of ERCP in order to match any risk budgets or return targets.

(11) Turning to historical portfolio statistics, Table 1.5 shows that the ERCP had about half the risk of the market cap portfolio paired with a quarter lower average return, yielding a 50% higher SR. This is due to overweighting Tsies and underweighting equities (see Table 1.10). The substantial tail wind of bonds over the past decades seriously biases backtests of ERCPs.

In their empirical study, Asness *et al.* [ASN 12] illustrate the historical outperformance of ERCPs (or IVP since they consider only two asset classes, US equity and bonds) over a market cap weighted portfolio over the period 1926–2010. As an explanation, they raise leverage aversion as the driving force behind the performance of ERCPs. This mechanism works as follows. (Some) investors are averse (or restricted) to applying leverage and they bid up the prices of high risk/high beta assets in order to fill their risk budget. As a result, the risk premium offered on high-risk assets is reduced, implying that low beta (risk) assets offer higher risk-adjusted returns and high beta (risk) assets offer lower risk-adjusted returns. A less than average leverage-averse (or constrained) investor can benefit from overweighting low beta (low risk) assets and underweighting high beta (high risk) assets. Leverage is applied to fill the risk budget or to attain a targeted risk level. In addition to leverage aversion, the “lottery ticket effect” may be at work, in which investors with a propensity to “gamble” overbid for high-risk assets, thus reducing their risk premium. Finally, delegated portfolio management, centered around benchmarked portfolios, implies that low (high) risk stocks have a large (small) tracking error. As argued by Blitz and van Vliet [BLI 07], this introduces the low volatility anomaly, implying a flat or negative risk-return trade-off. Since low volatility assets outperform and ERCPs overweight low-risk assets, this may explain their outperformance.

Anderson *et al.* [AND 12] raise some serious backtest issues in the research by Asness *et al.* [ASN 12]. They note that the outperformance of the ERCP is not uniform over subperiods and they show that market frictions (borrowing costs and turnover-induced trading costs) eat into performance. In addition, they argue that Asness *et al.*'s [ASN 12] risk parity strategy is not an investable strategy since it uses unconditional leverage: they use a constant scale factor, computed from the full 1926–2010 period, to match the volatilities of the levered risk parity strategy and the market cap portfolio. Hence, their empirical setup suffers from a look-ahead bias. Implementing conditional leverage (where at each rebalancing date the volatility scale factor is derived from past 3 year trailing windows), Anderson *et al.* [AND 12]

show that this halves the cumulative total return of the risk parity strategy as reported by Asness *et al.* [ASN 12]. Realistic borrowing costs and trading costs further reduce the cumulative total return of the risk parity strategy. In all, these realistic adjustments made the performance difference between the risk parity strategy and the market cap portfolio disappear.

1.8. Inverse volatility portfolio (IVP): naive risk parity

Maillard *et al.* [MAI 10] discuss the IVP next to the ERCP, although volatility weighting (or “normalization”) has been applied for long by practitioners to improve cross-asset comparability and to reduce portfolio or strategy risk (this may be inspired by statistics, where inverse variance weighting is used to minimize the variance of the sum of two or more random variables).

Recipe: Set each weight proportional to the stand-alone volatility of the corresponding asset and next normalize so that the weights sum to unity. This volatility-weighting in the cross-section yields:

$$w_i = \frac{\frac{1}{\sigma_i}}{\sum_j \frac{1}{\sigma_j}} \quad [1.12]$$

The IVP is equivalent to the *ERCP* when there are only two assets (in the two-asset case, the correlation is irrelevant.) In all other cases, neglecting correlation information makes IVP a “naive” risk parity strategy.

Characteristics: (1) When correlations are uniform (or zero), the IVP is the *ERCP*. (In this case, all comments made for ERCPs also apply for IVPs). When everything else is equal, then compared to the IVP, the ERCP will be tilted toward low correlated assets. (2) When volatilities are uniform, the IVP is the *1/N* portfolio.

The S&P Low Volatility Index is composed of the 100 stocks from the S&P 500 index with the lowest (252 days past) volatility, where each stock is weighted with its IVP. The MSCI Risk Weighted Indices use inverse variance (and not volatility) to weight constituents. (3) Inverse variance weighting yields the *MVP* when all correlations are uniform (or zero).

Evaluation: Except for the impact of (markedly different) correlations, IVPs are quite similar to ERCPs. As shown in Table 1.11, the IVP assigns more weight to IG (was 20%) and less weight to Tsies (was 55%). The latter can be explained because the IVP ignores the negative correlation with equities and HY. This shift in weights translates into less balanced risk contributions.

	<i>Eq</i>	<i>Tsies</i>	<i>IG</i>	<i>HY</i>	<i>IVP</i>
<i>Weight</i>	12%	42%	29%	17%	100%
<i>Beta</i>	2.12	0.29	1.15	1.76	
<i>% Risk contribution</i>	25%	12%	33%	29%	100%
<i>Implied risk premium</i>	3.80	0.52	2.06	3.16	1.79
<i>Implied Sharpe ratio</i>	0.25	0.13	0.34	0.30	0.38

Table 1.11. Risk attribution with respect to IVP, and implied risk premia (%) and Sharpe ratios

On a historical basis, Table 1.5 shows that the IVP had a higher volatility and a somewhat higher average return than the ERCP. This combined effect is due to the lower weight of Tsies (which have the lowest average return, the lowest volatility, and negative correlations with equities and HY).

1.9. Volatility weighting over time

The risk control strategies as discussed before focus on risk in the cross-section, i.e. over portfolio constituents. Risk control at each point in time will also affect the portfolio's risk level (or more generally, its return distribution) over time. Volatility weighting over time, and specifically volatility targeting, is designed to explicitly control the portfolio risk level over time. Typically, when weighting or targeting a portfolio's risk level over time, the composition of a portfolio's risky part is not changed, only the weights of the risky and the risk-free part are adjusted.

Fleming, Kirby and Ost diek [FLE 01] document the empirical finding that volatility weighting improves the SR. Hallerbach [HAL 12] provides a formal proof that, under mild assumptions, volatility weighting over time indeed increases the SR or information ratio.

Recipe: (1) Set the risky portfolio's target volatility level V . (2) At the start of each period t , take a position w_t in the risky portfolio and $(1-w_t)$ in the risk-free asset with return r_{ft} . Since \tilde{r}_{pt} is the risky portfolio's excess return, this yields:

$$w_t \cdot \tilde{r}_{pt} + r_{ft} \quad [1.13]$$

(3) Estimate the volatility $\hat{\sigma}_t$ of the risky portfolio for period t , for example by using an adaptive exponentially weighted moving average (EWMA) volatility process. (4) Rescale the exposure to the risky portfolio to the target volatility level V by setting $w_t = V / \hat{\sigma}_t$. According to equation [1.13], this implies adding a cash position or borrowing (when allowed) at the suitable borrowing rate, subject to a leverage constraint. (5) Apply the leverage constraint. When the volatility target V is high or when the forecasted volatility is low, cap the implied borrowing by setting $w_t \leq L$, where the maximum leverage ratio satisfies $L \geq 1$. When $L = 1$, no borrowing is allowed.

Characteristics: (1) Volatility weighting and volatility targeting accomplish volatility smoothing over time. Volatility smoothing mitigates the volatility of the portfolio volatility over time. It can be shown that the lower the fluctuations of the temporal (“instantaneous”) portfolio volatility *within* some time period, the lower the aggregate volatility *over* the whole time period (this is a convexity effect, see [HAL 12]). (2) Note that volatility smoothing is different from return smoothing. Return smoothing aims at achieving a lower aggregate level of return volatility (and not a lower volatility of the volatility over time). Return smoothing thus implies less “variance slippage” in compounded returns. This variance slippage refers to the difference between the arithmetic mean and the geometric mean return. As an approximation, we have geometric mean \approx arithmetic mean $- \frac{1}{2}$ variance. Lowering the return variance by return smoothing thus increases the geometric mean of returns, *cet. par.* (3) Naive risk parity or the *IVP*, i.e. volatility weighting in cross-section, already establishes some volatility weighting in time-series. (4) Risk targeting or risk control indices have been introduced by S&P Dow Jones, MSCI, FTSE, and EURO STOXX.

Evaluation – or: why would volatility targeting work? (1) First, depending on the quality of our volatility forecasts, we should be able to target a portfolio’s volatility to some degree over time. (2) In addition, it can be shown that this volatility smoothing increases the SR or information ratio of the portfolio, *cet. par.* [HAL 12]. (3) Furthermore, the (risk-adjusted) return of a volatility-targeted portfolio benefits from an additional timing effect, due to the so-called asymmetric volatility phenomenon. The asymmetric volatility phenomenon is a stylized fact that is observed for most financial markets. Returns tend to be negatively correlated with volatility and especially surges in financial market volatility are mostly associated with negative returns. The volatility feedback mechanism is that higher expected volatility translates into a higher risk premium and consequently negative realized returns. Hence, under asymmetric volatility, there is a timing effect (in addition to the convexity effect of smoothing of volatility) that will boost performance. After

all, a volatility-weighting strategy takes large positions when volatility is low (and returns are high) and small positions when volatility is high (and returns are low); see also [ZAK 14]. (4) As a cautionary (and perhaps superfluous) note, we stress that implementing a volatility-weighted strategy calls for a strict risk-budgeting and risk-monitoring process. In particular, we may want to set limits to the maximum position size in order to mitigate the risk of blow-ups when the contemporaneous volatility is relatively low.

1.10. Evaluation

Conventional 60/40 portfolios or MSRPs are concentrated in risks and fail to offer diversification against losses. For this reason, the use of risk control techniques (and especially risk parity) as full-fledged investment criteria is sometimes coined the “new paradigm” in investing [INK 11, LEE 11, LEO 12, GOL 13, RON 14]. Indeed, risk control strategies, and risk parity in particular, can produce balanced portfolios and can offer various degrees of diversification. From a risk perspective, these techniques are indeed expected to deliver what they promise. The true value of risk control strategies is in analyzing and specifying a preferred risk contribution profile within the portfolio. This should be a part of any risk-budgeting process. Relevant questions are: what are the risk contributions of the portfolio components? Is the portfolio properly diversified or are there any hot spots? How much confidence do we have in risk premia views in order to shift risk contributions within the portfolio? Do we fully understand the sources and contributions of risk and return of the portfolio? And last but not least, risk is a multidimensional concept, so risk analyses should not only focus on volatility (standard deviation) but also take downside risk and event risk in consideration.

The catch is that risk control portfolios appear to have historically outperformed market cap weighted or mean-variance optimized portfolios. So, while ignoring risk premia information, risk control strategies seem to offer a better risk-return trade-off. Some critical comments are in place, however. First, several studies tune down the apparent outperformance of risk-based strategies by criticizing backtests, see section 1.7 and [GOL 13]. Second, when the underlying mechanism of outperformance is an implicit exposure to anomalies or factor premia such as value, size, low beta or low (residual) volatility (as shown by [SCH 11], [LEO 12] and [JUR 15]), then it makes much more sense to consider these factor exposures explicitly when forming portfolios. Factor investing [ANG 14] provides much more efficient and effective ways to tailor factor exposures on the portfolio level than applying risk control techniques. In the former case, factor exposures are taken intentionally and top-down, whereas in the latter case it is not

clear what factor exposures will percolate bottom-up and reveal themselves in the portfolio.

However, aside from their obvious value in a risk-budgeting context, risk control strategies can provide a sensible heuristic or starting point in the portfolio formation process if there is considerable uncertainty about the required asset attributes, *viz.* the risk premia and (co-) variance inputs. When the available information on an attribute allows for meaningfully differentiating between assets, then the portfolio formation process is steered by this attribute. Alternatively, when the required information on an attribute is lacking (or surrounded by substantial estimation error) then this attribute does not have discriminating power between assets. Consequently, it then makes sense to treat the assets as substitutes regarding this attribute. Indeed, we know that mean-variance optimized portfolios are error-maximizing [MIC 89] in the sense that their composition is very sensitive to especially risk premia inputs. So, the presence of estimation risk can justify the use of risk control techniques: when we do not have information to meaningfully differentiate between the assets' risk premia (or SRs), the recipe is to treat all assets as risk premia "substitutes" and focus only on their risk attributes.

To further illustrate this point, we introduce our portfolio decision pyramid, see Figure 1.1. This pyramid illustrates the increasing requirements that apply to portfolio optimization inputs when moving from a naively diversified portfolio to the MSRP. (1) Starting at the bottom of this inverted pyramid, we cannot indicate any meaningful differences among risk premia, standard deviations and correlations. The best we can do is to naively diversify and equate money weights within the portfolio, yielding the 1/N portfolio. (2) Having reliable trust in differences among standard deviations allows for shifting from naive money weight diversification to naive risk weight diversification by applying volatility-weighting. This yields the IVP. (3) On the third level, we have full risk information (reliable estimates of both volatilities and correlations, the full covariance matrix is available), so the MVP, MDP or ERCP (full risk parity portfolios) can be constructed. Of course, we have to take into account the relative shortcomings of these portfolios as noted in sections 1.5–1.7. Finally, at the top level, we are able to indicate meaningful differences between all relevant inputs, *i.e.* the *ex ante* covariance matrix and risk premia. In this case, we can perform a full-fledged mean-variance optimization and obtain the MSRP.

Of course, estimation risk does not necessarily dictate to ignore risk premia inputs altogether and to stop at a risk control portfolio. As outlined in sections 1.2 and 1.11, estimation risk surrounding risk premia can be tackled by the Black and Litterman [BLA 92] approach. We start from a risk control reference portfolio, calculate the implied risk premia and next use our views and the confidence we place in these views to (slightly) adjust the optimization inputs (for a detailed

exposition, see [HAE 14]). Depending on our convictions regarding risk premia, the resulting portfolio is situated between the risk control portfolio and MSRP.

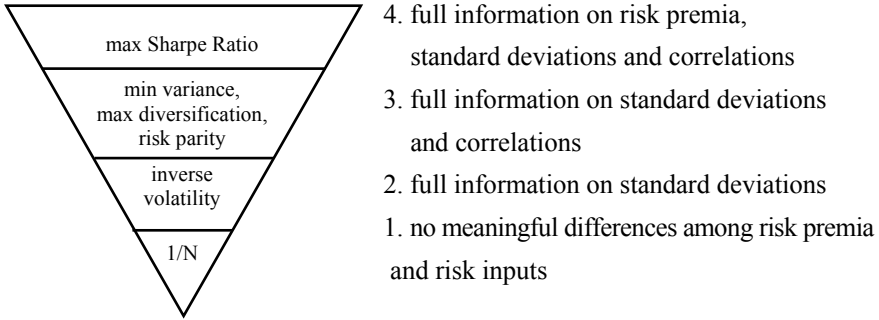


Figure 1.1. *The portfolio decision pyramid: the link between portfolio rules and the information burden placed on the investor when adopting that rule*

1.11. Appendix

1.11.1. Asset and portfolio (excess) returns

We start with an opportunity set of N securities with excess returns over the risk-free rate denoted by \tilde{r}_{it} . Tildes indicate random variables. For notational simplicity, we henceforth ignore the time index t . We consider a portfolio p defined by the investment weights $\{w_i\}_{i \in p}$, satisfying full investment $\sum_{i=1}^N w_i = 1$ and no short positions: $w_i \geq 0, \forall i \in p$. The portfolio excess return is given by:

$$\tilde{r}_p = \sum_i w_i \tilde{r}_i \tag{1.14}$$

1.11.2. Marginal and component contributions to portfolio (excess) return

It follows from equation [1.14] that the marginal contribution of asset i to portfolio excess return is given by \tilde{r}_i . This is the increase in portfolio excess return when the weight of asset i is increased marginally. The component (i.e. full) contribution of asset i to portfolio excess return is $w_i \tilde{r}_i$. The sum of component contributions to excess return equals the portfolio's excess return, see equation [1.14].

1.11.3. Portfolio risk premium

The average portfolio return over the risk-free rate, the portfolio risk premium, follows as:

$$\bar{r}_p = \sum_{i \in p} w_i \bar{r}_i \quad [1.15]$$

The marginal and component contributions of asset i to the portfolio risk premium are \bar{r}_i and $w_i \bar{r}_i$, respectively.

1.11.4. Portfolio variance

The variance of portfolio excess returns is defined by the double sum:

$$\sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_{ij} \quad [1.16]$$

By definition of the correlation ρ_{ij} , the covariance σ_{ij} can be expressed as $\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$.

Since (1) the variance of a variable is the covariance of that variable with itself and (2) the covariance is a linear operator (the covariance of a weighted sum is the weighted sum of covariances), we can write the variance of the portfolio excess return as:

$$\text{var}(\tilde{r}_p) \equiv \text{cov}(\tilde{r}_p, \tilde{r}_p) = \text{cov}\left(\sum_i w_i \tilde{r}_i, \tilde{r}_p\right) = \sum_i w_i \text{cov}(\tilde{r}_i, \tilde{r}_p) = \sum_i w_i \sigma_{ip}, \quad [1.17]$$

where σ_{ip} is the covariance between the excess returns on asset i and the portfolio p . So, although the portfolio variance is the quadratic sum of weights and covariances, we can express the portfolio variance as the weighted sum of the covariances of each asset with the portfolio: $\sigma_p^2 = \sum_i w_i \sigma_{ip}$.

1.11.5. Decomposing portfolio volatility

Dividing the previous expression by the portfolio volatility, we get:

$$\sigma_p = \sum_i w_i \frac{\sigma_{ip}}{\sigma_p} = \sum_i w_i \sigma_p \rho_{ip} \quad [1.18]$$

Indeed, it is not the decomposition of portfolio variance we are looking for, but the decomposition of portfolio volatility, as defined by equation [1.18]. To see why this is true, note that the portfolio volatility is linearly homogeneous in the portfolio weights: multiplying portfolio weights with a constant k multiplies the portfolio volatility with the same constant k . Euler's theorem then implies that $\sigma_p = \sum_i w_i \partial \sigma_p / \partial w_i$, where it can be checked from [1.17] that $\partial \sigma_p / \partial w_i = \sigma_{ip} / \sigma_p$. The term $\partial \sigma_p / \partial w_i$ is the marginal contribution of asset i to portfolio volatility. The term $w_i \sigma_{ip} / \sigma_p$ is the component contribution of asset i to portfolio volatility. The sum of all component contributions to volatility equals total portfolio volatility, see equation [1.18]. The portfolio volatility is the cake and each component contribution is a separate piece of that cake. Dividing [1.18] by σ_p yields the relative risk contributions of the assets, summing to 100%:

$$1 = \sum_i w_i \frac{\sigma_{ip}}{\sigma_p} \quad [1.19]$$

To gain further insight into this decomposition, consider the Ordinary Least Squares (OLS) time-series regression of asset i 's excess returns on the portfolio excess returns:

$$\tilde{r}_i = \alpha_i + \beta_{ip} \tilde{r}_p + \tilde{\varepsilon}_i. \quad [1.20]$$

In this regression, the expected (or average) value of the disturbances is zero and the disturbances and the portfolio excess return are uncorrelated, hence $E(\tilde{\varepsilon}_i) = E(\tilde{r}_p \tilde{\varepsilon}_i) = 0$. The regression slope or beta is defined as:

$$\beta_{ip} = \frac{\sigma_{ip}}{\sigma_p^2} = \rho_{ip} \frac{\sigma_i}{\sigma_p} \quad [1.21]$$

Substituting the expression for beta in [1.19] gives:

$$1 = \sum_i w_i \beta_{ip} \quad [1.22]$$

So, β_{ip} is the relative marginal contribution of asset i to portfolio volatility (or the relative marginal risk contribution):

$$\beta_{ip} = \frac{\partial \sigma_p / \sigma_p}{\partial w_i} \quad [1.23]$$

and $w_i\beta_{ip}$ is the asset's relative component contribution to portfolio volatility. So, given the assets' betas, the decomposition of portfolio volatility is a piece of cake. When $w_i\beta_{ip}$ is comparatively large, this identifies a "hot spot" in the portfolio, or a pocket of risk concentration, indicating that asset's i contribution to portfolio risk is large. Hence, this position is likely to contribute heavily to any loss that may be realized on the portfolio. In short, $\{w_i\}$ defines money allocation and $\{w_i \cdot \beta_i\}$ defines risk allocation. To go from money allocation to risk allocation, each investment weight is multiplied with the corresponding beta (note that the average value of beta is unity).

1.11.6. Portfolio optimality: maximize the Sharpe ratio

From equation [1.20], it follows that the expected excess return or risk premium of asset i is related to the portfolio's risk premium as:

$$\bar{r}_i = \alpha_i + \beta_{ip}\bar{r}_p \quad [1.24]$$

Now, consider the mean-variance optimal portfolio, this is the portfolio that maximizes the SR:

$$\max_{\{w_i\}_{i \in p}} SR_p = \frac{\bar{r}_p}{\sigma_p} \quad [1.25]$$

The first-order conditions for optimality imply the following relation between risk premia and betas:

$$\bar{r}_i = \beta_{ip}\bar{r}_p \quad [1.26]$$

In other words, for the MSRP, the risk premia of all constituents are proportional to their betas. Considering equation [1.24], this implies that for all assets included in the MSRP p^* , the alpha α_i equals zero, $\alpha_i = 0 \quad \forall i \in p^*$. To provide some intuition, note that for each asset comprised in an MSRP the relative marginal contribution to excess return must equal the relative marginal contribution to risk, or:

$$\frac{\bar{r}_i}{\bar{r}_p} = \beta_{ip} \quad [1.27]$$

This can be rephrased as requiring equal ratios of marginal return and risk contributions⁵:

$$\frac{\bar{r}_i}{\beta_{ip}} = \frac{\bar{r}_j}{\beta_{jp}} = \bar{r}_p \quad [1.28]$$

If this does not hold, the SR of the portfolio can be improved by increasing the weight of the assets with higher contributions to return (or lower contributions to risk) and decreasing the weight of assets with lower contributions to return (or higher contributions to risk).

In other words, referring to equation [1.24], when an asset's alpha is positive, $\alpha_i > 0$, this asset shows outperformance against the portfolio and the SR of the portfolio can be increased by increasing the weight of this asset. Conversely, when an asset's alpha is negative, $\alpha_i < 0$, this asset shows underperformance and the portfolio's SR can be increased by decreasing the weight of this asset.

Two additional comments are in order. First, in equation [1.26], we recognize the infamous "Security Market Line" of the capital asset pricing model (CAPM). However, the above results apply to *any* MSRP, whereas the CAPM applies to the equally infamous market portfolio (the overall global market cap weighted portfolio containing all assets) under the heroic equilibrium assumption that this portfolio is mean-variance efficient. Hence, the results presented above are completely general.

Second, using the second definition of beta in equation [1.21] allows us to rewrite [1.26] as $\bar{r}_i / \sigma_i = \rho_{ip} \bar{r}_p / \sigma_p$. Using the definition of the SR, this boils down to:

$$SR_i = \rho_{ip} SR_p \quad [1.29]$$

In other words, for any MSRP, any constituent's stand-alone SR equals the product of (1) its correlation with this portfolio and (2) the SR of the portfolio. Because of diversification contribution, a weakly correlated asset can have a lower SR. Conversely, the SR of a perfectly correlated asset should match the portfolio's SR. When an asset's SR is larger (smaller) than given by [1.29], this implies that the asset's alpha is positive (negative). This also applies to assets not comprised in the portfolio. If $\alpha_i > 0$, or equivalently $SR_i > \rho_{ip} SR_p$, then the SR of the portfolio is increased by adding that asset to the portfolio (and *vice versa*).

⁵ Note that \bar{r}_i / β_{ip} is the Treynor's [TRE 66] ratio of risk-adjusted performance.

1.11.7. Reverse optimization: implied risk premia

In conventional mean-variance portfolio optimization, the asset's risk premia and their covariance matrix are used to calculate the weights of the MSRP. In reverse portfolio optimization, it is assumed that the portfolio at hand actually *is* the MSRP. Together with the covariance matrix of excess returns, this allows us to derive the “imputed” risk premia [SHA 74]. Using these implied risk premia together with the asset's standard deviations, we can then compute the implied SRs. Hence, given a particular portfolio, these implied risk premia (or implied SRs) would make this portfolio the MSRP.

How do we derive these implied risk premia? We start from the historical risk premium of the market cap portfolio. Assuming that this portfolio p^* is mean-variance efficient, we can calculate the implied coefficient of relative risk aversion λ^* from $\lambda^* = \bar{r}_{p^*} / \sigma_{p^*}^2$ [SHA 74]. Using the historical average excess return and volatility of the market cap portfolio in Table 1.1, this yields $\lambda^* = 8.0$. Switching to an alternative portfolio p with historical volatility σ_p , then assuming that this portfolio is mean-variance efficient implies that its corresponding risk premium is $\bar{r}_p^* = \lambda^* \cdot \sigma_p^2$. Given this implied portfolio risk premium, we finally use the first-order condition for the MSRP in equation [1.26] together with the asset betas to compute the implied risk premium \bar{r}_i^* as the product of the beta and the implied portfolio risk premium:

$$\bar{r}_i^* = \beta_{ip} \cdot \bar{r}_p^* \quad [1.30]$$

The implied SR then readily follows as $SR_i^* = \bar{r}_i^* / \sigma_i$.

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Smart Beta: Managing Diversification of Minimum Variance Portfolios

In this chapter, we consider a new framework for understanding risk-based portfolios (global minimum variance (GMV), equally weighted (EW), equal risk contribution (ERC) and most diversified portfolio (MDP)). This framework is similar to the constrained minimum variance model of Jurczenko *et al.* [JUR 15], but with another definition of the diversification constraint. The corresponding optimization problem can then be solved using the cyclical coordinate descent (CCD) algorithm. This allows us to extend the results of Cazalet *et al.* [CAZ 14] and to better understand the trade-off relationships between volatility reduction, tracking error and risk diversification. In particular, we show that the smart beta portfolios differ because they implicitly target different levels of volatility reduction. We also develop new smart beta strategies by managing the level of volatility reduction and show that they present appealing properties compared to the traditional risk-based portfolios.

2.1. Introduction

The capital asset pricing model (CAPM) of Sharpe [SHA 64] and the empirical study of Jensen [JEN 69] have been the backbone of passive management based on capitalization-weighted (CW) portfolios. In this approach, there is a single market risk premium, measured by the beta, and this risk premium compensates investors for holding non-diversifiable risk. In the CAPM theory, an investor can capture the market risk premium by holding the market portfolio. Applied to the universe of stocks, this justifies the strong development of CW equity indices. But, since CAPM was introduced, academic research has put forward convincing evidence that CW portfolios are poorly diversified (criticism 1) and there are systematic sources of

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return in the equity markets other than simply the market beta (criticism 2). This justifies the strong development of smart beta in recent years. In fact, the term “smart beta” refers to two different approaches. First, it includes alternative-weighted portfolios, whose purpose is to be more diversified than CW portfolios. This smart beta approach corresponds to criticism 1 and it is also known as risk-based investing. Second, it refers to portfolios that are designed intentionally to capture alternative risk premia other than the market risk premium, such as value, size, momentum, low beta and quality. This second approach corresponds to criticism 2 and it is also known as factor-based investing.

In this chapter, we focus on risk-based portfolios¹. The main objective of this smart beta approach is to manage the risk more effectively than a CW index, and achieve a better performance. At first sight, risk-based portfolios seem to be heterogeneous because there are several notions of risk and each method considers one specific aspect of diversification. However, we can show that both approaches aim to reduce the volatility compared to the CW portfolio. This means that they are the solution to a minimum variance optimization problem, but with a different weight constraint. Our chapter highlights, therefore, the central role of the minimum variance portfolio. Nevertheless, it is impossible to define a unique minimum variance portfolio. In fact, there are as many minimum variance portfolios as there are smart beta products. In this situation, it is essential to have some metrics in order to understand their differences. Using a global optimization program, we can measure the different trade-off relationships between volatility reduction, tracking error, weight diversification and risk concentration. In particular, we can show that these minimum variance portfolios behave differently because they do not target the same volatility reduction. Some of them are very aggressive, whereas others are closer to the CW portfolio. But, once we impose the same level of volatility reduction, the differences between smart beta portfolios vanish even if they consider different weight constraints.

In risk-based investing, the key variable is then the level of volatility reduction. Because the objective of the investor is (almost) always to obtain a better performance, the choice of this parameter is crucial. This is why we also investigate how the performance of the portfolio is related to the volatility reduction. We show that this relationship depends strongly on the level of the market risk premium. Using this result, we can then build minimum variance strategies by targeting a time-varying volatility reduction, which depends on the market conditions.

The chapter is divided as follows. In section 2.2, we show how the different risk-based portfolios can be cast in a minimum variance problem. In section 2.3, we propose a unique optimization program in order to compare the diversification profile

¹ Even if the boundary between risk-based investing and factor-based investing is blurred in practice.

of smart beta strategies. Section 2.4 then analyzes their behavior and proposes new smart beta strategies by dynamically managing the objective of volatility reduction. Section 2.5 offers some concluding remarks.

2.2. Risk-based investing and variance minimization

Risk-based investing is generally associated with the concept of diversification. Because diversification cannot be measured by a single number, practitioners consider different approaches. The most popular are the equally weighted (EW) portfolio, the equal risk contribution (ERC) portfolio and the most diversified portfolio (MDP). Each of these portfolios maximizes a diversification measure. For instance, the MDP uses the diversification ratio. The EW portfolio minimizes the concentration in terms of weights, whereas the ERC portfolio minimizes the concentration in terms of risk contributions.

Besides these three risk-based approaches, practitioners also consider the minimum variance (MV) portfolio². In this case, the goal is to explicitly manage the volatility rather than the diversification of the portfolio. But, as shown by Maillard *et al.* [MAI 10], the EW and ERC portfolios can also be interpreted as constrained MV portfolios. Jurczenko *et al.* [JUR 13] proposed a similar approach, which also encompasses the MDP. In particular, they consider the following optimization problem:

$$\begin{aligned}
 x^*(\delta, \gamma) &= \arg \min \frac{1}{2} x^\top \Sigma x & [2.1] \\
 \text{u.c. } &\begin{cases} \sum_{i=1}^n \sigma_i^\delta (x_i^{1-\gamma} - 1) \geq c \\ 1^\top x = 1 \end{cases}
 \end{aligned}$$

where Σ is the covariance matrix of asset returns and σ_i is the volatility of asset i . In this optimization program, $\delta \geq 0$ and $\gamma \geq 0$ are two given parameters and c is a scalar to be determined. They obtain the following correspondence between the parameters (δ, γ) and the risk-based portfolios $x^*(\delta, \gamma)$:

Portfolio	GMV	EW	ERC	MDP
δ	0	0	0	1
γ	0	∞	1	0

In what follows, we consider an extension of the original optimization problem of Maillard *et al.* [MAI 10]. Our model is related to the approach of Cazalet *et al.* [CAZ 14] and helps us to understand that risk-based portfolios are in fact

² We note GMV the long-only global (or unconstrained) MV portfolio. This portfolio plays a special role in limit cases of portfolio optimization.

minimum variance portfolios with a diversification constraint. The goal of risk-based portfolios is then to reach a lower volatility than the volatility of the capitalized-weighted portfolio. However, because each approach considers a specific definition of the diversification, there is a trade-off between these different measures of diversification.

2.2.1. MV portfolio

Global minimum variance (GMV) portfolios are never used by practitioners because they correspond to mathematical corner solutions that are concentrated in a few number of assets. This is why minimum variance portfolios are always implemented by considering a constrained optimization problem:

$$x^* = \arg \min \frac{1}{2} x^\top \Sigma x \quad [2.2]$$

$$\text{u.c.} \begin{cases} x \in \mathcal{C} \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$

The constraints $x \geq \mathbf{0}$ and $\mathbf{1}^\top x = 1$ imply that the portfolio is long-only. The management of the weight concentration is specified by the constraint $x \in \mathcal{C}$. There are of course different ways to specify \mathcal{C} . One of the popular approaches consists of using the Herfindahl index defined by:

$$\mathcal{H}(x) = \sum_{i=1}^n x_i^2$$

$\mathcal{H}(x)$ takes the value 1 if the portfolio is perfectly concentrated in one asset. Conversely, $\mathcal{H}(x)$ takes the value $1/n$ if the portfolio is EW. We can, therefore, define the weight diversification as:

$$\mathcal{D}_w(x) = \frac{\mathcal{H}^{-1}(x)}{n} = \frac{1}{n \sum_{i=1}^n x_i^2} \quad [2.3]$$

Using this diversification definition, the previous optimization problem becomes:

$$x^*(c) = \arg \min \frac{1}{2} x^\top \Sigma x \quad [2.4]$$

$$\text{u.c.} \begin{cases} \mathcal{D}_w(x) \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$

with $c \in [1/n, 1]$. We have $x^*(1/n) = x_{\text{gmv}}$ and $x^*(1) = x_{\text{ew}}$. Because $\sigma(x^*(c))$ is an increasing function of the parameter c , we deduce that:

$$\sigma(x_{\text{gmv}}) \leq \sigma(x^*(c)) \leq \sigma(x_{\text{ew}})$$

REMARK 2.1.– We notice that the optimization program [2.4] is equivalent to solving this Lagrange problem:

$$\begin{aligned} y^*(\lambda) &= \frac{1}{2} y^\top \Sigma y + \lambda y^\top y & [2.5] \\ \text{u.c. } &\begin{cases} \mathbf{1}^\top y = 1 \\ y \geq \mathbf{0} \end{cases} \end{aligned}$$

with $\lambda \geq 0$. In this case, the optimal solution $x^*(c)$ is equal to $y^*(\lambda)$ with the following relationship:

$$c = \frac{1}{n \sum_{i=1}^n y_i^*(\lambda)^2}$$

If $c \leq c_{\text{gmv}} = (n x_{\text{gmv}}^\top x_{\text{gmv}})^{-1}$, $x^*(c) = x_{\text{gmv}}$.

2.2.2. ERC portfolio

Let $\sigma(x) = \sqrt{x^\top \Sigma x}$ be the portfolio volatility. The risk contribution of asset i is defined by:

$$\mathcal{RC}_i = x_i \cdot \frac{\partial \sigma(x)}{\partial x_i}$$

These risk contributions are a key when performing risk allocation because the sum of risk contributions is exactly equal to the portfolio volatility:

$$\sum_{i=1}^n \mathcal{RC}_i = \sigma(x)$$

In the ERC portfolio, the risk contributions are the same for all assets:

$$\mathcal{RC}_i = \mathcal{RC}_j$$

Maillard *et al.* [MAI 10] show that the ERC portfolio can be found by using the following optimization program:

$$y^*(c') = \arg \min \frac{1}{2} y^\top \Sigma y$$

$$\text{u.c. } \begin{cases} \sum_{i=1}^n \ln y_i \geq c' \\ y \geq \mathbf{0} \end{cases}$$

where c' is a scalar. The ERC portfolio is then equal to the normalized portfolio $y^*(c')$:

$$x_{\text{erc}} = \frac{y^*(c')}{\mathbf{1}^\top y^*(c')}$$

Let us now consider this second optimization program:

$$x^*(c) = \arg \min \frac{1}{2} x^\top \Sigma x \tag{2.6}$$

$$\text{u.c. } \begin{cases} \sum_{i=1}^n \ln x_i \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$

where $c \in]-\infty, n \ln n]$. Maillard *et al.* [MAI 10] demonstrated that there exists a value of c such that the optimized portfolio is the ERC portfolio. In this case, we have the following relationship:

$$c_{\text{erc}} = c' - n \ln \sum_{i=1}^n y_i^*(c')$$

Roncalli [RON 13] also deduces that the optimized portfolio $y^*(c')$ is a leveraged version of the ERC portfolio:

$$y^*(c') = \exp\left(\frac{c' - c_{\text{erc}}}{n}\right) \cdot x_{\text{erc}}$$

Because $\sigma(x^*(c))$ is an increasing function of the parameter c , we obtain the same inequality as in the case of constrained minimum variance portfolios:

$$\sigma(x_{\text{gmV}}) \leq \sigma(x^*(c)) \leq \sigma(x_{\text{ew}})$$

We deduce that:

$$\sigma(x_{\text{gmV}}) \leq \sigma(x_{\text{erc}}) \leq \sigma(x_{\text{ew}})$$

REMARK 2.2.– The Lagrange formulation of the optimization problem [2.6] is:

$$\begin{aligned}
 y^*(\lambda) &= \frac{1}{2} y^\top \Sigma y - \lambda \sum_{i=1}^n \ln y_i & [2.7] \\
 \text{u.c.} &\begin{cases} \mathbf{1}^\top y = 1 \\ y \geq \mathbf{0} \end{cases}
 \end{aligned}$$

with $\lambda \geq 0$. In this case, the optimal solution $x^*(c)$ corresponds to the portfolio $y^*(\lambda)$ with:

$$c = \sum_{i=1}^n \ln y_i^*(\lambda)$$

According to this framework, a natural way to measure the diversification is to consider the Herfindahl index applied to risk contributions:

$$\mathcal{D}_{rc}(x) = \frac{1}{n \sum_{i=1}^n \mathcal{R}C_i^2(x)} \quad [2.8]$$

Let us consider portfolios with positive risk contribution³. It follows that $\mathcal{D}_{rc}(x) \in [1/n, 1]$ and we have $\mathcal{D}_{rc}(x_{erc}) = 1$. This means that the ERC portfolio is then the one that maximizes the risk diversification.

2.2.3. Most diversified portfolio

Choueifaty and Coignard [CHO 08] introduce the concept of diversification ratio, which corresponds to the following expression:

$$\mathcal{DR}(x) = \frac{x^\top \sigma}{\sqrt{x^\top \Sigma x}}$$

By construction, $\mathcal{DR}(x)$ is equal to 1 if the portfolio is fully invested in one asset or if the correlations $\rho_{i,j}$ are all equal to 1. In the other cases, we have $\mathcal{DR}(x) > 1$. The MDP is then the portfolio which maximizes the diversification ratio:

$$\begin{aligned}
 x_{\text{mdp}} &= \arg \max \mathcal{DR}(x) & [2.9] \\
 \text{u.c.} &\begin{cases} \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}
 \end{aligned}$$

³ It is always the case if the cross-correlations $\rho_{i,j}$ are positive.

2.2.3.1. A first route toward variance minimization

Let ρ be the correlation matrix deduced from Σ . We note $x_{\text{gmv}}(\rho)$ the long-only minimum variance portfolio based only on the correlation matrix. The MDP is then a rescaled version of the GMV portfolio:

$$x_{\text{mdp},i} \propto \frac{x_{\text{gmv},i}(\rho)}{\sigma_i}$$

2.2.3.2. A second route

Let us consider the following optimization problem:

$$\begin{aligned} y^*(c') &= \arg \min \frac{1}{2} y^\top \Sigma y \\ \text{u.c. } &\begin{cases} \sum_{i=1}^n y_i \sigma_i \geq c' \\ y \geq \mathbf{0} \end{cases} \end{aligned} \quad [2.10]$$

with $c' > 0$. We can demonstrate that the MDP corresponds to the normalized portfolio⁴:

$$x_{\text{mdp}} = \frac{y^*(c')}{\mathbf{1}^\top y^*(c')}$$

It follows that the MDP is the solution of the following optimization program for a specific value of c :

$$\begin{aligned} x^*(c) &= \arg \min \frac{1}{2} x^\top \Sigma x \\ \text{u.c. } &\begin{cases} \sum_{i=1}^n x_i \sigma_i \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases} \end{aligned} \quad [2.11]$$

where $c \in [0, \max_i \sigma_i]$. Indeed, we have:

$$c_{\text{mdp}} = \frac{c'}{\sum_{i=1}^n y_i^*(c')}$$

⁴ Because we have the following property:

$$\frac{y^*(c')}{c'} = \frac{y^*(c'')}{c''}$$

It follows that:

$$\sigma(x_{\text{gmv}}) \leq \sigma(x_{\text{mdp}}) \leq \max_i \sigma_i$$

REMARK 2.3.– The Lagrange formulation of the optimization problem [2.11] is:

$$y^*(\lambda) = \frac{1}{2} y^\top \Sigma y - \lambda y^\top \sigma \quad [2.12]$$

$$\text{u.c.} \begin{cases} \mathbf{1}^\top y = 1 \\ y \geq \mathbf{0} \end{cases}$$

with $\lambda \geq 0$. In this case, the optimal solution $x^*(c)$ corresponds to the portfolio $y^*(\lambda)$ with:

$$c = \sum_{i=1}^n y_i^*(\lambda) \sigma_i$$

If we delete the constraint $\mathbf{1}^\top y = 1$, we obtain the solution $y^*(c')$ given by the optimization program [2.10]. Let us consider the restricted universe of invested assets, that is the assets i such that $x_{\text{mdp},i} > 0$. It follows that the MDP weights of this restricted universe are:

$$\tilde{x}_{\text{mdp}} = \frac{\tilde{\Sigma}^{-1} \tilde{\sigma}}{\mathbf{1}^\top \tilde{\Sigma}^{-1} \tilde{\sigma}}$$

where $\tilde{\Sigma}$ is the covariance matrix of the invested assets.

2.2.4. Comparing the trade-off relationships

Following Cazalet *et al.* [CAZ 14], we compare the different optimization programs [2.4], [2.6] and [2.11] by changing the value of c . We consider the Eurostoxx 50 index and the 1-year empirical covariance matrix estimated in February 2013. The results are reported in Figures 2.1, 2.2 and 2.3. In each figure, the first panel represents the tracking error volatility $\sigma(x | x_{\text{cw}})$ with respect to the volatility reduction $\mathcal{VR}(x | x_{\text{cw}})$ defined by:

$$\mathcal{VR}(x | x_{\text{cw}}) = \frac{\sigma(x_{\text{cw}}) - \sigma(x)}{\sigma(x_{\text{cw}})}$$

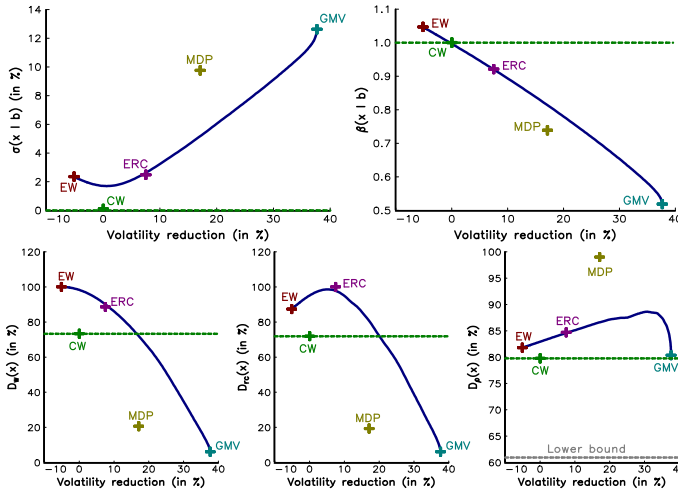


Figure 2.1. Trade-off relationships of problem [2.4] (MV). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

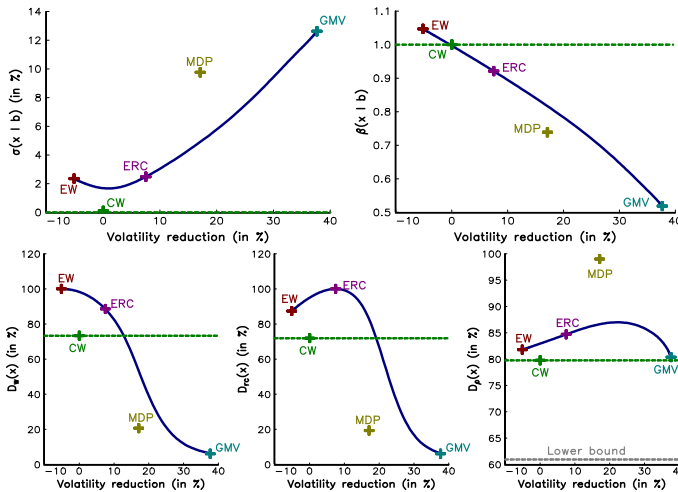


Figure 2.2. Trade-off relationships of problem [2.6] (ERC). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

In the second panel, we consider the beta $\beta(x | x_{cw})$ of the portfolio with respect to the CW portfolio. The three panels at the bottom show the impact of the volatility

reduction on the diversification measures⁵. These results show that investors have to puzzle out the trade-off between volatility, tracking error and diversification. However, we notice that the trade-off relationships are very similar when comparing MV and ERC portfolios (Figures 2.1 and 2.2), which is not the case when considering MDP (Figure 2.3).

2.3. Managing the diversification

2.3.1. Mixing the constraints

When we consider Figure 2.3, we observe that solutions are not very interesting because we cannot manage the diversification in terms of weights or risk contributions. This is why we can introduce these constraints into problem [2.11]. For instance, the MDP optimization problem with the weight diversification becomes:

$$x^*(c_1, c_2) = \arg \min \frac{1}{2} x^\top \Sigma x \quad [2.13]$$

$$\text{u.c.} \begin{cases} \sum_{i=1}^n x_i \sigma_i \geq c_1 \\ \mathcal{D}_w(x) \geq c_2 \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$

In this case, we can build smart beta portfolios between the MDP ($c_1 = c_{\text{mdp}}$ and $c_2 = 0$) and the EW portfolio ($c_1 = c_{\text{mdp}}$ and $c_2 = 1$). An example is given in Figure 2.4 by setting $c_1 = c_{\text{mdp}}$. If we prefer to consider the risk diversification, we obtain:

$$x^*(c_1, c_2) = \arg \min \frac{1}{2} x^\top \Sigma x \quad [2.14]$$

$$\text{u.c.} \begin{cases} \sum_{i=1}^n x_i \sigma_i \geq c_1 \\ \sum_{i=1}^n \ln x_i \geq c_2 \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$

⁵ The two measures $D_w(x)$ and $D_{rc}(x)$ correspond to the weight and risk diversifications defined in equations [2.3] and [2.8]. The diversification measure $D_\rho(x)$ is the ratio between the diversification ratio $\mathcal{DR}(x)$ of the portfolio and the diversification ratio $\mathcal{DR}(x_{\text{mdp}})$ of the MDP.

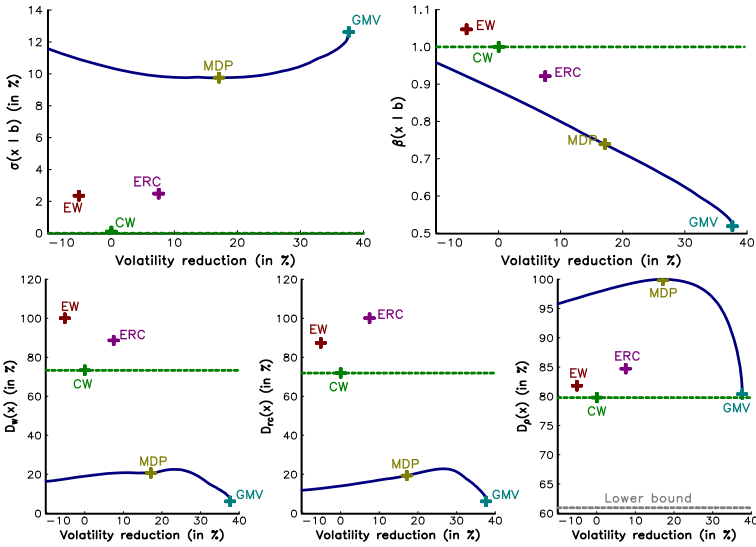


Figure 2.3. Trade-off relationships of problem [2.11] (MDP). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

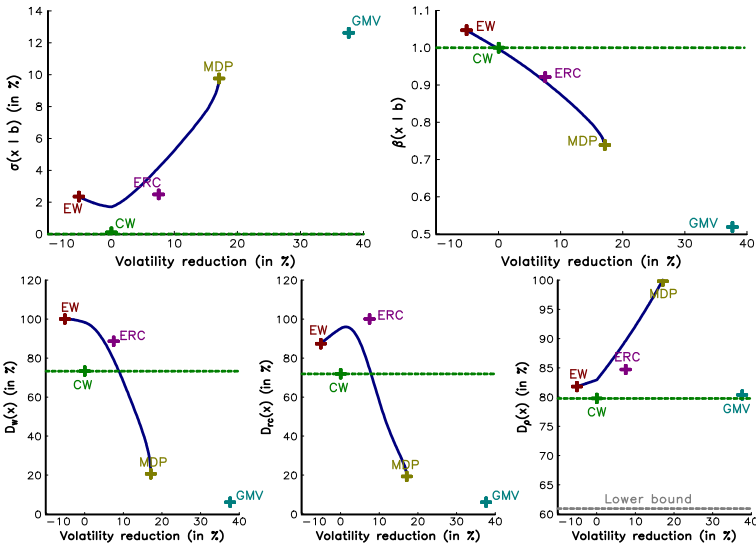


Figure 2.4. Trade-off relationships of problem [2.13] with $c_1 = c_{mdp}$. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

2.3.2. A unified optimization framework

In fact, we can combine these different constraints in a unique variance minimization problem with the following set of constraints:

$$\begin{cases} \mathbf{1}^\top x = 1 \\ \sum_{i=1}^n x_i^2 \leq c_1 \\ \sum_{i=1}^n \ln x_i \geq c_2 \\ \sum_{i=1}^n x_i \sigma_i \geq c_3 \end{cases}$$

The first and fourth constraints allow the GMV portfolio and the MDP, respectively, to be obtained. The second and third constraints manage the diversification in terms of weights (using the Herfindahl index) and risk contributions. Therefore, we can write the constrained problem using Lagrange multipliers:

$$\begin{aligned} x^* = \arg \min & \frac{1}{2} x^\top \Sigma x - & [2.15] \\ & \lambda_{\text{gmv}} \left(\sum_{i=1}^n x_i \right) + \lambda_{\text{h}} \left(\sum_{i=1}^n x_i^2 \right) - \\ & \lambda_{\text{erc}} \left(\sum_{i=1}^n \ln x_i \right) - \lambda_{\text{mdp}} \left(\sum_{i=1}^n x_i \sigma_i \right) \\ \text{u.c. } & x \geq \mathbf{0} \end{aligned}$$

with $\lambda_{\text{h}} \geq 0$ and $\lambda_{\text{erc}} \geq 0$. From a technical point of view, there are no restrictions on λ_{gmv} and λ_{mdp} even if some cases are more relevant ($\lambda_{\text{gmv}} \geq 0$ and $\lambda_{\text{mdp}} \geq 0$).

REMARK 2.4.– The previous framework can be extended by replacing the variance minimization problem by the tracking error minimization problem. In section 2.6.1, we show that it is equivalent to introducing a constraint in the form $x^\top \Sigma x_{\text{cw}} \geq c_4$. In this case, problem [2.15] must include a new penalty function which is equal to:

$$-\lambda_{\text{te}} \left(\sum_{i=1}^n x_i (\Sigma x_{\text{cw}})_i \right)$$

The benefits of using the formulation [2.15] are twofold. First, this optimization problem is very easy to solve using the cyclical coordinate descent (CCD) algorithm. This numerical method was used by Griveau-Billion *et al.* [GRI 13] to find the solution of the ERC portfolio. In section 2.6.2, we extend this analysis to the general problem [2.15]. The second interest lies in the explicit trade-off relationships contained in the optimization problem. If the aim is to emphasize one specific diversification measure, we have to use a larger value for the corresponding Lagrange coefficient, but it is not

possible to match all the different diversification constraints. This means that even if there is no restriction between the Lagrange multipliers, only a subset of them is interesting from a financial point of view. This is equivalent to imposing a structure between the different constraints. Let us consider this specific problem, for instance:

$$x^* = \arg \min \frac{1}{2} x^\top \Sigma x \tag{2.16}$$

$$\text{u.c.} \begin{cases} \mathcal{D}(x; \gamma) \geq c_1 \\ \mathcal{B}(x; \delta) = c_2 \\ x \geq \mathbf{0} \end{cases}$$

where $\mathcal{D}(x; \gamma) = \gamma \sum_{i=1}^n \ln x_i - (1 - \gamma) \sum_{i=1}^n x_i^2$ is a diversification constraint and $\mathcal{B}(x; \delta) = \delta \sum_{i=1}^n x_i + (1 - \delta) \sum_{i=1}^n x_i \sigma_i$ is a budget constraint. The parameter $\gamma \in [0, 1]$ controls the trade-off between weights and risk diversification, whereas the parameter $\delta \in [0, 1]$ controls the budget allocation. We can then restrict (c_1, c_2) by considering this optimization problem:

$$x^*(\lambda, \gamma, \delta) = \arg \min \frac{1}{2} x^\top \Sigma x - \lambda \mathcal{D}(x; \gamma) + (\lambda - 1) \mathcal{B}(x; \delta) \tag{2.17}$$

$$\text{u.c. } x \geq \mathbf{0}$$

where $\lambda \geq 0$ controls the impact on the diversification. Problem [2.17] is a special case of problem [2.15], but it is wide enough to include most of the solutions⁶.

REMARK 2.5.– If we include a tracking error constraint, the budget constraint becomes $\mathcal{B}(x; \delta, \kappa) = \sum_{i=1}^n x_i (\delta + \kappa (\Sigma x_{cw})_i + (1 - \delta - \kappa) \sigma_i)$ with $0 \leq \kappa + \gamma \leq 1$.

Parameters	GMV	EW	ERC	MDP	RP	BP	CW
λ	0	$+\infty$	1	0	$+\infty$	$+\infty$	0
γ		0/1	1		1	1	1
δ	1	1		0	1	0	
κ	0	0		0	0	1	1

Table 2.1. Limits of the smart beta portfolio $x^*(\lambda, \gamma, \delta, \kappa)$

In Table 2.1, we indicate the parameters that give the different smart beta portfolios⁷. For instance, $(\lambda, \gamma, \delta, \kappa) = (1, 1, 0, 0)$ gives the ERC portfolio, while $(\lambda, \gamma, \delta, \kappa) = (0, 1, 1, 0)$ gives the GMV portfolio.

⁶ It is equivalent to impose that $\lambda_{\text{gmv}} - \lambda_{\text{h}} + \lambda_{\text{erc}} + \lambda_{\text{mdp}} = 1$ and $\lambda_{\text{erc}} = \lambda_{\text{h}} + \lambda$.

⁷ The risk parity (RP) and beta parity (BP) portfolios are defined in section 2.6.2.3. They correspond to inverse-volatility and inverse-beta weighting schemes.

With problem [2.17], we can explore new risk-based portfolios by mixing different constraints. We consider the following set of parameters:

Set	Achievable Portfolios	λ	γ	δ	κ
(1)	RP-ERC-MDP	$\in \mathbb{R}_+$	1	0	0
(2)	CW-ERC-MDP	0	1	0	$\in [0, 1]$
(3)	CW-ERC	$\in \mathbb{R}_+$	1	0	1
(4)	CW-GMV	0	0	$1 - \kappa$	$\in [0, 1]$
(5)	EW-MDP	$\in \mathbb{R}_+$	1	$1 - e^{-\lambda}$	0

For each set, we indicate the achievable portfolios. For instance, if $\kappa = 1$ (and $\delta = 0$), we obtain the CW portfolio. Depending on the values of λ and γ , we can then build risk-based portfolios between CW and another smart beta portfolio. For instance, if $\lambda \in \mathbb{R}_+$ and $\gamma = 1$, we obtain solutions between the CW portfolio and the ERC portfolio. In Figure 2.5, we have reported the paths of the different parameter sets.

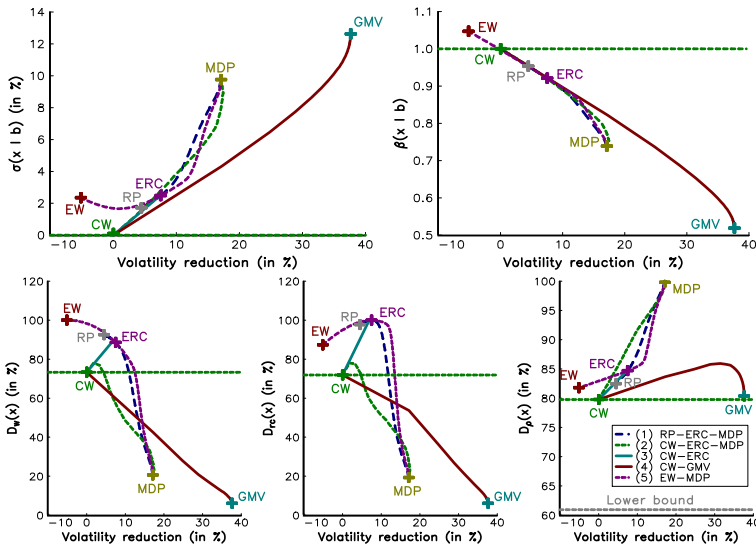


Figure 2.5. Trade-off relationships of problem [2.17]. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

2.3.3. Diversification profile of risk-based portfolios

Radar charts of the different objectives are reported in Figure 2.6. Each hexagonal chart (represented by dashed lines) corresponds to an improvement of the measure by

15%. In order to compare the different profiles, we use a benchmark profile which has the GMV volatility reduction, a zero tracking error, a beta equal to 1, the diversification ratio of the MDP, and the weight and risk diversifications of the EW and ERC portfolios. The GMV portfolio focuses on minimizing the volatility but presents a poor diversification in terms of weights and risk contributions. It is also the portfolio with the highest beta risk and tracking error risk. The EW portfolio performs well to maximize the beta and minimize the tracking error while diversifying the weights. But, this is done with no volatility reduction. The ERC portfolio has a similar profile but pays more attention to volatility reduction. Finally, the MDP profile is similar to the GMV profile, but has a lower beta risk and tracking error risk.

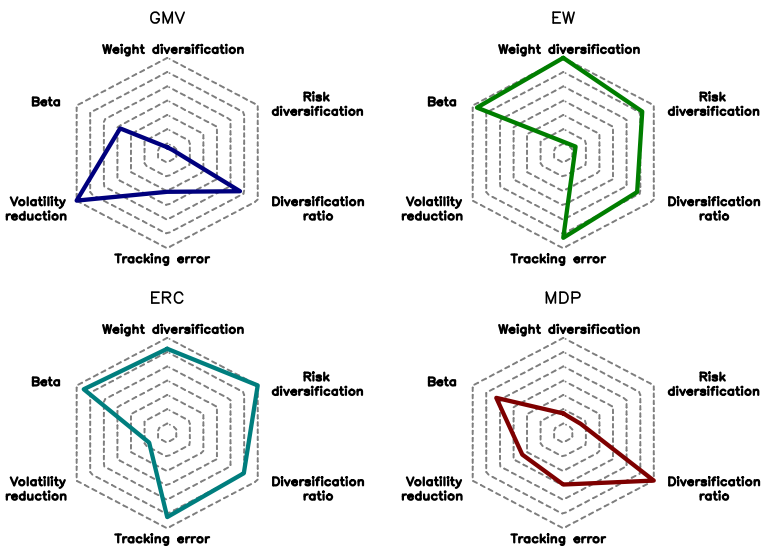


Figure 2.6. *Diversification profile of smart beta portfolios. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip*

Figure 2.7 illustrates the role of the parameter λ . κ is equal to zero and we fix the other parameters in a balanced manner: $\gamma = \delta = 0.5$. We observe that the volatility reduction is done at the expense of the diversification. Moreover, the weight diversification decreases more quickly than the risk diversification. Indeed, the volatility of the ERC portfolio is always lower than the volatility of the EW portfolio. This means that the impact of the volatility reduction on the diversification is weaker for the ERC.

In Figure 2.8, we have reported the diversification profile when we specifically target a volatility reduction (5, 10, 20 and 30%). In this example, we confirm that the weight diversification decreases more quickly than the risk diversification. The

diversification ratio is the less impacted measure by the change in the volatility reduction.

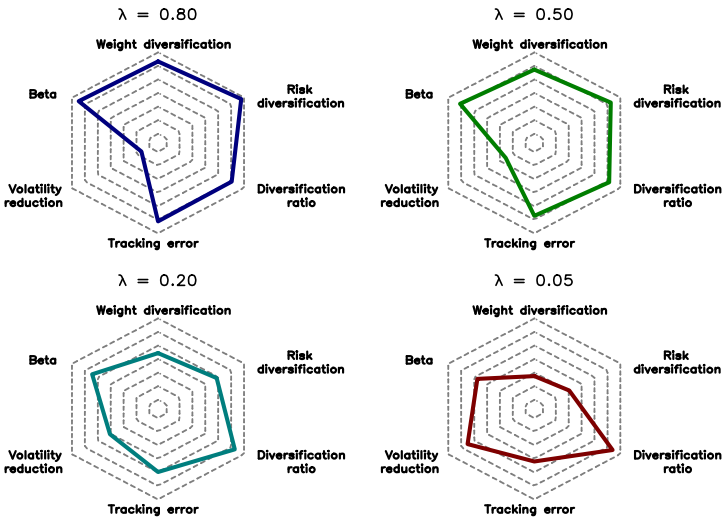


Figure 2.7. Diversification profile and weight diversification. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

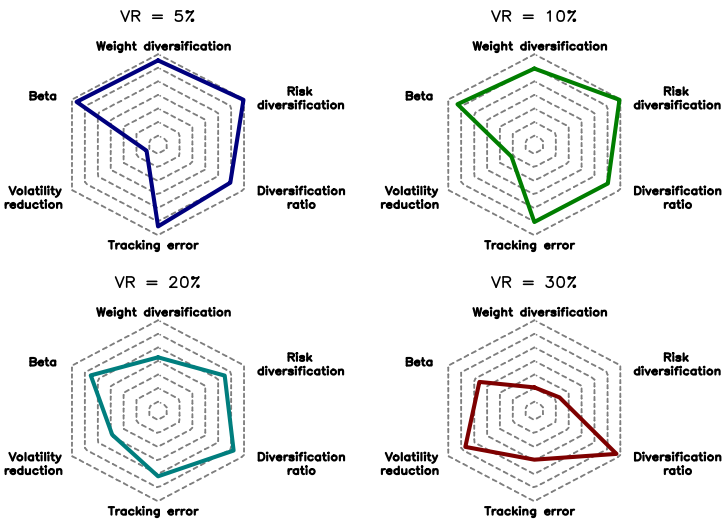


Figure 2.8. Diversification profile and volatility reduction. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

2.4. Understanding the behavior of smart beta portfolios

We consider here real-life applications with four different stock universes: the Eurostoxx 50 index (SX5E), the Topix 100 index (TPX100), the S&P 500 index (SPX) and the MSCI EM index (MXEF). We have chosen these stock indices because they correspond to different regions and different sizes of the universe. For each universe, we compute smart beta portfolios by using the 1-year empirical covariance matrix of stock returns. The allocation is rebalanced at a monthly frequency. We conduct backtests from January 2001 to December 2014⁸. Empirical results confirm that there are some trade-off principles. In particular, we obtain a first rule of smart beta indexing:

RULE 2.1.— There is no free lunch in smart beta. In particular, it is not possible to target high volatility reduction, to be highly diversified and to take low beta risk.

In Figure 2.9, we have reported the relationship between the volatility reduction and the beta for the four universes and the four smart beta portfolios (EW, GMV, ERC and MDP). Each point corresponds to a rebalancing date. By reducing the volatility, the smart beta portfolios increase the beta risk. We observe similar results for the other risk measures: tracking error, weight diversification, risk diversification and diversification ratio.

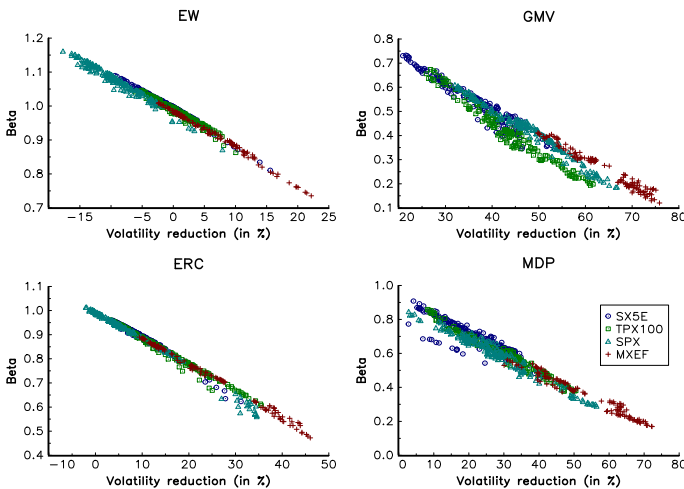


Figure 2.9. Relationship between the volatility reduction and the beta. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

⁸ For the MSCI EM index, the starting date is February 2005.

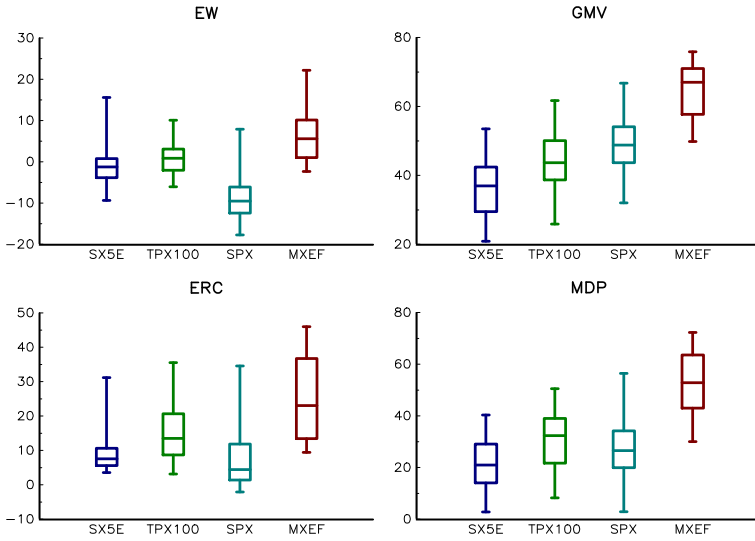


Figure 2.10. Boxplot of the volatility reduction (in %). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

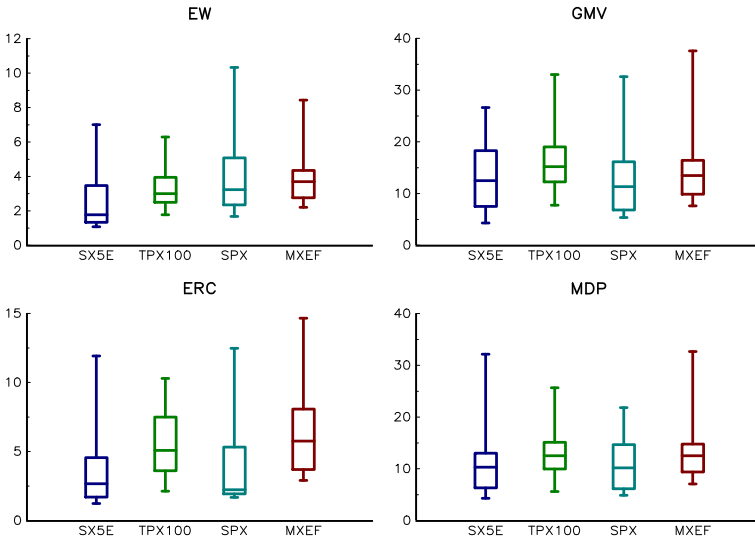


Figure 2.11. Boxplot of the tracking error (in %). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

2.4.1. Volatility reduction

RULE 2.2.—The smart beta portfolios have a time-varying objective of volatility reduction and tracking error.

This rule shows that the behavior of traditional smart beta portfolios (EW, GMV, ERC and MDP) is not homogeneous across time in terms of volatility reduction and tracking error. We have reported the boxplots in Figures 2.10 and 2.11. The bottom and top of the box indicate the first and third quartiles of the statistics, the line inside the box corresponds to the median, whereas the ends of whiskers are the minimum and the maximum. We notice that the volatility reduction depends on the underlying index. However, we do not observe a strong relationship with the size of the universe, except for the GMV portfolio. For instance, the EW portfolio has a higher volatility, on average, than the CW portfolio in the case of the S&P 500 index, but the volatility reduction is maximal for the MSCI EM index. If we consider the tracking error, the behavior is even more complex with respect to the underlying index. For instance, the Topix 100 universe presents the highest tracking error in the case of ERC and GMV portfolios.

A statistical analysis shows that the level of the volatility reduction and the tracking error, as well as their variations, cannot be explained by the level or the variation of the volatility of the CW index. We can, therefore, obtain all the possible existing configurations:

$\sigma(x_{cw})$	High	Low
$\mathcal{V}\mathcal{R}(x x_{cw})$ or $\sigma(x x_{cw})$	High/Low	High/Low
$\Delta_t \sigma(x_{cw})$	+	-
$\Delta_t \mathcal{V}\mathcal{R}(x x_{cw})$ or $\Delta_t \sigma(x x_{cw})$	+/-	+/-

Even if there is no obvious relationship between the volatility of the CW portfolio and the volatility or the tracking error of the smart beta portfolios, there is some similar behavior between the smart beta portfolios themselves. For instance, we report the correlation between the variations during two consecutive rebalancing dates of the volatility reduction in Table 2.2. We compute the statistic $\rho_{i,j}^{\Delta_t \mathcal{V}\mathcal{R}} = \rho(\Delta_t \mathcal{V}\mathcal{R}(x^{(i)} | x_{cw}), \Delta_t \mathcal{V}\mathcal{R}(x^{(j)} | x_{cw}))$ for all pairs (i, j) of smart beta portfolios. We notice that the cross-correlations are high except for the EW portfolio in the case of the S&P 500 universe. Results for the tracking error correlation $\rho_{i,j}^{\Delta_t \text{TE}} = \rho(\Delta_t \sigma(x^{(i)} | x_{cw}), \Delta_t \sigma(x^{(j)} | x_{cw}))$ are also reported in Table 2.2. Like the volatility reduction, the tracking error cross-correlations are high especially for the pairs (GMV,MDP) and (ERC,MDP).

If we consider the other risk statistics, we obtain similar results for the beta⁹ and diversification ratio, but not for the weight and risk diversifications. For these last two

9 We have $\rho_{i,j}^{\Delta_t \beta} \simeq \rho_{i,j}^{\Delta_t \mathcal{V}\mathcal{R}}$.

statistics, the average correlation is close to zero. We report the results for $\rho_{i,j}^{\Delta_t \mathcal{D}_{rc}}$ and $\rho_{i,j}^{\Delta_t \mathcal{D}_\rho}$ in Table 2.3. We notice that the cross-correlation \mathcal{D}_ρ is extremely high¹⁰.

(i, j)	Volatility reduction				Tracking error			
	SX5E	TPX100	SPX	MXEF	SX5E	TPX100	SPX	MXEF
(EW,GMV)	16.7	44.1	-0.5	37.1	25.0	44.4	44.7	76.3
(EW,ERC)	66.4	70.2	33.1	79.8	26.6	33.1	11.5	76.0
(EW,MDP)	26.9	44.3	5.6	37.9	38.1	37.6	42.0	75.5
(GMV,ERC)	69.2	77.7	49.2	64.5	62.8	41.1	48.1	87.4
(GMV,MDP)	74.7	79.9	45.6	80.3	76.3	90.7	75.5	98.2
(ERC,MDP)	71.9	79.2	66.0	64.8	82.4	53.0	63.0	90.2

Table 2.2. Empirical correlations $\rho_{i,j}^{\Delta_t \mathcal{V}\mathcal{R}}$ and $\rho_{i,j}^{\Delta_t \text{TE}}$ (in %)

(i, j)	Risk diversification				Diversification ratio			
	SX5E	TPX100	SPX	MXEF	SX5E	TPX100	SPX	MXEF
(EW,GMV)	2.1	-35.4	-2.4	-20.8	73.7	77.1	54.9	87.5
(EW,ERC)	-6.7	8.7	-8.2	-15.5	93.2	93.6	84.8	96.1
(EW,MDP)	0.5	-38.1	-22.5	-29.2	79.2	85.8	81.0	91.1
(GMV,ERC)	34.5	-14.2	-17.9	-7.4	75.5	85.4	65.7	92.4
(GMV,MDP)	25.6	12.9	14.1	42.6	75.3	86.2	74.0	96.2
(ERC,MDP)	23.0	-3.2	15.9	23.5	92.8	92.9	89.7	96.5

Table 2.3. Empirical correlations $\rho_{i,j}^{\Delta_t \mathcal{D}_{rc}}$ and $\rho_{i,j}^{\Delta_t \mathcal{D}_\rho}$ (in %)

2.4.2. Normalizing the smart beta portfolios

In Table 2.4, we report the average correlation between the three smart beta portfolios (GMV, ERC and MDP) for the different statistics¹¹. We notice that the average correlation between returns is less than 90%, implying that the behavior of these smart beta portfolios may be very different in some specific periods.

Index	$\mathcal{V}\mathcal{R}$	TE	β	\mathcal{D}_w	\mathcal{D}_{rc}	\mathcal{D}_ρ	R_t
SX5E	67.4	81.9	73.0	39.2	26.5	95.8	89.6
TPX100	88.2	81.1	87.6	28.9	27.7	93.6	92.8
SX5E	79.9	80.2	82.9	21.3	32.6	97.3	83.2
MXEF	89.3	93.2	93.1	2.4	34.5	97.8	88.5
Average	81.2	84.1	84.1	23.0	30.3	96.1	88.5

Table 2.4. Average correlation between GMV, ERC and MDP portfolios (in %)

¹⁰ It is equal to 84% on average.

¹¹ For the risk statistics ($\mathcal{V}\mathcal{R}$, TE, β , \mathcal{D}_w , \mathcal{D}_{rc} and \mathcal{D}_ρ), we consider the monthly series. The correlation between returns R_t is computed using the daily series of the 1-year performance.

We can ask if these differences come from the implied intrinsic constraint of each model, or from the level of volatility reduction targeted by each model. This is why we investigate the behavior of smart beta portfolios when we normalize them by targeting the same level of volatility reduction. Therefore, we calibrate the set of parameters $(\lambda_{\text{gmv}}, \lambda_h, \lambda_{\text{erc}}, \lambda_{\text{mdp}}, \lambda_{\text{te}})$ such that:

$$\mathcal{VR}(x^*(\lambda_{\text{gmv}}, \lambda_h, \lambda_{\text{erc}}, \lambda_{\text{mdp}}, \lambda_{\text{te}}) \mid x_{\text{cw}}) = \eta^*$$

where η^* is the targeted volatility reduction. For each smart beta portfolio, the calibration is done with one parameter (it is underlined), whereas the other parameters are fixed:

GMV $\lambda_{\text{gmv}} = 1$, $\lambda_h \in [0, +\infty)$, $\lambda_{\text{erc}} = 0$, $\lambda_{\text{mdp}} = 0$ and $\lambda_{\text{te}} = 0$;

ERC $\lambda_{\text{gmv}} = -\infty$, $\lambda_h = 0$, $\lambda_{\text{erc}} \in (0, +\infty)$, $\lambda_{\text{mdp}} = 0$ and $\lambda_{\text{te}} = 0$;

MDP $\lambda_{\text{gmv}} = 0$, $\lambda_h = 0$, $\lambda_{\text{erc}} = 1$, $\lambda_{\text{mdp}} \in (-\infty, +\infty)$ and $\lambda_{\text{te}} = 0$;

For instance, the calibration is done using the Herfindahl parameter λ_h in the case of the GMV portfolio. Results are reported in Table 2.5. We notice that the average correlation between the three smart beta methods has highly increased. This is particularly true for the 1-year performance, for which the average correlation is close to 100%. We conclude that the differences between the smart beta methods (GMV, ERC and MDP) are mainly explained by the different levels of targeted volatility reduction, which is a consequence of their intrinsic constraints. These results are also valid when we target a level of tracking error. Therefore, we obtain a third rule of smart beta indexing:

Index	η^*	\mathcal{VR}	TE	β	\mathcal{D}_w	\mathcal{D}_{rc}	\mathcal{D}_p	R_t
SX5E	5%	100.0	99.2	100.0	99.3	99.5	99.8	100.0
	10%	100.0	92.1	99.5	86.7	71.6	98.9	99.8
	15%	100.0	91.5	97.4	88.6	76.4	97.2	99.2
TPX100	5%	100.0	99.8	100.0	99.7	99.8	99.9	100.0
	10%	100.0	88.3	98.9	89.1	65.0	98.2	100.0
	15%	100.0	91.5	97.6	92.7	78.4	97.5	99.9
SPX	5%	100.0	96.8	99.8	86.4	63.6	98.2	99.8
	10%	100.0	86.9	97.1	88.4	69.7	93.4	99.0
	15%	100.0	85.6	90.8	88.9	77.6	88.4	97.6
MXEF	5%	100.0	100.0	100.0	99.9	100.0	100.0	100.0
	10%	100.0	100.0	100.0	98.2	99.5	99.8	100.0
	15%	100.0	99.9	100.0	96.1	95.0	99.5	100.0
Average		100.0	94.3	98.4	92.8	83.0	97.6	99.6

Table 2.5. Average correlation between GMV, ERC and MDP portfolios (in %)

RULE 2.3.– When we impose an objective of volatility reduction or tracking error, the smart beta portfolios become comparable.

2.4.3. Performance of the smart beta portfolios

RULE 2.4.– The performance of smart beta portfolios depends on the market risk premium. When this is high, it is better to consider an objective of low volatility reduction (or tracking error volatility). Conversely, it is preferable to target a high volatility reduction when the market risk premium is weak or negative.

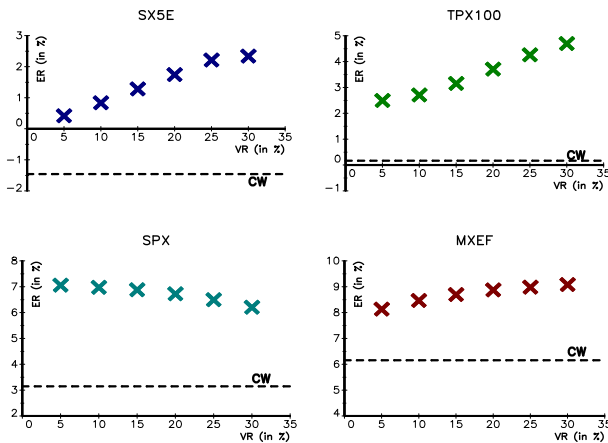


Figure 2.12. Relationship between volatility reduction and excess return (2001–2014). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

This rule is very logical and easy to understand. Indeed, when the performance of stocks is high, it is better to invest in a more diversified portfolio than the CW portfolio, but with a limited tracking error in order to fully benefit from the bull market. Conversely, in a bear market, a concentrated portfolio of low volatility stocks will do a better job. In Figures 2.12, 2.13 and 2.14, we have reported the relationship¹² between the volatility reduction (in %) of smart beta portfolios and their excess return (in %) measured as the difference between the annualized return and the risk-free rate. The excess return for the CW index corresponds to the horizontal dashed line. Results for

¹² Because of the third rule, we know that the calibration method to target a given volatility reduction has little impact, particularly on the performance. This is why we only report the results with the GMV approach when the calibration is done by estimating λ_h . We obtain the same results if we consider other calibration schemes.

the entire study period (January 2001 – December 2014) are given in Figure 2.12. We notice that the rule is satisfied except for the MSCI EM index. If we consider the financial crisis (July 2007 – February 2009), we observe a positive relationship between volatility reduction and excess return (Figure 2.13). For this period, it is, therefore, better to target a high volatility reduction. The opposite is true if we consider the recent recovery period (March 2009 – December 2013).

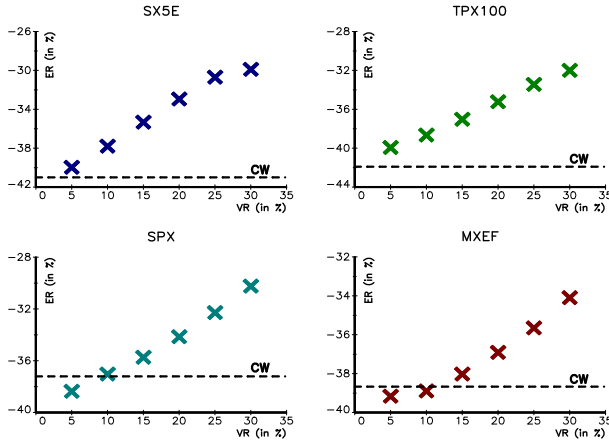


Figure 2.13. Relationship between volatility reduction and excess return (July 2007–February 2009). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

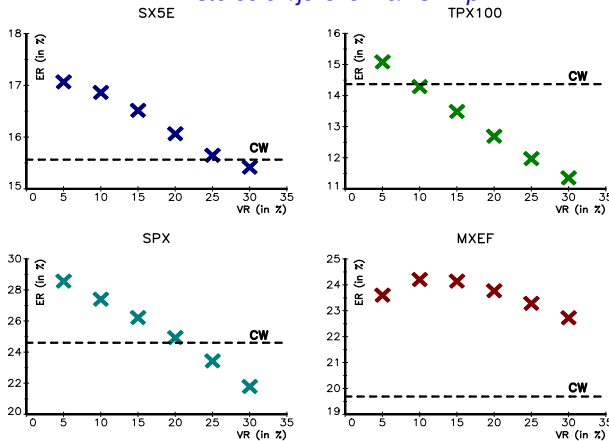


Figure 2.14. Relationship between volatility reduction and excess return (March 2009–December 2013). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

2.4.4. Dynamic smart beta strategies

The previous rule can be used to build dynamic smart beta strategies. The idea is to fix the level of volatility reduction with respect to market conditions. If the risk sentiment is high, we would like to have an aggressive portfolio or to target a high level of volatility reduction. If the risk sentiment is low, it is better to have a more diversified portfolio with low tracking error with respect to the CW index.

We consider the optimization problem [2.17] with $\lambda \in [0, 1]$, $\gamma = 1$ and $\delta = 1$. In this case, we obtain smart beta portfolios between the GMV portfolio ($\lambda = 0$) and the ERC portfolio ($\lambda = 1$). At each date t , we estimate the market sentiment by computing the cross-sectional volatility σ_t^{CSV} of stocks which belong to the CW index¹³. We then consider the following rule to fix λ :

$$\lambda = 1 - \phi \frac{\sigma_t^{\text{CSV}} - \sigma_t^-}{\sigma_t^+ - \sigma_t^-}$$

where σ_t^- and σ_t^+ are the minimum and maximum values of σ_t^{CSV} observed for the window period $[t-h; t]$ and ϕ is a scalar between 0 and 1. We consider two strategies. $D_{\#1}$ corresponds to the case $\phi = 1$ and $\lambda \in [0, 1]$. For the second strategy $D_{\#2}$, ϕ is equal to 0.85 meaning that $\lambda \in [0.15, 1]$. Backtests¹⁴ for the study period 2001–2014 are reported in Table 2.6. For each strategy, we calculate the annualized return $\mu(x)$, the annual volatility $\sigma(x)$, the corresponding Sharpe ratio $\text{SR}(x)$, the maximum drawdown $\mathcal{DD}(x)$ and the turnover $\tau(x)$ of the allocation. We notice that the dynamic smart beta strategies $D_{\#1}$ and $D_{\#2}$ improve the performance of the GMV and ERC portfolios for three indices (Eurostoxx 50, S&P 500 and MSCI EM). In particular, the second strategy $D_{\#2}$ is a considerable improvement on the ERC strategy with a higher return, a lower volatility, a reduced drawdown and a limited turnover¹⁵. Even this application is a toy model as it gives some indications about the benefit of dynamically managing the volatility reduction with respect to the market sentiment.

2.5. Conclusion

Smart beta indexing is becoming increasingly popular with institutional investors and pension funds. It is perceived as a method of reducing risk and increasing performance with respect to CW indexing. However, there are many ways to build a

¹³ In order to reduce the noise, we also apply an exponentially weighted moving average with a smoothing coefficient of 0.98 to the cross-sectional volatility.

¹⁴ The lag window h is equal to 1 year.

¹⁵ On average, its turnover is twice the turnover of the ERC portfolio, but half that of the GMV portfolio.

smart beta portfolio. One interesting property is that these different alternative-weighted portfolios belong to the same optimization problem family. They are minimum variance portfolios and differ because of the implied constraint they consider. In this chapter, we develop a unified analytical framework based on the CCD algorithm in order to show the trade-off between the volatility reduction and the risks of such alternative-weighted solutions. Using this approach, we can illustrate and understand the behavioral differences of smart beta portfolios. We can also develop new smart beta strategies by explicitly targeting a level of volatility reduction or by dynamically linking this level to the market sentiment.

	CW	GMV	ERC	D _{#1}	D _{#2}	CW	GMV	ERC	D _{#1}	D _{#2}
	SX5E					TPX100				
$\mu(x)$	0.6	3.8	3.4	5.1	4.7	0.4	6.3	3.3	3.6	3.2
$\sigma(x)$	24.5	19.1	23.1	21.3	22.4	24.4	16.3	21.3	18.9	19.8
SR(x)	-0.1	0.1	0.1	0.1	0.1	0.0	0.4	0.1	0.2	0.1
$\mathcal{DD}(x)$	-59.6	-52.4	-54.4	-50.7	-51.5	-62.8	-49.4	-57.4	-51.1	-54.2
$\tau(x)$	0.2	3.4	0.8	3.0	1.9	0.3	3.8	1.0	2.9	1.8
	SPX					MXEF				
$\mu(x)$	5.0	8.3	9.9	11.5	10.5	8.0	12.0	10.8	14.3	12.6
$\sigma(x)$	20.1	12.2	19.2	16.2	18.2	21.6	9.4	16.3	13.0	14.3
SR(x)	0.2	0.5	0.4	0.6	0.5	0.3	1.1	0.6	1.0	0.8
$\mathcal{DD}(x)$	-55.3	-33.3	-55.9	-44.7	-52.5	-65.1	-29.9	-53.8	-34.9	-44.9
$\tau(x)$	0.1	5.9	1.0	3.5	1.6	0.5	5.6	1.6	4.2	2.8

Table 2.6. Comparing GMV, ERC and dynamic smart beta strategies (2001–2014)

2.6. Appendix

2.6.1. Managing the tracking error volatility

Let x_{cw} be the CW portfolio. The tracking error variance of the portfolio x is:

$$\begin{aligned}\sigma^2(x | x_{cw}) &= (x - x_{cw})^\top \Sigma (x - x_{cw}) \\ &= x^\top \Sigma x - 2x^\top \Sigma x_{cw} + x_{cw}^\top \Sigma x_{cw}\end{aligned}$$

Because $x_{cw}^\top \Sigma x_{cw}$ is constant, the optimization problem becomes:

$$x^*(c_1, c_2) = \arg \min \frac{1}{2} x^\top \Sigma x - x^\top \Sigma x_{cw} \quad [2.18]$$

We recognize a Markowitz's optimization problem where the expected returns μ are equal to Σx_{cw} . We notice that these expected returns are exactly the implied expected

returns in the Black–Litterman model¹⁶. Following Roncalli [RON 13], we can transform the optimization problem [2.18] into a μ -problem:

$$x^*(c) = \arg \min \frac{1}{2} x^\top \Sigma x \quad [2.19]$$

$$\text{u.c.} \begin{cases} x \in \mathcal{C} \\ \sum_{i=1}^n x_i (\Sigma x_{cw})_i \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$

with $c \in [0, c^+]$ with $c^+ = x_{cw}^\top \Sigma x_{cw}$. This problem is precisely the formulation [2.2] used in this chapter by adding the constraint $\sum_{i=1}^n x_i (\Sigma x_{cw})_i \geq c$. The limit cases are $x^*(0) = x_{\text{gmV}}$ and $x^*(c^+) = x_{cw}$.

We can use this framework to introduce the tracking error constraint in the different optimization problems considered in this study. For instance, we can mix this constraint with the ERC constraint. In this case, we will obtain optimized portfolios with a trade-off between the tracking error volatility and the diversification in terms of risk contributions.

2.6.2. Solving the general optimization problem using the CCD algorithm

We consider the following optimization problem:

$$x^*(\lambda_{\text{gmV}}, \lambda_h, \lambda_{\text{erc}}, \lambda_{\text{mdp}}, \lambda_{\text{te}}) = \arg \min \frac{1}{2} x^\top \Sigma x - \quad [2.20]$$

$$\lambda_{\text{gmV}} \left(\sum_{i=1}^n x_i \right) + \lambda_h \left(\sum_{i=1}^n x_i^2 \right) -$$

$$\lambda_{\text{erc}} \left(\sum_{i=1}^n \ln x_i \right) - \lambda_{\text{mdp}} \left(\sum_{i=1}^n x_i \sigma_i \right) -$$

$$\lambda_{\text{te}} \left(\sum_{i=1}^n x_i (\Sigma x_{cw})_i \right)$$

$$\text{u.c. } x \geq \mathbf{0}$$

¹⁶ See [RON 13] on page 23.

This formulation encompasses the different optimization problems presented in this chapter. We notice that problem [2.20] is of the form:

$$x^*(\lambda) = \arg \min \frac{1}{2} x^\top \Sigma x + \lambda \mathcal{P}(x)$$

u.c. $x \geq \mathbf{0}$

where $\mathcal{P}(x)$ is a penalty function combining different norms. This penalized optimization is frequent in machine learning and is generally solved using the cyclical coordinate descent algorithm.

2.6.2.1. CCD algorithm

The main idea behind the CCD algorithm is to minimize a function $f(x_1, \dots, x_n)$ by minimizing only one direction at each step, whereas classical descent algorithms consider all the directions at the same time. In this case, we find the value of x_i which minimizes the objective function by considering the values taken by x_j for $j \neq i$ as fixed. The procedure is repeated for each direction until the global minimum is reached. This method uses the same principles as Gauss–Seidel or Jacobi algorithms for solving linear systems. The main objective is then to find the update rule.

Convergence of coordinate descent methods requires that $f(x)$ is strictly convex and differentiable. However, Tseng [TSE 01] has extended the convergence properties to a non-differentiable class of functions:

$$f(x_1, \dots, x_n) = f_0(x_1, \dots, x_n) + \sum_{k=1}^m f_k(x_1, \dots, x_n)$$

where f_0 is strictly convex and differentiable and the functions f_k are non-differentiable.

Some properties make this algorithm very attractive. First, it is very simple to understand and implement. Second, the method is efficient for solving large-scale problems. This is why it is used in machine learning theory for computing constrained regressions or supporting vector machine problems [FRI 10]. A further advantage is that the method does not need stepsize descent tuning as opposed to gradient-based methods.

2.6.2.2. Application to the smart beta problem

In problem [2.20], $f_0(x) = \frac{1}{2} x^\top \Sigma x$ is strictly convex and the functions f_k are non-differentiable, meaning that we can apply the CCD algorithm. Let $\mathcal{L}(x)$ be the Lagrange function [2.20]. We have:

$$\frac{\partial \mathcal{L}(x)}{\partial x_i} = (\Sigma x)_i - \lambda_{\text{gmv}} + 2\lambda_{\text{h}} x_i - \frac{\lambda_{\text{erc}}}{x_i} - \lambda_{\text{mdp}} \sigma_i - \lambda_{\text{te}} (\Sigma x_{\text{cw}})_i$$

Let us assume that $\lambda_{\text{erc}} > 0$. At the optimum, we have $\hat{Q}_i \mathcal{L}(x) = 0$ or:

$$x_i (\Sigma x)_i - \lambda_{\text{gmv}} x_i + 2\lambda_{\text{h}} x_i^2 - \lambda_{\text{erc}} - \lambda_{\text{mdp}} x_i \sigma_i - \lambda_{\text{te}} x_i (\Sigma x_{\text{cw}})_i = 0 \quad [2.21]$$

It follows that:

$$x_i^2 (\sigma_i^2 + 2\lambda_{\text{h}}) + x_i \left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{gmv}} - \lambda_{\text{mdp}} \sigma_i - \lambda_{\text{te}} (\Sigma x_{\text{cw}})_i \right) - \lambda_{\text{erc}} = 0$$

we notice that the polynomial function is convex because we have $\sigma_i^2 + 2\lambda_{\text{h}} > 0$. Since the product of the roots is negative¹⁷, we always have two solutions with opposite signs. We deduce that the solution is the positive root of the second-degree equation:

$$x_i^* = \frac{\lambda_{\text{gmv}} + \lambda_{\text{mdp}} \sigma_i + \lambda_{\text{te}} (\Sigma x_{\text{cw}})_i - \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{2(\sigma_i^2 + 2\lambda_{\text{h}})} + \sqrt{\frac{\left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{gmv}} - \lambda_{\text{mdp}} \sigma_i - \lambda_{\text{te}} (\Sigma x_{\text{cw}})_i \right)^2 + 4(\sigma_i^2 + 2\lambda_{\text{h}}) \lambda_{\text{erc}}}{2(\sigma_i^2 + 2\lambda_{\text{h}})}} \quad [2.22]$$

If the values of (x_1, \dots, x_n) are strictly positive, it follows that x_i^* is strictly positive. The positivity of the solution is then achieved after each iteration if the starting values are positive. The coordinate-wise descent algorithm consists of iterating equation [2.22] until convergence and normalizing the solution at the final step.

REMARK 2.6.— When the correlation of the assets is equal to zero, we obtain a closed-form expression:

$$x_i^* = \frac{\lambda_{\text{gmv}} + \lambda_{\text{mdp}} \sigma_i + \lambda_{\text{te}} x_{\text{cw},i} \sigma_i^2}{2(\sigma_i^2 + 2\lambda_{\text{h}})} + \sqrt{\frac{(\lambda_{\text{gmv}} + \lambda_{\text{mdp}} \sigma_i + \lambda_{\text{te}} x_{\text{cw},i} \sigma_i^2)^2 + 4(\sigma_i^2 + 2\lambda_{\text{h}}) \lambda_{\text{erc}}}{2(\sigma_i^2 + 2\lambda_{\text{h}})}}$$

¹⁷ We have $-(\sigma_i^2 + 2\lambda_{\text{h}}) \lambda_{\text{erc}} < 0$.

REMARK 2.7.— We can deduce the risk contributions of risk-based portfolios from equation [2.21]:

$$\mathcal{RC}_i \propto \lambda_{\text{gmV}} x_i - \lambda_{\text{h}} x_i^2 + \lambda_{\text{erc}} + \lambda_{\text{mdp}} \sigma_i x_i + \lambda_{\text{te}} x_i (\Sigma x_{\text{CW}})_i$$

We retrieve the different well-known results. For instance, the risk contributions are equal for the ERC portfolio, correspond to the weights for the GMV portfolio and are proportional to $x_i \sigma_i$ for the MDP, etc.

2.6.2.3. Special cases

In this section, we derive limit cases of problem [2.20] by using the CCD formulation of the solution.

2.6.2.3.1. EW portfolio

If we assume that $\lambda_{\text{mdp}} = \lambda_{\text{te}} = \lambda_{\text{h}} = 0$ and $\lambda_{\text{erc}} > 0$, the solution is reduced to:

$$x_i^* = -\frac{\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{gmV}}}{2\sigma_i^2} + \frac{\sqrt{\left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{gmV}}\right)^2 + 4\sigma_i^2 \lambda_{\text{erc}}}}{2\sigma_i^2}$$

We have $\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{gmV}} \approx |\lambda_{\text{gmV}}|$ when $\lambda_{\text{gmV}} = -\infty$. Using a first-order Taylor expansion in the neighborhood of zero, we obtain:

$$\begin{aligned} \lim_{\lambda_{\text{gmV}} \rightarrow -\infty} x_i^* &= \lim_{\lambda_{\text{gmV}} \rightarrow -\infty} \frac{-|\lambda_{\text{gmV}}| + |\lambda_{\text{gmV}}| \sqrt{1 + \frac{4\sigma_i^2 \lambda_{\text{erc}}}{|\lambda_{\text{gmV}}|^2}}}{2\sigma_i^2} \\ &\approx \lim_{\lambda_{\text{gmV}} \rightarrow -\infty} \frac{-|\lambda_{\text{gmV}}| + |\lambda_{\text{gmV}}| \left(1 + \frac{2\sigma_i^2 \lambda_{\text{erc}}}{|\lambda_{\text{gmV}}|^2}\right)}{2\sigma_i^2} \\ &= \frac{\lambda_{\text{erc}}}{|\lambda_{\text{gmV}}|} \end{aligned}$$

This means that all the weights are constant and equal. We finally obtain the EW portfolio:

$$\lim_{\lambda_{\text{gmV}} \rightarrow -\infty} x^* = x_{\text{ew}} = \frac{1}{n}$$

There is another way to find the EW portfolio due to the Herfindahl index. If we assume that $\lambda_{\text{mdp}} = \lambda_{\text{te}} = \lambda_{\text{gmv}} = 0$ and $\lambda_{\text{erc}} > 0$, the solution is reduced to:

$$x_i^* = -\frac{\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{2(\sigma_i^2 + 2\lambda_h)} + \frac{\sqrt{\left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right)^2 + 4(\sigma_i^2 + 2\lambda_h) \lambda_{\text{erc}}}}{2(\sigma_i^2 + 2\lambda_h)}$$

When $\lambda_h \gg +\infty$, we obtain:

$$x_i^* \approx \frac{\sqrt{\lambda_{\text{erc}}}}{\sqrt{2\lambda_h}}$$

Again, all the weights are constant and we obtain the EW portfolio.

2.6.2.3.2. RP portfolio

If we assume that $\lambda_{\text{gmv}} = \lambda_{\text{te}} = \lambda_h = 0$ and $\lambda_{\text{erc}} > 0$, we have $\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{mdp}} \sigma_i \approx |\lambda_{\text{mdp}}| \sigma_i$ when $\lambda_{\text{mdp}} = -\infty$ and:

$$\lim_{\lambda_{\text{mdp}} \rightarrow -\infty} x_i^* = \frac{\lambda_{\text{erc}}}{|\lambda_{\text{mdp}}| \sigma_i}$$

This means that the weight is inversely proportional to the asset volatility. We then obtain the RP portfolio:

$$\lim_{\lambda_{\text{mdp}} \rightarrow -\infty} x^* = x_{\text{rp}} = \frac{\sigma^{-1}}{\mathbf{1}^\top \sigma^{-1}}$$

2.6.2.3.3. BP portfolio

If we assume that $\lambda_{\text{gmv}} = \lambda_{\text{mdp}} = \lambda_h = 0$ and $\lambda_{\text{erc}} > 0$, we have¹⁸ $\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda_{\text{te}} (\Sigma x_{\text{cw}})_i \approx |\lambda_{\text{te}}| (\Sigma x_{\text{cw}})_i$ when $\lambda_{\text{te}} = -\infty$ and:

$$\lim_{\lambda_{\text{te}} \rightarrow -\infty} x_i^* = \frac{\lambda_{\text{erc}}}{|\lambda_{\text{te}}| (\Sigma x_{\text{cw}})_i}$$

The beta $\beta_i(x_w)$ of the asset i with respect to the CW portfolio x_w is:

$$\beta_i(x_{\text{cw}}) = \frac{(\Sigma x_{\text{cw}})_i}{x_{\text{cw}}^\top \Sigma x_{\text{cw}}}$$

¹⁸ We must also have $(\Sigma x_{\text{cw}})_i \geq 0$.

We deduce that the weight is inversely proportional to the asset beta. We finally obtain the BP portfolio:

$$\lim_{\lambda_{te} \rightarrow -\infty} x^* = x_{bp} = \frac{\beta^{-1}(x_{cw})}{\sum_{j=1}^n \beta_j^{-1}(x_{cw})}$$

2.6.2.3.4. Summary

Finally, the different limit cases are reported in Table 2.7 where $\lambda \geq 0$ is an arbitrary constant.

Parameters	GMV	EW		ERC	MDP	RP	BP	CW
λ_{gmv}	$+\infty$	$-\infty$	0	0	0	0	0	0
λ_h	0	0	$+\infty$	0	0	0	0	0
λ_{erc}	λ	$+\infty$	λ	λ	λ	$+\infty$	$+\infty$	λ
λ_{mdp}	0	0	0	0	$+\infty$	$-\infty$	0	0
λ_{te}	0	0	0	0	0	0	$-\infty$	$+\infty$

Table 2.7. Limits of the smart beta portfolio x^* ($\lambda_{gmv}, \lambda_h, \lambda_{erc}, \lambda_{mdp}, \lambda_{te}$)

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Trend-Following, Risk-Parity and the Influence of Correlations

Trend-following¹ strategies take long positions in assets with positive past returns and short positions in assets with negative past returns. They are typically constructed using futures contracts across all asset classes, with weights that are inversely proportional to volatility, and have historically exhibited great diversification features, especially during dramatic market downturns. However, following an impressive performance in 2008, the trend-following strategy has failed to generate strong returns in the post-crisis period (2009–2013). This period has been characterized by a large degree of co-movement even across asset classes, with the investable universe being roughly split into the so-called risk-on and risk-off subclasses. We examine whether the inverse-volatility weighting scheme, which effectively ignores pairwise correlations, can turn out to be suboptimal in an environment of increasing correlations. By extending the conventionally long-only risk-parity (equal risk contribution) allocation, we construct a long-short trend-following strategy that makes the use of risk-parity principles. Not only do we significantly enhance the performance of the strategy, but we also show that this enhancement is mainly driven by the performance of the more sophisticated weighting scheme in extreme average correlation regimes.

3.1. Introduction

Trend-following is a simple trading strategy that consists of long positions for upward trending assets and short positions for falling assets. This strategy profits when

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assets continue performing in-line with their most recent performance. In other words, this strategy aims to take advantage of return autocorrelation empirical patterns².

Trend-following strategies are largely employed by systematic funds, such as commodity trading advisor (CTA) and managed futures funds³ (see [COV 09] for a broad overview), and are typically constructed using futures contracts across all asset classes⁴ in an effort to increase diversification. The benefit from using futures contracts is twofold: first, taking long and short positions using futures contracts is equally straightforward (in contrast, for instance, to using cash equity instruments), and second, the use of futures contracts allows the inclusion of non-equity contracts in the portfolio (e.g. trading commodities for investment purposes is typically done using futures).

The construction of a trend-following portfolio involves an important challenge, which is the choice of the weighting scheme that should be employed, given that contracts from different asset classes have very different risk-return profiles (a typical commodity or equity index contract is much more volatile than a government bond contract). An equal-weight allocation would result in a portfolio that would be dominated in terms of risk by the higher volatility assets, i.e. equities and commodities. Instead, the weighting scheme should make use of the relative riskiness of the contracts in order to allocate risk as evenly as possible across all constituents.

2 Trend-following (also known as time-series momentum) is structurally different from the conventional cross-sectional winners-minus-losers momentum strategy of Jegadeesh and Titman [JEG 93, JEG 01]. The former is a strategy that takes a position in every asset of the investable universe, is not cash-neutral and, at the extreme, can be in a long-only or short-only state (if all assets have a positive or negative past return, respectively); hence, it is a clear bet on the serial correlation of returns. Instead, the latter invests only in the extremes of the cross-section (e.g. top vs. bottom decile), it is – in theory – cash-neutral and its profitability can be either attributed to cross-sectional return dispersion premia or time-series return correlation (the recent paper by Asness, Moskowitz and Pedersen [ASN 13] documents cross-sectional momentum patterns “everywhere”). For an analysis of the relationship between the two momentum strategies, see [MOS 12] and [CLA 14a].

3 Baltas and Kosowski [BAL 13b] show that futures-based trend-following strategies can explain large part of the return variation of CTA benchmark indices.

4 Szakmary, Shen and Sharma [SZA 10] study trend-following strategies in commodity markets, Burnside, Eichenbaum and Rebelo [BUR 11] study carry and trend-following strategies in currency markets and finally in two recent papers Clare, Seaton, Smith and Thomas [CLA 14a, CLA 14b] study both cross-sectional momentum and trend-following strategies in commodity markets and across broad market indices of different asset classes (equities, bonds, commodities and real estate) from a global asset allocation point of view.

The typical choice is to employ inverse-volatility weights, so that all assets enter the portfolio with the same *ex ante* volatility. For obvious reasons, this scheme is known as the volatility-parity scheme. This approach has been followed by the large majority of academic research papers focusing on the topic [MOS 12, HUR 12, HUR 13, BAL 13b, BAL 15]. Most importantly as long as all pairwise correlations are equal, this weighting scheme splits the total portfolio volatility equally across all portfolio constituents.

Using a broad dataset of 35 futures contracts from all asset classes (energy, commodities, fixed income, foreign exchange and equities), we construct a volatility-parity trend-following strategy and document its superior performance relative to a long-only equivalent over a long history of more than 25 years (1988–2013). By employing long and short positions, the trend-following strategy benefits from (either upward or downward) trending markets and achieves in neutralizing (at least unconditionally) the exposure to standard benchmark indices such as the MSCI World Index or the S&P GSCI Index. The strategy benefits from the combination of different asset classes and delivers a Sharpe ratio of 1.31 compared to one of 0.70 for the long-only equivalent over the entire sample period.

Contrary to its historical superior performance, and following an impressive performance in 2008, the trend-following strategy has consistently delivered very poor performance in the post-crisis period (see also [HUR 12] and [BAL 13b]). Between January 2009 and December 2013, a volatility-parity trend-following strategy delivered a Sharpe ratio of 0.31 against a Sharpe ratio of 0.59 for the long-only counterpart. What could have possibly gone wrong?

Following the introduction of the Commodity Futures Modernization Act (CFMA) in 2000, commodities have started becoming more correlated to each other as futures markets became accessible to investors as a way to hedge commodity price risk in what is often referred to as the “*financialization of commodities*”⁵. More generally and more aggressively, following the recent financial crisis in 2008, assets from different asset classes (and not just commodities) have started exhibiting stronger co-movement patterns, with the diversification benefits being dramatically diminished.

In an environment of increased asset co-movement, the volatility-parity weighting scheme can be deemed a suboptimal choice. By ignoring the covariation between assets, volatility-parity fails to allocate equal amount of risk to each portfolio constituent. This is the reason why volatility-parity is also often referred to

⁵ The financialization of commodities has recently been a very active research field. Indicatively, see the recent papers by Falkowski [FAL 11], Irwin and Sanders [IRW 11], Tang and Xiong [TAN 12], Basak and Pavlova [BAS 14], Boons, deRoos and Szymanowska [BOO 14], Cheng and Xiong [CHE 14] and Henderson, Pearson and Wang [HEN 15] as well as references therein.

as naive risk-parity [BHA 12]. Following these observations, one possible reason for the recent lacklustre performance of trend-following can be the suboptimal weighting scheme that ignores pairwise correlations (see, for example, [BAL 15]). Our aim is to address this particular feature of the strategy and construct a portfolio that formally accounts for pairwise correlations.

At this stage, it is important to stress that the profitability of a trend-following strategy depends on two factors: (1) the existence of serial-correlation in the return series and (2) the efficient combination of assets from various asset classes. It is obvious that the first factor is of utmost importance for the profitability of the strategy; non-existence of persistent price trends cannot be alleviated by a more robust weighting scheme. By amending the volatility-parity scheme in a way that accounts for pairwise correlations, we can only address any inefficiency in the risk allocation between portfolio constituents. However, it is reasonable to argue that a different portfolio allocation technique can only do so much.

In principle, an optimal allocation to risk that would also account for correlations would optimally overweight assets, which correlate less with the rest of the universe and underweight assets that correlate more with the rest of the universe in an effort to improve the overall portfolio diversification. This is the principle of the risk-parity portfolio construction methodology (also known as the equal risk contribution scheme), that is, to equate the contribution to risk from each portfolio constituent, after accounting for any pairwise correlation dynamics. Risk-parity has been a very popular portfolio construction technique during the last decades due to its remarkable performance (see, for example, [AND 12] and [ASN 12]⁶) and has, therefore, been a topic of extensive research⁷.

Applying the risk-parity methodology directly to a trend-following strategy is not admissible because risk-parity is defined as a long-only allocation framework⁸. Inspired by Jessop *et al.* [JES 13], we contribute to the literature by extending the conventional long-only risk-parity framework, in a way that also allows for short positions. To achieve this extension, we first generalize the conventional long-only

6 Both papers by Anderson *et al.* [AND 12] and Asness *et al.* [ASN 12] employ inverse-volatility weights (volatility-parity), which they call “risk-parity” weights, for a stocks and bonds portfolio (2-asset portfolio). To avoid confusion, a risk-parity allocation for two assets degenerates mathematically into a volatility-parity allocation. Along these lines, their claim for “risk-parity” is valid as a special 2-asset case.

7 The long list of papers includes [MAI 12, BHA 11, INK 11, LEE 11, CHA 11, CHA 12, BHA 12, LEO 12, LOH 12, BER 13, LOH 14, FIS 15, JUR 15].

8 It is worth highlighting that two recent papers by Clare, Seaton, Smith and Thomas [CLA 14a, CLA 14b] claim to combine risk-parity with trend-following strategies, but in practice, they only employ conventional volatility-parity schemes that they call “risk-parity”. Similarly, Fisher *et al.* [FIS 15] identify “risk-parity” portfolios which are effectively volatility-parity portfolios and reserve the term “equal risk contribution” for what we call “risk-parity”.

risk-parity scheme into a risk-budgeting scheme, which, in turn, allows for the introduction of long and short positions in the overall portfolio.

The empirical question is whether this more sophisticated scheme can overcome the limitations of volatility-parity and consequently hedge against drawdowns experienced in high-pairwise-correlation states. Our findings show that the trend-following portfolio that employs risk-parity principles constitutes a genuine improvement to the traditional volatility-parity variant of the strategy. The Sharpe ratio of the strategy increases from 1.31 to 1.48 over the entire sample period (April 1988–December 2013), but most importantly it more than doubles over the post-crisis period (January 2009–December 2013), from 0.31 to 0.78. The improvement is both economically and statistically significant. A correlation event study shows that the improvement is mainly driven by the superior performance of the risk-parity variant of the strategy in extreme average correlation conditions.

This chapter is divided as follows. Section 3.2 describes the construction of a typical trend-following portfolio using a volatility-parity weighting scheme. Section 3.3 presents the futures dataset that is used for the empirical analysis and section 3.4 presents a thorough empirical evaluation of trend-following strategies, while highlighting their recent lacklustre performance. Section 3.5 provides the theoretical framework under which a long-only risk-parity framework can be extended into a long-short allocation and section 3.6 reports the empirical benefits when this allocation is used for the trend-following strategy. Finally, section 3.7 concludes the Chapter.

3.2. Methodology

This section describes the steps for constructing a typical trend-following portfolio that employs a volatility-parity weighting scheme.

3.2.1 Constructing a trend-following strategy

Let N_t denote the number of available assets at time t . A trend-following (*TF*, henceforth) strategy involves taking a long or short position on each asset i , based on the sign of the past excess return over a prescribed lookback period that is typically equal to 12 months⁹. Let $w_t^{i,Gross}$ denote the gross (absolute) weight invested in asset i at time t . Trivially, $\sum_{i=1}^{N_t} w_t^{i,Gross} = 100\%$ and the return of the strategy is given by:

$$r_{t,t+1}^{TF} = \sum_{i=1}^{N_t} \underbrace{\text{sign}(r_{t-12,t}^i) \cdot w_t^{i,Gross}}_{w_t^{i,Net}} \cdot r_{t,t+1}^i \quad [3.1]$$

9 In unreported results, we find that a 12-month horizon generates the largest Sharpe ratio for trend-following strategies across each asset class in line with Moskowitz *et al.* [MOS 12] and Baltas and Kosowski [BAL 13b]. See also [BAL 13a].

The net weights, denoted by $w_t^{i,Net}$, do not in practice add up to 100%, since they can take either positive or negative values.

Trend-following strategies are typically implemented using a constant-volatility¹⁰ (*CV*, henceforth) overlay by targeting *ex ante* a prescribed level of volatility σ_{TGT} . This requires employing dynamic leverage that is equal to the ratio between the running realized volatility of the unlevered trend-following strategy of equation [3.1], denoted by σ_t^{TF} , and the target level, σ_{TGT} . The generalized formulation of a constant-volatility trend-following (*CVTF*) strategy is, therefore, given by:

$$r_{t,t+1}^{CVTF} = \frac{\sigma_{TGT}}{\sigma_t^{TF}} \cdot \sum_{i=1}^{N_t} w_t^{i,Net} \cdot r_{t,t+1}^i \quad [3.2]$$

The dynamic leverage equals the ratio $\sigma_{TGT}/\sigma_t^{TF}$. As an example, if $\sigma_{TGT} = 10\%$ and at the end of some month the running volatility of the unlevered strategy is 5%, then all positions for the forthcoming month should be doubled (a 2x leverage ratio).

The final step is to define the functional form of the portfolio weights in equation [3.2].

3.2.2. Volatility-parity scheme

Given that trend-following strategies are formed across multiple asset classes, whose assets have very different risk profiles (as presented later in Figure 3.2), it is critical to determine a weighting scheme that assigns a weight to every asset that is a function of its underlying riskiness, in an effort to construct a strategy with a fairly balanced distribution of risk across assets and asset classes.

The obvious and simplest choice is to employ inverse-volatility weights, so that all assets enter the portfolio with the same *ex ante* volatility. For this reason, this weighting scheme is also called volatility-parity (*VP*, henceforth):

$$w_t^{i,Gross,VP} = \frac{1/\sigma_t^i}{\sum_{j=1}^{N_t} 1/\sigma_t^j}, \quad \forall i \quad [3.3]$$

This weighting scheme has been used extensively in every academic study that focuses on trend-following strategies; see [MOS 12, HUR 12, HUR 13, BAL 13b, BAL 15]. It can be shown that *VP* can split the portfolio volatility equally across all portfolio constituents as long as all pairwise correlations are equal. In practice, as we

¹⁰ A similar technique has been employed by Barroso and Santa-Clara [BAR 14] and Daniel and Moskowitz [DAN 14], who focus on cross-sectional winners-minus-losers momentum strategies. See also [HAL 12, HAL 14].

discuss in a later section of the chapter, the pairwise correlations between assets and asset classes are neither equal nor constant over time. Under such conditions, the distribution of risk of a *VP* scheme is not uniform and for this reason the scheme is also called naive risk-parity [BHA 12].

Going back to the construction of our benchmark trend-following strategy, substituting the *VP* weights back into equation [3.2] yields the return series of the volatility-parity trend-following (*VPTF*) strategy:

$$r_{t,t+1}^{VPTF} = \frac{\sigma_{TGT}}{\sigma_{TF}^2} \sum_{i=1}^{N_t} \text{sign}(r_{t-12,t}^i) \cdot \frac{(\sigma_t^i)^{-1}}{\sum_{j=1}^{N_t} (\sigma_t^j)^{-1}} \cdot r_{t,t+1}^i \quad [3.4]$$

For comparison purposes, we also construct a long-only (*VPLO*, henceforth) strategy benchmark as the inverse-volatility weighted portfolio of the assets, targeted again at the desired level of overall portfolio volatility:

$$r_{t,t+1}^{VPLO} = \frac{\sigma_{TGT}}{\sigma_{LO}^2} \sum_{i=1}^{N_t} (+1) \cdot \frac{(\sigma_t^i)^{-1}}{\sum_{i=1}^{N_t} (\sigma_t^i)^{-1}} \cdot r_{t,t+1}^i \quad [3.5]$$

3.3. Data description

In order to construct trend-following strategies, we use Bloomberg daily closing futures prices for 35 contracts across all asset classes: six energy contracts, ten commodity contracts, six government bond contracts, six FX contracts and seven equity index contracts. The choice of the cross-section of contracts is presented in Table 3.1 and is considered to be fairly dispersed both across asset classes and global regions.

We restrict the sample period to start from January 1987, when all asset classes have at least one contract traded and the cross-section is relatively diverse with 20 contracts being traded in total. Figure 3.1 presents the evolution in the number of contracts for each asset class over time until the end of the sample period in December 2013.

It is important to note that futures contracts have, by their nature, two idiosyncratic features, which do not characterise spot cash equity instruments. First, futures contracts have finite life and are only traded for a short period of time before expiration. Second, futures contracts are zero-cost investments and, in theory, no capital is required to initiate a (long or short) position. In practice, entering into a new futures position implies posting collateral in form of an initial margin payment that is typically a small fraction of the prevailing futures price and a function of the contemporaneous riskiness (measured by volatility or VaR) of the underlying entity.

Energy		Commodities		Fixed income		FX		Equities	
Brent crude	July-1988	Cocoa	January-1987	German Bobl 5 Yr	November-1991	AUD	February-1987	Dax	December-1990
Gas oil	August-1989	Copper	January-1989	German Bund 10 Yr	December-1990	CAD	January-1987	EuroStoxx 50	July-1998
Gasoline	November-2005	Corn	January-1987	JGB 10 Yr	January-1987	CHF	January-1987	FTSE 100	March-1988
Heating oil #2	January-1987	Cotton #2	January-1987	US T-Notes 5 Yr	June-1988	EUR	June-1998	Kospi 200	June-1996
Light crude	January-1987	Gold	January-1987	US T-Notes 10 Yr	January-1987	GBP	January-1987	Nasdaq	May-1996
Natural gas	May-1990	Live cattle	January-1987	US T-Notes 30 Yr	January-1987	JPY	January-1987	Nikkei	October-1988
		Silver	January-1987					S&P 500	January-1987
		Soybeans	January-1987						
		Sugar #11	January-1987						
		Wheat	January-1987						

Table 3.1. The futures contracts that we use including the first month that each series is available. The dataset is retrieved from Bloomberg and the sample period ends in December 2013

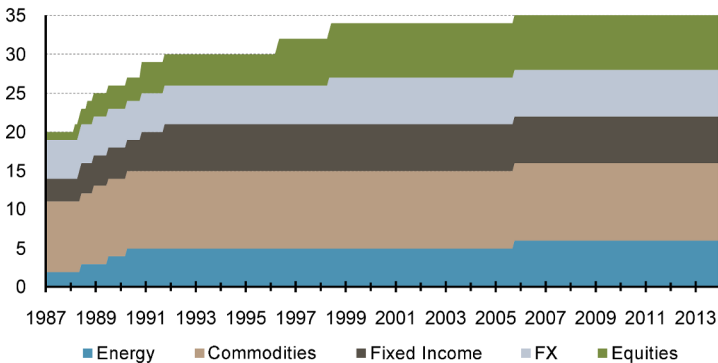


Figure 3.1. The number of available futures contracts per asset class over time. The dataset is retrieved from Bloomberg and the sample period is January 1987–December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

These specific features of futures contracts complicate the back-testing of futures-based trading strategies as continuous price-series have to be constructed and specific assumptions have to be put in place for the calculation of holding period returns as illustrated in [BAL 13a] and [BAL 15]. We address these issues by using the generic continuous-price series provided by Bloomberg, which are constructed in such a way so that we always trade the most liquid contract (typically the “front” contract), and calculate for each futures contract fully collateralized monthly returns in excess of the prevailing risk-free rate using the formula¹¹:

$$r_{t,t+1} = \frac{F_{t+1} - F_t}{F_t} \quad [3.6]$$

where F_t and F_{t+1} denote the futures prices at the end of months t and $t + 1$.

Figure 3.2 presents annualized volatilities of all the assets in our dataset. What easily stands out is the large cross-sectional dispersion in volatilities, with fixed income contracts exhibiting traditionally the lowest volatilities. At the other end of the distribution, energy contracts are the most volatile contracts in the cross-section.

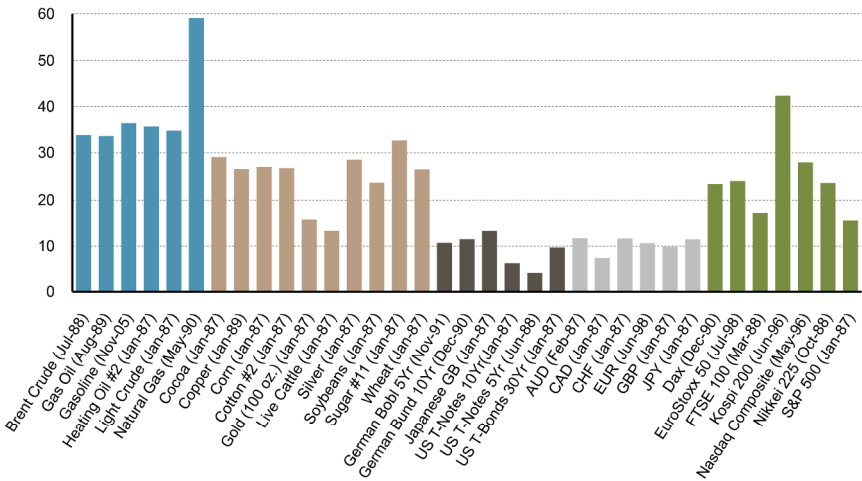


Figure 3.2. The unconditional volatility for each asset of the dataset. The coloring scheme separates the five asset classes (energy, commodities, fixed income, FX rates and equities). The legend states the starting month for each asset. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

¹¹ This approach in estimating returns of futures contracts is fairly standard in the academic literature. Indicatively, see [DER 00, MOS 12] and [BAL 13, BAL 15].

3.4. Performance evaluation of trend-following

We start our empirical analysis by evaluating the performance of the trend-following strategy over the entire sample period and then focus on the most recent post-crisis period (2009–2013). The portfolio is rebalanced every month, when new trend-following signals are generated using the past 12-month performance of the assets. The weighting scheme (volatility-parity) is estimated using a window of the most recent 90 days until the end of each month. Finally, the strategy targets a 10% level of volatility; this requires an estimate of the running volatility of the unlevered strategy, which is estimated using the most recent 60 daily returns. The initial training period of the strategy is 12 months (for the signal generation) and another 60 business days for the calculation of the initial leverage ratio. Overall, this amounts to 15 months and therefore our sample period starts in April 1988.

Figure 3.3 presents the cumulative returns of a VP trend-following strategy and its long-only variant across the two periods. Full-sample performance statistics as well as correlations with major benchmark indices of asset classes are reported in Table 3.2.

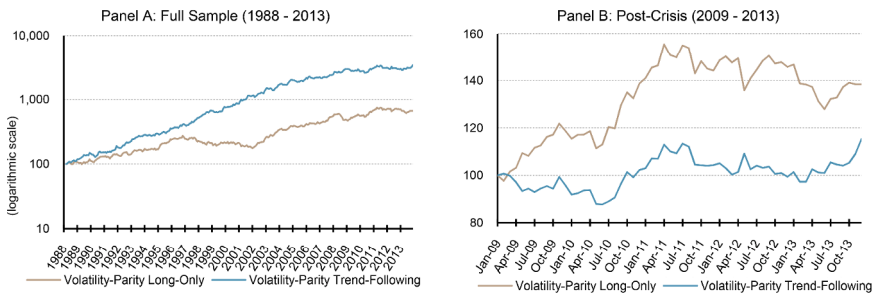


Figure 3.3. The cumulative returns of a long-only strategy and a trend-following strategy that employ a volatility-parity weighting scheme estimated using the past 90 days. The sample period is from April 1988 to December 2013 in Panel A and from January 2009 to December 2013 in Panel B. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Over the entire sample period, the outperformance of the trend-following strategy is largely pronounced. The strategy exhibits a Sharpe ratio that is twice as big as that of the long-only strategy (1.31 vs. 0.70). In unreported results, we find that trend-following strategies within each asset class deliver Sharpe ratios between 0.58 and 0.71 over the same sample period, which means that the combination of different asset classes leads to a substantial improvement in the performance of the strategy. A detailed examination of trend-following patterns across various asset classes is presented in [MOS 12] and [BAL 13a].

Performance statistics	VPLO	VPTF	Correlations	VPLO	VPTF
<i>Ann. geometric mean (%)</i>	7.63	14.67	<i>Commodity benchmark:</i>		
<i>t-statistic (Newey–West)</i>	3.25	6.71	<i>– S&P GSCI Commodity Index</i>	0.58	0.11
<i>Ann. volatility (%)</i>	11.54	10.96	<i>Fixed income benchmark:</i>		
<i>Skewness</i>	-0.12	0.38	<i>– JPM Global Bond Index</i>	0.69	0.31
<i>Kurtosis</i>	3.09	3.27	<i>FX benchmarks:</i>		
<i>Maximum drawdown (%)</i>	35.70	14.20	<i>– Trade-Weighted USD Index</i>	-0.54	-0.14
<i>Sharpe ratio (annualized)</i>	0.70	1.31	<i>– USD/JPY</i>	-0.45	-0.18
<i>Sortino ratio (annualized)</i>	1.13	2.81	<i>Equity benchmarks:</i>		
<i>Calmar ratio</i>	0.21	1.03	<i>– MSCI World (DM)</i>	0.52	-0.01
<i>Monthly turnover (%)</i>	11.51	31.69	<i>– MSCI Emerging Markets</i>	0.37	-0.05

Table 3.2. Performance statistics and correlations with various benchmark indices (retrieved from Bloomberg) using monthly returns for the volatility-parity long-only strategy (VPLO) and for the volatility-parity trend-following strategy (VPTF) across all contracts. The *t*-statistic of the mean return is calculated using Newey and West [NEW 87] standard errors. The Sortino ratio is defined as the annualized arithmetic mean return over the annualized downside volatility. The Calmar ratio is defined as the annualized geometric mean return over the maximum drawdown. The sample period is from April 1988 to December 2013

Most importantly, the trend-following strategy exhibits very low correlations with the various benchmark indices¹², whereas the long-only strategy bears, by construction, strong directional bets and therefore large correlations with these indices. This piece of evidence justifies the use of trend-following strategies as diversification vehicles.

Contrary to the above evidence, when we shift our attention to the most recent post-crisis period, between January 2009 and December 2013, we find that the trend-following strategy has dramatically underperformed (see Figure 3.3, Panel B).

¹² It is worth commenting that the trend-following strategy exhibits an almost zero unconditional correlation with the MSCI World Index (point estimate is -0.01). In unreported results, we find that this seemingly uncorrelated pair bears interesting nonlinear (higher order) correlation dynamics and, in particular, the trend-following strategy exhibits strong positive returns in large positive or negative states of the market in line with Moskowitz *et al.* [MOS 12].

The Sharpe ratio in the aftermath of the global financial crisis has been 0.31, compared to 0.59 of the long-only strategy. This recent lacklustre performance for a strategy that has historically delivered very strong returns across (both up and down) markets over several decades has been largely highlighted both in academic studies (e.g. [BAL 13b]) as well as in the press. What could have possibly gone wrong?

One of the most prevalent claims for this recent lacklustre performance of trend-following has been the post-crisis increased level of pairwise correlations (see, for example, [BAL 15]). In an environment of increased correlations, diversification benefits diminish and assets are clustered into “Risk-On” and “Risk-Off” subsets. Figure 3.4 presents a 90-day rolling estimate of the average pairwise correlation across all futures contracts of our dataset. It is obvious that the level of pairwise correlations has significantly shifted during the last decade of the sample period, exhibiting one of the most prevalent increases in record during 2008.

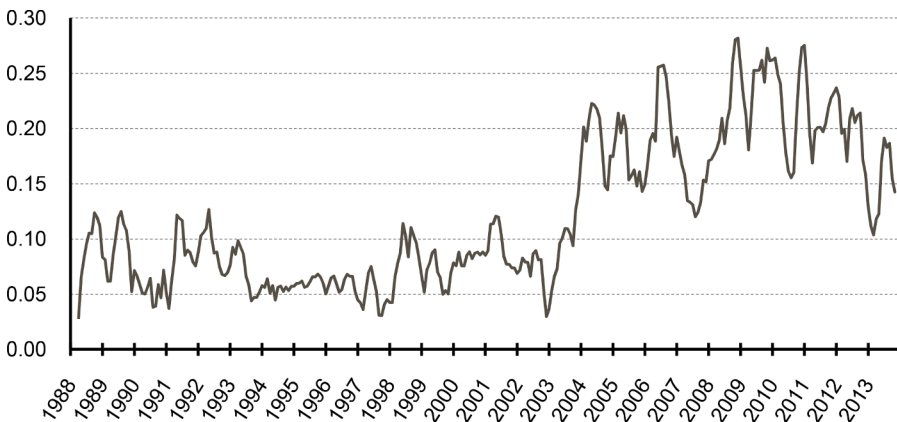


Figure 3.4. Monthly average pairwise correlation across all contracts using a 90-day rolling estimation window. The sample period is from April 1988 to December 2013

In order to comprehend these dramatic shifts in asset pairwise correlations, Figure 3.5 presents the average correlation between assets of the same asset class (intra-asset class correlations) and the average correlation between assets of one asset class with the assets of all other asset classes (inter-asset class correlations). The evidence shows that the recent dramatic shift in average pairwise correlation is largely driven by the fact that all inter-asset class correlations have significantly increased during the most recent decade. The only exception is the correlation of fixed income assets with the rest of the asset classes. These plots clearly illustrate the risk-off (fixed income) and risk-on (remaining asset classes) regrouping of assets after the recent financial crisis.

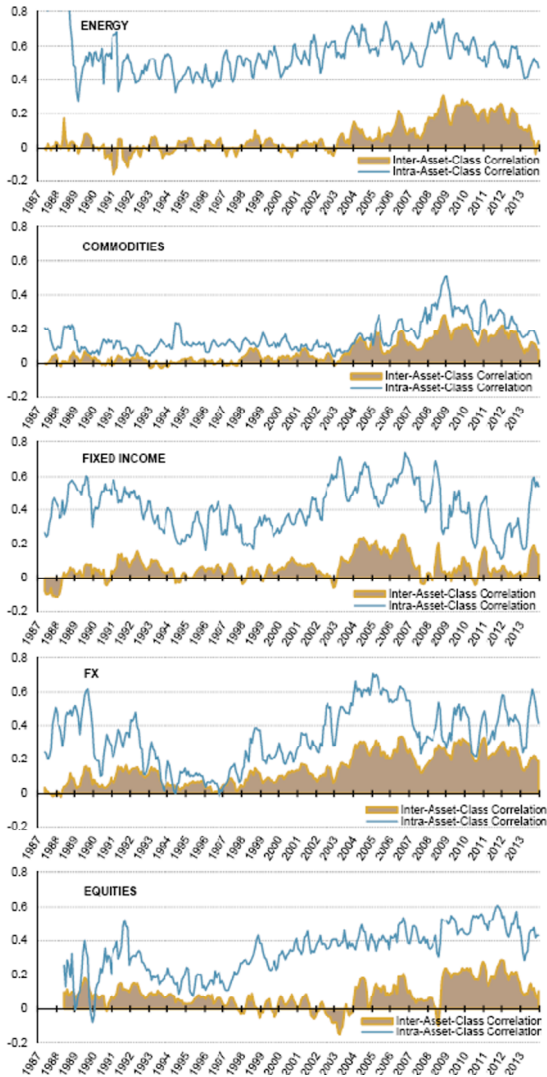


Figure 3.5. 90-day average pairwise correlation at the inter-class and intra-asset class level

3.4.1. Volatility-parity ignores pairwise correlations

Following the documentation of the correlation patterns, we return to the performance of the trend-following strategy. The weighting scheme that has been employed so far for the analysis is the volatility-parity scheme, which, as already

highlighted, completely ignores the correlation of the assets. We, therefore, hypothesize that under an increased correlation environment, such as the post-crisis era, volatility-parity is not an optimal risk allocation methodology.

Before evaluating this hypothesis, we should first illustrate how volatility-parity has allocated weights and risk across the five asset classes over time. The gross weight per asset class is simply calculated by summing up the individual gross weights of the constituents of each asset class at the end of each month, i.e. $\sum_j |w_t^j|$, where the summation is performed only across the assets within each asset class.

In order to calculate the percentage risk allocation per asset class, we should first define the so-called marginal contribution to risk (*MCR*, henceforth) for each asset. This is defined as the partial derivative of portfolio volatility at any point in time, σ_t^P , with respect to the contemporaneous weight of each asset, w_t^i , or in other words the change in portfolio volatility for a small (hence, marginal) change in the asset weight:

$$MCR_t^i = \frac{\partial \sigma_t^P}{\partial w_t^i} \quad [3.7]$$

It is easy to show that the *MCRs* satisfy the following identity¹³:

$$\sum_{j=1}^{N_t} w_t^j \cdot MCR_t^j = \sigma_t^P \quad [3.8]$$

Given this definition, the percentage contribution to risk from each asset class at the end of each month is calculated by summing the weighted *MCR* of each asset in the asset class, normalized by the total portfolio volatility, i.e. $\sum_j w_t^j \cdot MCR_t^j / \sigma_t^P$, where the summation is performed only across the assets within the same asset class. It is critical to note that if all pairwise correlations are equal, then the percentage contribution of each asset is simply $1/N_t$ and therefore the percentage contribution of each asset class would be proportional to the asset class size. Given that we have fairly balanced asset classes (six energy contracts, ten commodity contracts, six government bond contracts, six FX contracts and seven equity index contracts), this would result in almost equal asset class contributions to portfolio risk.

Figure 3.6 presents the time evolution of the weight and risk allocation across the five asset classes. The former is fairly stable over time, due to the large persistence of volatility measures; the strategy allocates on average 33% in fixed income, 27% in FX, 21% in commodities, 12% in equities and 7% in energy. However, when

13 Contrast this with the fact that the weighted sum of volatilities typically exceeds the overall portfolio volatility due to the diversification benefit: $\sum_{j=1}^{N_t} w_t^j \cdot \sigma_t^j \geq \sigma_t^P$.

translated into risk terms, this allocation is nowhere near an equal distribution across constituents. The risk allocation per asset class is largely unstable over time and, more importantly, there are times during which some asset classes (with the given *VP* weights) have negative risk contribution (i.e. diversify risk away), like for fixed income during 2007. At such times, it would be reasonable to increase the allocation to these asset classes, as this would lower portfolio risk.

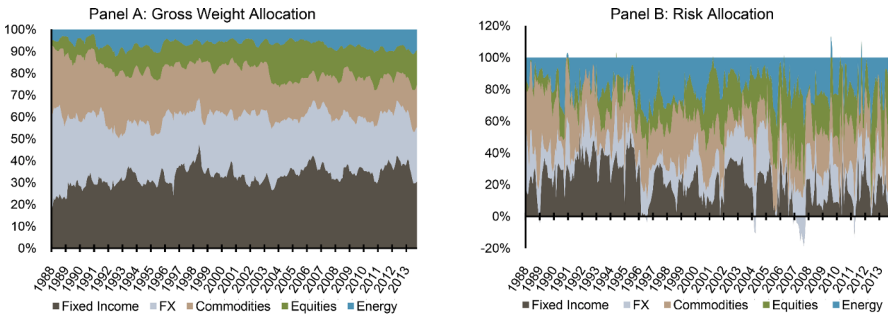


Figure 3.6. The sum of gross weights (Panel A) and the percentage constitution to risk (Panel B) for each asset class over time when a volatility-parity weighting scheme is employed for a trend-following strategy. The sample period is from April 1988 to December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Evidently, the *VP* allocation does not equate the risk contribution from each asset class and this is solely due to the time-varying nature of the correlations between assets and asset classes as documented in Figure 3.5.

Going back to the hypothesis that the recent poor performance of trend-following could be potentially related to the suboptimal allocation of risk from the *VP* scheme, we next conduct a correlation event study that is presented in Figure 3.7. In particular, we first calculate the average pairwise correlation of the universe of all assets for each calendar month in our sample, using only the daily returns of the assets within each particular month. Next, we group all months of the dataset in four correlation regimes: low (less than 5%), medium (5–10%), high (10–20%) and extreme (more than 20%). For each correlation regime, we estimate the Sharpe ratio of the trend-following strategy. The evidence is overwhelming. The performance of the *VP* trend-following strategy drops dramatically when the level of average pairwise correlation deviates significantly away from zero and into the positive territory and the drop is most dramatic as we move from a high correlation regime (Sharpe ratio of 1.28) to an extreme one (Sharpe ratio of 0.27).

This performance drop can be attributed to two possible reasons: (1) the absence of strong price trends in high correlation regimes and/or (2) the suboptimal distribution of risk to portfolio constituents by the volatility-parity weighting scheme. The objective of this chapter is to focus solely on the portfolio construction implications for trend-following strategies and therefore address the extent to which a more sophisticated portfolio construction methodology that accounts for the time-varying nature of correlations can improve the diversification of the portfolio in higher correlation regimes and therefore improve its risk-adjusted performance.

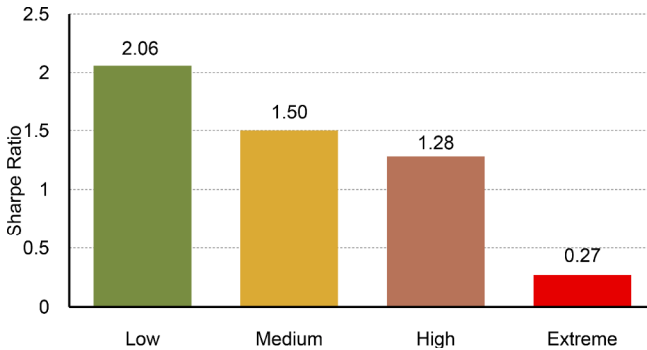


Figure 3.7. The annualized Sharpe ratio of a volatility-parity trend-following strategy for four different regimes of average pairwise correlation: low (less than 5%; 49 months), medium (5–10%; 109 months), high (10–20%; 92 months) and extreme (above 20%; 59 months). The sample period is from April 1988 to December 2013

3.5. Risk-parity principles

This section outlines the steps to extend the risk-parity principles to a long-short allocation. This involves an intermediate step of defining a long-only and, subsequently, a long-short risk-budgeting framework.

3.5.1. Risk-parity

Risk-parity (*RP*, henceforth) constitutes the extension to the volatility-parity weighting scheme and its objective is to distribute the total portfolio risk (volatility) equally across the portfolio constituents, after accounting for pairwise correlations. Using the property of equation [3.8], this objective is equivalent to equating the weighted marginal contribution to risk of each constituent:

$$w_t^{i,RP} \cdot MCR_t^i = \text{constant}, \quad \forall i \quad [3.9]$$

where MCR is defined as in equation [3.7]. The constant in the above equation can be trivially shown to be equal to the $\frac{1}{N_t}$ th of the portfolio volatility.

The solution to the RP objective does not come in closed-form (unless all pairwise correlations are equal, in which case the solution boils down to VP), but instead through an optimization. Following [JES 13], this can be attained by maximizing the sum of logarithmic weights, subject to a risk constraint of target volatility σ_{TGT} for the whole portfolio¹⁴:

Long-Only Risk-Parity:

$$\text{Maximize: } \sum_{i=1}^{N_t} \log(w_t^i) \quad [3.10]$$

$$\text{Subject to: } \sqrt{\mathbf{w}_t' \cdot \Sigma_t \cdot \mathbf{w}_t} \leq \sigma_{TGT}$$

where \mathbf{w} denotes the vector of weights and Σ denotes the variance-covariance matrix of the universe, both evaluated at time t . It is easy to show (see the appendix in section 3.8) that the Lagrangian of the optimization results in the risk-parity objective of equation [3.9]. This optimization is solved in practice, using a nonlinear optimizer with the initial guess for the weight vector being the VP solution, because this is exactly the point of convergence of RP weights when all correlations are equal:

$$w_t^{i,RP,initial} = w_t^{i,VP} = \frac{1/\sigma_t^i}{\sum_{j=1}^{N_t} 1/\sigma_t^j}, \quad \forall i \quad [3.11]$$

The final point that should be made is that the risk-parity portfolio weights that come out of the optimization do not typically add up to 100%. Rescaling the resulting weights post-optimization is admissible, because volatility, as a measure of risk, exhibits positive homogeneity (see the appendix in section 3.8).

Risk-parity constitutes one of the most popular risk-based portfolio construction methodologies, however, it has only been defined for a long-only allocation; the objective function in the optimization does not allow for negative weights (due to the logarithm) and, in fact, the optimization sets a natural bound at zero for all weights. The risk-parity portfolio will always have strictly positive weights for *all* the assets.

¹⁴ Logarithmic weights for the formulation of risk-parity portfolios are also used by Kaya [KAY 12a], Kaya and Lee [KAY 12b] and Roncalli [RON 14].

For our purpose, which is the application of risk-parity principles to a trend-following strategy, the above formulation cannot be used. Using *VP* weights, it is straightforward to calculate gross weights that are inversely proportional to the asset volatilities and then invert the weights for these assets that we require a short position. However, solving a long-only risk-parity framework and subsequently inverting these weights is not admissible. The correlation structure of a long-only universe is very different from the correlation structure of a long-short universe. A short position on a particular asset means that all correlations of this asset with the rest of the universe switch sign. It is, therefore, *signed* correlations that should be used for the determination of a risk-parity long-short portfolio. Simply inverting the long-only solution for the assets with a short position is completely incorrect. We need a proper long-short framework. In order to introduce this, we first present the concept of long-only risk-budgeting.

3.5.2. Risk-budgeting

The long-only risk-parity paradigm can be generalized to a long-only risk-budgeting (*RB*) framework, under which each portfolio constituent contributes an amount to the overall portfolio volatility that is proportional to a certain positive asset-specific score, denoted by s_t^i ; these scores can be, for instance, the ranks from a customized screen of the universe. As an example, if $s_t^i = 2s_t^j$ for two assets i and j , then the objective is for asset i to have a weighted *MCR* that is twice as big as that of asset j . Hence, the *RB* objective is:

$$w_t^{i, RB} \cdot MCR_t^i \propto s_t^i, \quad \forall i \quad [3.12]$$

This objective can be attained by solving a modified version of the optimization [3.10] as also shown in [KAY 12b]:

Long-Only Risk-Budgeting:

$$\text{Maximize: } \sum_{i=1}^{N_t} s_t^i \cdot \log(w_t^i) \quad [3.13]$$

$$\text{Subject to: } \sqrt{\mathbf{w}_t' \cdot \boldsymbol{\Sigma}_t \cdot \mathbf{w}_t} \leq \sigma_{TGT}$$

Indeed, the Lagrangian of this optimization coincides with equation [3.12]. The key difference to the original RP framework is the introduction of the asset-specific score in the objective function. Solving this optimization is again fairly

straightforward, using an initial guess that is a score-adjusted variant of the VP weights:

$$w_t^{i, RB, initial} = \frac{s_t^i / \sigma_t^i}{\sum_{j=1}^{N_t} s_t^j / \sigma_t^j}, \quad \forall i \quad [3.14]$$

The long-only risk-budgeting framework is the intermediate step that we need in order to introduce the long-short risk-parity portfolio construction methodology.

3.5.3. Long-short risk-budgeting

The interesting question is whether we can allow the risk-specific scores to be negative. Motivated by the arguments of Jessop *et al.* [JES 13], if the scores are estimates or views of expected returns of the assets, then these should be allowed to be negative. In other words, a negative view for a certain asset can be used as a signal for a short position. Based on this, we extend the risk-budgeting framework to a long-short risk-budgeting framework by allowing the asset-specific scores to take negative values and therefore instruct short positions. Along these lines:

$$\text{sign}(w_t^{i, Net, RB}) = \text{sign}(s_t^i), \quad \forall i \quad [3.15]$$

Given that the calculation of MCR encompasses the sign of the weight (a long position with a positive MCR implies that a negative position of the same size will have a negative MCR), the objective of the long-short risk-budgeting becomes:

$$w_t^{i, Net, RB} \cdot MCR_t^i \propto |s_t^i|, \quad \forall i \quad [3.16]$$

The only obvious difference to equation [3.12] is the introduction of an absolute value to the scores. However, there exist more fundamental, yet subtle, differences. Both the asset weight and the MCR are now *signed*, i.e. contain information about the type of position that is prescribed for the asset by the sign of the score.

In order to solve for this objective, we reformulate the optimization as follows:

Long-Short Risk-Budgeting:

$$\text{Maximize: } \sum_{i=1}^{N_t} |s_t^i| \cdot \log(|w_t^i|)$$

$$\text{Subject to: } \sqrt{\mathbf{w}'_t \cdot \Sigma_t \cdot \mathbf{w}_t} \leq \sigma_{TGT} \quad [3.17]$$

As before, the Lagrangian of this optimization coincides with equation [3.16]. In this formulation, both the score and the weight that feed into the objective function bear an absolute value. The absolute value of the weight is necessary so that the logarithm is also defined for short positions. The most important point is that the asset-specific score and the respective weight always agree in their signs. Along these lines, the objective function pushes the positive weights away from zero toward the positive territory, with the relative effects being more aggressive for assets with larger (positive) scores and equivalently pushes the negative weights away from zero toward the negative territory, with the effects being more aggressive for assets with larger (in absolute value, yet negative) scores.

In order for the above methodology to operate as explained, the initial weights for each position must match the sign of the respective scores. As long as this is the case, then the signs of the weights will not flip during the optimization; instead, the signs of the weights are preserved due to the mathematical formulation of the optimization and the use of the logarithm. The role of optimization is only to scale the weights, following risk-budgeting principles, while preserving their signs. The initial net weights can be deduced from the long-only *RB* approach after incorporating an absolute value in the denominator, so that the gross weights sum up to 100%:

$$w_t^{i,Net,RB,initial} = \frac{s_t^i / \sigma_t^i}{\sum_{j=1}^{N_t} |s_t^j| / \sigma_t^j}, \quad \forall i \quad [3.18]$$

As required, these initial weights will be positive for assets with positive scores and negative for assets with negative scores.

3.5.4. Trend-following meets risk-parity

The long-short risk-budgeting framework is exactly what we need to introduce risk-parity principles to a trend-following strategy. Given that the sign of the asset-specific score instructs the type of position (long or short), we can set it equal to the trend-following signal, which is the sign of the past 12-month return of the asset at the end of each month:

$$s_t^i = \text{sign}(r_{t-12,t}^i), \quad \forall i \quad [3.19]$$

This choice achieves two goals at the same time. Not only does it formally incorporate the trend-following signals in the portfolio optimization, but it also

achieves the transition from the risk-budgeting framework back to risk-parity (i.e. equal risk contributions). This is because $|s_t^i| = |\text{sign}(r_{t-12,t}^i)| = 1$ and therefore the long-short risk-budgeting objective of equation [3.16] boils down to the conventional risk-parity objective:

$$w_t^{i,Net,RP} \cdot MCR_t^i \propto 1 \Rightarrow w_t^{i,Net,RP} \cdot MCR_t^i = \text{constant}, \quad \forall i \quad [3.20]$$

The difference to the original formulation is that now the optimization allows for both long and short positions. In particular, the optimization problem [3.17] simplifies into:

Long-Short Risk-Parity:

$$\text{Maximize: } \sum_{i=1}^{N_t} \log(|w_t^i|)$$

$$\text{Subject to: } \sqrt{\mathbf{w}_t' \cdot \boldsymbol{\Sigma}_t \cdot \mathbf{w}_t} \leq \sigma_{TGT} \quad [3.21]$$

and the initial guess for the solution, which follows naturally from equation [3.18] coincides with the net long-short *VP* scheme that we have used so far for trend-following strategy of equation [3.4]:

$$w_t^{i,Net,RP,initial} = w_t^{i,Net,VP} = \text{sign}(r_{t-12,t}^i) \cdot \frac{1/\sigma_t^i}{\sum_{j=1}^{N_t} 1/\sigma_t^j}, \quad \forall i \quad [3.22]$$

Using the net weights that come out of the risk-parity trend-following optimizer, we finally get the risk-parity trend-following (*RPTF*) strategy:

$$r_{t,t+1}^{RPTF} = \frac{\sigma_{TGT}}{\sigma_{TF}^i} \cdot \sum_{i=1}^{N_t} w_t^{i,Net,RP} \cdot r_{t,t+1}^i \quad [3.23]$$

3.6. Performance evaluation of risk-parity trend-following

Figure 3.8 presents the cumulative returns of a trend-following strategy that uses either a *VP* scheme or the just introduced *RP* long-short scheme. Both weighting schemes are dynamically estimated using a window of 90 business days up until the end of each month and both strategies employ a 10% target level of volatility. Full-sample performance statistics as well as correlations with major benchmark indices are reported in Table 3.3.

Visually, these two variants of trend-following exhibit a large degree of co-movement, and a full-sample correlation of 0.82, given that they employ the same set of long and short positions at the end of each month. They only differ in the weighting scheme. Importantly enough, up until 2003, the two lines are one and the same. This period has been characterized by relatively low and insignificant correlations between the asset classes (as shown in Figure 3.4), and therefore volatility-parity and risk-parity solutions are expected to be statistically and numerically indistinguishable.

However, post-2004, when correlations between different asset classes increase almost uniformly, the *RP* allocation system underweights in relative terms the assets that are more correlated on average with the universe and accordingly overweights the assets that have low average correlation with the universe and therefore exhibit larger diversification properties. This differentiation in the weight allocation appears to be driving the outperformance of the risk-parity solution over the most recent decade and, most importantly, during the post-crisis period.

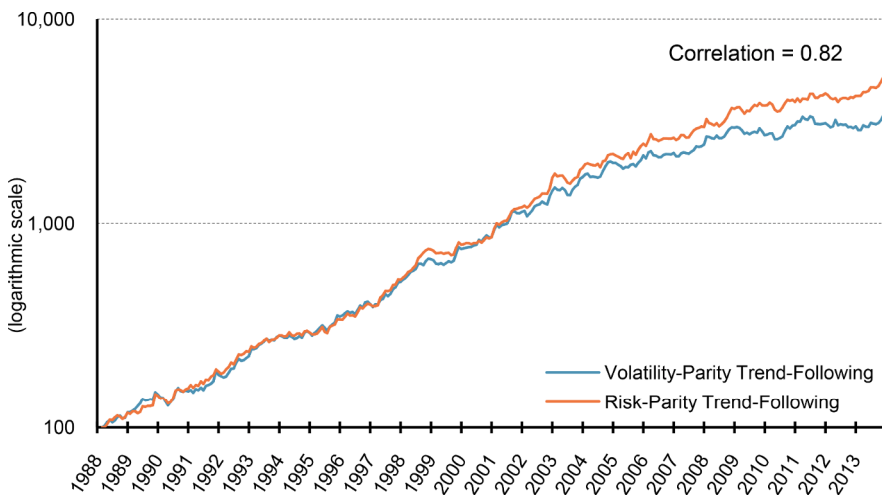


Figure 3.8. The cumulative returns of a volatility-parity trend-following strategy and a risk-parity trend-following strategy. The weighting schemes are estimated using the past 90 days. Both strategies target a volatility of 10%. The scale is logarithmic. The sample period is between April 1988 and December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Performance statistics	VPTF	RPTF	Correlations	VPTF	RPTF
<i>Annualized geometric mean (%)</i>	14.67	16.61	<i>Commodity benchmark:</i>		
<i>t-statistic (Newey–West)</i>	6.71	8.05	<i>– S&P GSCI Commodity Index</i>	0.11	0.07
<i>Annualized volatility (%)</i>	10.96	10.86	<i>Fixed income benchmark:</i>		
<i>Skewness</i>	0.38	0.57	<i>– JPM Global Bond Index</i>	0.31	0.21
<i>Kurtosis</i>	3.27	3.83	<i>FX benchmarks:</i>		
<i>Maximum drawdown (%)</i>	14.20	10.67	<i>– Trade-Weighted USD Index</i>	-0.14	-0.09
<i>Sharpe ratio (annualized)</i>	1.31	1.48	<i>– USD/JPY</i>	-0.18	-0.13
<i>Sortino ratio (annualized)</i>	2.81	3.47	<i>Equity benchmarks:</i>		
<i>Calmar ratio</i>	1.03	1.56	<i>– MSCI World (DM)</i>	-0.01	0.00
<i>Monthly turnover (%)</i>	31.69	57.74	<i>– MSCI Emerging Markets</i>	-0.05	0.00

Table 3.3. Various performance statistics and correlations with various benchmark indices (retrieved from Bloomberg) using monthly returns for the volatility-parity trend-following strategy (VPTF) and for the risk-parity trend-following strategy (RPTF). The *t*-statistic of the mean return is calculated using Newey and West [NEW 87] standard errors. The Sortino ratio is defined as the annualized arithmetic mean return over the annualized downside volatility. The Calmar ratio is defined as the annualized geometric mean return over the maximum drawdown. The sample period is from April 1988 to December 2013

Over the entire sample period, the RP allocation improves the performance of the trend-following strategy, with the Sharpe ratio increasing from 1.31 to 1.48. The performance improvement becomes more pronounced for performance ratios that account for downside risk, such as the Sortino ratio (arithmetic mean return over annualized downside volatility) and the Calmar ratio (geometric mean return over maximum drawdown). Interestingly, the already low correlations with the various benchmark indices fall even further in absolute value.

In order to test whether this improvement in the performance is genuine and statistically strong, we calculate two-sided and one-sided p-values for a paired signed-rank Wilcoxon [WIL 45] test. The equality in the average return between the two variants of the trend-following strategy is rejected with a two-sided p-value of 6.92% and more strongly with a one-sided p-value (in favor of the risk-parity

variant) of 3.46%. In other words, the different weighting scheme results in a genuine and statistically strong improvement for the trend-following strategy.

The outperformance of the *RP* trend-following strategy against its *VP* variant becomes significantly more pronounced when we focus on the post-crisis period, 2009–2013. Table 3.4 reports performance statistics for the two strategies over this 5-year period. In short, *RP* revives trend-following, achieving a statistically significant average return (t-statistic of 1.73 compared to the insignificant t-statistic of 0.78 for the *VP* variant) and Sharpe ratio of 0.78 compared to just 0.31 for the *VP* variant¹⁵. Focusing on the downside, the benefit is even more pronounced with the Calmar ratio (ratio between the geometric mean return and the maximum drawdown) almost quadrupling from 0.20 to 0.77.

	<i>VPTF</i>	<i>RPTF</i>
Annualized geometric mean (%)	2.82	7.27
t-statistic (Newey–West)	0.78	1.73
Annualized volatility (%)	10.77	9.60
Skewness	0.12	-0.06
Kurtosis	3.01	2.48
Maximum drawdown (%)	14.20	9.49
Sharpe ratio (annualized)	0.31	0.78
Sortino ratio (annualized)	0.49	1.35
Calmar ratio	0.20	0.77

Table 3.4. Various performance statistics for the volatility-parity trend-following strategy (*VPTF*) and the risk-parity trend-following strategy (*RPTF*) over the period between January 2009 and December 2013

In order to draw parallels against volatility-parity, Figure 3.9 presents the absolute weight and risk allocation across the five asset classes for the risk-parity strategy. This figure should be studied alongside Figure 3.6.

¹⁵ The two-sided and one-sided p-values of a paired Wilcoxon [WIL 45] signed-rank test on the equality between the mean return of the *RP* trend-following strategy and the *VP* trend-following strategy during the 2009–2013 period is 5.80 and 2.90%, respectively, showing that the return of the *RP* strategy is statistically larger.

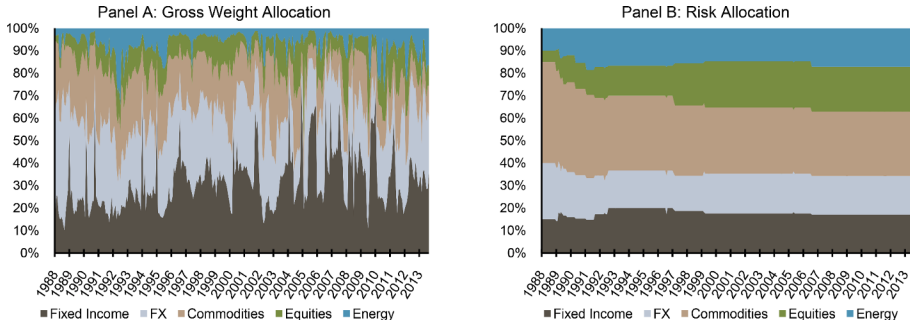


Figure 3.9. The sum of gross weights (Panel A) and the percentage constitution to risk (Panel B) for each asset class over time when a risk-parity weighting scheme is employed for a trend-following strategy. The sample period is from April 1988 to December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Evidently, by taking into account the pairwise correlations, risk-parity succeeds in *ex ante* equating the risk contribution of each asset. The only reason that Panel B of Figure 3.9 exhibits (any) risk allocation shifts between asset classes is due to the fact that the overall size of the number of available contracts N_t is not constant over time.

In order to maintain the equal-risk-contribution objective, the *RP* scheme shifts radically between asset classes, as shown in Panel A of Figure 3.9, due to the fast-changing correlation environment. As an example, the gross allocation in fixed income contracts ranges between 10 and 76% over time, contrary to the more stable allocation under *VP* that ranges between 18 and 48% (from Panel A of Figure 3.6); the average allocation over time is very similar in both schemes (32 and 33% for *RP* and *VP*, respectively). This clearly results in larger turnover for the *RP* variant of trend-following. Table 3.3 reports a full-sample monthly two-way turnover of 58% compared to 32% for the *VP* scheme. The additional turnover (and therefore additional transaction costs) should not subsume the genuine improvement in the risk allocation of the trend-following strategy, even though it might indeed reduce the after-costs benefit of the strategy.

Finally, in order to illustrate that the more sophisticated portfolio construction methodology can lead to superior performance especially in higher correlation environments, we augment the correlation event study of Figure 3.7 with the performance of the risk-parity trend-following strategy under the four different correlation regimes. Figure 3.10 presents the results. It is evident that if there is any significant improvement in the performance of the strategy, this is mainly pronounced in extreme correlation environments, with the Sharpe ratio increasing

from 0.27 up to 0.86. In such states of the market, the volatility-parity scheme is rendered suboptimal in its risk allocation across assets, whereas risk-parity, by taking into account the pairwise correlations, succeeds in delivering a more diversified allocation and therefore in delivering superior performance.

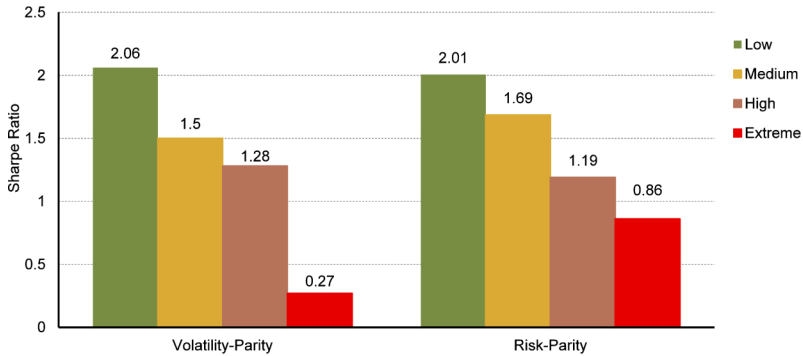


Figure 3.10. The annualized Sharpe ratio of a volatility-parity and a risk-parity trend-following strategy for four different regimes of average pairwise correlation: low (less than 5%; 49 months), medium (5–10%; 109 months), high (10–20%; 92 months) and extreme (above 20%; 59 months). The sample period is from April 1988 to December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

A recent report by Baltas, Jessop, Jones and Zhang [BAL 14] highlights that the added value of risk-parity in higher correlation environments is, in fact, due to larger dispersion of pairwise correlations between asset classes.

As a final point, it should be highlighted that even with the use of a risk-parity allocation, Figure 3.10 documents that trend-following suffers in higher correlation environments. Evidently, this must be related directly to the non-existence of strong price trends in high correlation environments. As it has been shown, a more sophisticated weighting scheme can only do so much as far as portfolio diversification is concerned. Any further improvement of the strategy should look into the reasons why price trends might be less persistent in high correlation environments. Future research should address these claims.

From a different perspective, this performance drop can be rationalized using the Grinold and Kahn's [GRI 00] fundamental law of active management under which the information ratio equals the product of the manager's skill (information coefficient) and the square root of the breadth of the investable universe. Assuming a constant skill, when asset correlations increase, the breadth of the investable universe falls and this should expectedly cause a fall in the attainable information ratio.

3.7. Conclusion

Trend-following strategies have been very profitable historically and have constituted great diversification vehicles against market downturns, such as recently during the financial crisis of 2008. Their main source of profitability is related to the diversification benefit that arises from combining futures contracts across different asset classes, which have exhibited historically very low, if not insignificant, cross-correlations. A simple volatility-parity (inverse-volatility) weighting scheme has been historically considered appropriate in order to combine assets from different asset classes with very diverse risk profiles. Such a weighting scheme would distribute risk equally among constituents, should their correlations were constant over time.

Following an impressive performance in 2008, trend-following strategies have performed poorly over the most recent period (2009–2013), which has been characterized by dramatic increases in the correlations between different asset classes. We hypothesize that the volatility-parity weighting scheme leads to uneven and therefore suboptimal risk allocation under such conditions and this could be one of the reasons for the recent underperformance of trend-following.

A risk-parity weighting scheme can succeed in equating the contribution to the total portfolio risk from all portfolio constituents, after also accounting for correlations. However, its conventional formulation can only be applied to a long-only portfolio. Inspired by Jessop *et al.* [JES 13], we contribute to the literature by extending the conventional long-only risk-parity framework, in a way that also allows for short positions. This extension is necessary, if such a weighting scheme were to be employed in a trend-following portfolio.

The risk-parity trend-following portfolio constitutes a genuine improvement to the traditional volatility-parity variant of the strategy. The Sharpe ratio of the strategy increases from 1.31 to 1.48 over the entire sample period (April 1988–December 2013), and more than doubles over the post-crisis period (January 2009–December 2013) from 0.31 to 0.78. A correlation event study shows that the improvement is mainly driven from the superior performance of the strategy in extreme average correlation environments.

3.8. Appendix: solving for risk parity

The risk-parity methodology solves for portfolio weights at the end of each month, so that every asset is contributing the same amount of risk to the overall portfolio, i.e.:

$$w_t^{RP,i} \cdot MCR_t^i = \text{constant}, \quad \forall i \quad [3.24]$$

We can solve for this objective using the following nonlinear constrained optimization problem:

- maximize the sum of logarithmic weights: $\sum_{i=1}^{N_t} \log(w_t^i)$;
- subject to a risk constraint of target volatility $\sigma_{\mathbf{w}_t} \equiv \sqrt{\mathbf{w}_t' \cdot \boldsymbol{\Sigma}_t \cdot \mathbf{w}_t} \leq \sigma_{TGT}$.

We first form the Lagrangian (we denote the Lagrange multiplier by λ):

$$L(\mathbf{w}_t) = \sum_{i=1}^{N_t} \log(w_t^i) - \lambda \cdot (\sigma_{\mathbf{w}_t} - \sigma_{TGT}) \quad [3.25]$$

We then calculate all partial derivatives with respect to each weight w_t^i :

$$\frac{\partial L(\mathbf{w}_t)}{\partial w_t^i} = \frac{1}{w_t^i} - \lambda \cdot \underbrace{\frac{\partial \sigma_{\mathbf{w}_t}}{\partial w_t^i}}_{\equiv MCR_t^i}, \quad \forall i \quad [3.26]$$

Setting the above expression equal to zero leads to equation [3.24]:

$$w_t^i \cdot MCR_t^i = \frac{1}{\lambda} = \text{constant}, \quad \forall i \quad [3.27]$$

Post-optimization, all weights are rescaled so that they sum up to 1. The fact that volatility exhibits positive homogeneity (for a scaling constant κ , $\sigma_{\kappa \cdot \mathbf{w}} = \kappa \cdot \sigma_{\mathbf{w}}$) renders the *MCR* scale-invariant as is easily deduced by:

$$\frac{\partial \sigma_{\kappa \cdot \mathbf{w}_t}}{\partial (\kappa w_t^i)} = \frac{\partial \sigma_{\mathbf{w}_t}}{\partial w_t^i} = MCR_t^i, \quad \forall i \quad [3.28]$$

This means that the rescaled weights satisfy the risk-parity objective of equation [3.24], as the normalization constant (the sum of unadjusted weights) is absorbed by the constant of equation [3.24]. Hence, delaying the application of the “fully-invested” constraint until after the optimization helps computationally and does not alter the end result in terms of the risk-parity objective.

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Diversifying Risk Parity: In Today, Out Tomorrow?

Striving for maximum diversification, we follow Meucci [MEU 09] in measuring and managing a multi-asset class portfolio. Under this paradigm, the maximum diversification portfolio is equivalent to a risk parity strategy with respect to the uncorrelated risk sources embedded in the underlying portfolio assets. We characterize the mechanics and properties of this diversified risk parity strategy. Moreover, we explore the risk and diversification characteristics of traditional risk-based asset allocation techniques such as $1/N$, minimum-variance and risk parity and demonstrate the diversified risk parity strategy to be quite meaningful when benchmarked against these alternatives. Finally, we demonstrate the benefits of diversification when backtesting risk-based investment strategies in a simulated environment of rising interest rates.

Diversification pays. This insight is at the heart of most portfolio construction paradigms like the seminal one of Markowitz [MAR 52]. Under his mean-variance optimization, diversifying portfolio weights is key to obtaining efficient portfolios with an optimal risk and return trade-off. Unfortunately, mean-variance optimization is typically confounded by estimation risk, especially the one embedded in estimates of expected returns (see [ZIE 93])¹. One way to circumvent this problem is to simply refrain from estimating expected returns and to resort to risk-based allocation

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¹ Of course, estimation risk in portfolio management is at the heart of many works, such as [HER 06, JOR 86, KAN 07, KLE 76, MER 80, MIC 89].

techniques². Within the framework of Markowitz [MAR 52], this approach leads to the well-known minimum-variance portfolio. However, minimum-variance portfolios are designed to load on low-volatility assets which render them rather concentrated in a few assets. Thus, minimum-variance portfolios are hardly diversified in terms of a homogenous weights distribution.

In fact, there are different views on the meaning of portfolio diversification. For instance, Woerheide *et al.* [WOH 93] focus on portfolio weights, while Frahm and Wiechers [FRA 13] focus on portfolio return variance. In striving for well-diversified portfolios, Meucci [MEU 09] builds on principal component analysis (PCA) of the portfolio assets to extract the main drivers of the assets' variability. These principal components can be interpreted as principal portfolios representing the uncorrelated risk sources inherent in the portfolio assets. For a portfolio to be well diversified, its overall risk should, therefore, be evenly distributed across these principal portfolios. Condensing this risk decomposition into a single diversification metric, Meucci [MEU 09] opts for the exponential of this risk decomposition's entropy because of its intuitive interpretation as the number of uncorrelated bets.

The contribution of this chapter is to apply the framework of Meucci [MEU 09] in an empirical multi-asset allocation study. Under this paradigm, the maximum diversification portfolio emerges from a risk parity strategy that is budgeting risk with respect to the extracted principal portfolios rather than the underlying portfolio assets. Therefore, we think of this approach as a diversified risk parity (DRP) strategy which turns out to be a reasonable alternative when it comes to risk-based asset allocation. Moreover, the framework allows for a litmus test of competing techniques, such as $1/N$, minimum-variance and risk parity. While minimum-variance is fairly well known for picking up rather concentrated risks, we find the traditional risk parity strategy to be more balanced. However, benchmarking risk parity against diversified risk parity we observe a degeneration in its diversification characteristics over time, rendering the traditional risk parity strategy a rather concentrated bet in the current environment. Moreover, many of the risk-based strategies have been profiting from a huge bond exposure in an environment of constantly falling interest rates. Simulating an increase in interest rates over the next 5 years, we provide evidence that portfolio construction techniques striving for diversification are highly suitable in such a challenging environment.

The chapter is organized as follows. Section 4.1 reviews the approach of Meucci [MEU 09] for managing and measuring diversification. Section 4.2 presents the data and further rationalizes the concept of principal portfolios. Section 4.3 is devoted to

² This argument has also been elaborated by [FRA 10a], or [FRA 10b], among others.

contrasting the diversified risk parity strategy to alternative risk-based asset allocation strategies. Section 4.4 concludes the chapter.

4.1. Managing diversification

According to standard portfolio theory, diversification is geared at eliminating unsystematic risk. In addition, investors' and portfolio managers' common notion of diversification is the desire to avoid exposure to single shocks or risk factors. In either case, diversification especially pays when combining low-correlated assets. Taking this idea to extremes, Meucci [MEU 09] constructs uncorrelated risk sources by applying a principal component analysis (PCA) to the variance-covariance matrix of the portfolio assets. In particular, he considers a portfolio consisting of N assets with return vector \mathbf{R} . Given weights \mathbf{w} , the resulting portfolio return is $R_w = \mathbf{w}'\mathbf{R}$. According to the spectral decomposition theorem, the covariance matrix Σ can be expressed as a product:

$$\Sigma = \mathbf{E}\mathbf{\Lambda}\mathbf{E}' \quad [4.1]$$

where $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_N)$ is a diagonal matrix consisting of Σ 's eigenvalues that are assembled in descending order, $\lambda_1 \geq \dots \geq \lambda_N$. The columns of matrix \mathbf{E} represent the eigenvectors of Σ . These eigenvectors define a set of N *principal portfolios*³ whose returns given by $\tilde{\mathbf{R}} = \mathbf{E}'\mathbf{R}$ are uncorrelated and their variances equal $\lambda_1, \dots, \lambda_N$. As a result, a given portfolio can be either expressed in terms of its weights \mathbf{w} in the original assets or in terms of its weights $\tilde{\mathbf{w}} = \mathbf{E}'\mathbf{w}$ in the principal portfolios. Since the principal portfolios are uncorrelated by design, the total portfolio variance emerges from simply computing a weighted average over the principal portfolios' variances λ_i using weights \tilde{w}_i^2 :

$$\text{Var}(R_w) = \sum_{i=1}^N \tilde{w}_i^2 \lambda_i \quad [4.2]$$

Normalizing the principal portfolios' contributions by the portfolio variance then yields the *diversification distribution*:

$$p_i = \frac{\tilde{w}_i^2 \lambda_i}{\text{Var}(R_w)}, \quad i = 1, \dots, N \quad [4.3]$$

³ Note that [PRA 04] coined the term "principal portfolios": in their recasting of the efficient frontier in terms of these principal portfolios.

Note that the diversification distribution is always positive and that the p_i s sum to 100%. Building on this concept, Meucci [MEU 09] conceives a portfolio to be well diversified when the p_i are “approximately equal and the diversification distribution is close to uniform”. This definition of a well-diversified portfolio coincides with allocating equal risk budgets to the principal portfolios. Therefore, we dub this approach *diversified risk parity*. Conversely, portfolios loading on a specific principal portfolio display a peaked diversification distribution. It is thus straightforward to apply a dispersion metric to the diversification distribution to obtain a single diversification metric. Meucci [MEU 09] chooses the exponential of its entropy⁴:

$$\mathcal{N}_{Ent} = \exp \left(- \sum_{i=1}^N p_i \ln p_i \right) \quad [4.4]$$

The reason for choosing \mathcal{N}_{Ent} relates to its intuitive meaning as the number of uncorrelated bets. To rationalize this interpretation, consider two extreme cases. For a completely concentrated portfolio, we have $p_i = 1$ for one i and $p_j = 0$ for $i \neq j$ resulting in an entropy of 0 which implies $\mathcal{N}_{Ent} = 1$. Conversely, $\mathcal{N}_{Ent} = N$ holds for a portfolio that is completely homogenous in terms of uncorrelated risk sources. In this case, $p_i = p_j = 1/N$ holds for all i, j implying an entropy equal to $\ln(N)$.

Taking the above approach to the extreme, we can especially obtain the maximum diversification portfolio or the diversified risk parity weights \mathbf{w}_{DRP} by solving:

$$\mathbf{w}_{DRP} = \operatorname{argmax}_{\mathbf{w} \in \mathcal{C}} \mathcal{N}_{Ent}(\mathbf{w}) \quad [4.5]$$

where the weights \mathbf{w} may possibly be restricted according to a set of constraints \mathcal{C} .

Note that optimization [4.5] does not allow for a unique solution in the absence of further constraints. Obviously, an inverse volatility portfolio along the principal portfolios is a solution maximizing objective function [4.5]. However, if we multiply a given number of eigenvectors by -1, the ensuing matrix will still be orthogonal. The economic intuition of this result is straight-forward: because all of the principal portfolios are uncorrelated, buying and selling the same amount of a given principal portfolio gives rise to the same risk exposure. Acknowledging all possible variations of the long and short variants of the underlying principal portfolios, we collect 2^N optimal inverse volatility portfolios.

⁴ The entropy has been used before in portfolio construction, see, for example, [WOH 93], or more recently [BER 08]. However, these studies consider the entropy of portfolio weights thus disregarding the dependence structure of portfolio assets.

The issue of multiple solutions applies to traditional risk parity strategies as well. Investigating general risk-budgeting strategies, Bruder and Roncalli [BRU 12] and Roncalli and Weisang [RON 12] show unique risk-budgeting strategies when imposing positivity constraints with respect to the underlying risk factors. Therefore, imposing sign constraints with respect to the principal portfolios instead of the underlying assets is key for obtaining a unique DRP strategy. Thus, the optimal portfolio weights can be computed analytically given the eigenvector decomposition of the covariance matrix Σ . However, these weights might not be feasible for a given set of investment constraints. For instance, we will enforce positive asset weights and a full investment constraint later on. In this case, we numerically maximize objective function [4.5] using a sequential quadratic (SQP) algorithm⁵. For anchoring the numerical solution, we feed the optimizer with the unconstrained analytic solution as a starting value.

Note that there are many linear transformations (or torsions) of the original assets corresponding to many sets of portfolios with uncorrelated returns. Among these transformations, the principal component decomposition is a natural choice, however, there may be situations in which we may like to depart from this choice. For instance, the PCA approach might give rise to principal components that are hard to interpret. In this regard, Meucci *et al.* [MEU 13] extend the framework of Meucci [MEU 09] by choosing the linear transformation that is closest to the original factors (or assets). Going forward, we will nevertheless stick to the original approach given that our subsequent analysis establishes a quite close and stable connection between principal portfolios and the underlying asset classes.

4.2. Rationalizing principal portfolios

4.2.1. Data and descriptive statistics

In building risk-based asset allocation strategies, we focus on five broad asset classes as represented by the following indices. We use the JPM Global Bond Index for government bonds, the MSCI World Total Return Index for developed equities, the MSCI Emerging Markets Total Return Index for emerging equities, the DJ UBS Commodity Index for commodities and the Barclays U.S. Aggregates Credit Index. All indices are measured in monthly local currency returns, and we report total return figures from the perspective of a U.S. investor by employing the 3-month U.S. Treasury Rate.

Table 4.1 conveys the descriptive statistics of the above asset classes. Over the whole sample period, from December 1987 to September 2013, we observe an

⁵ In particular, we build on a variant of Meucci's [MEU 09] implementation which is available on his webpage, see <http://www.symmys.com/node/199>.

annualized bond return of 6.6% at a volatility of 3.9%, which happens to be the lowest figure across asset classes. During this period, developed equities have fared slightly better in terms of return (7.4%), however, their volatility is considerably higher (14.3%). For emerging equities, return and volatility figures are higher when compared to developed equities. Conversely, commodities are quite similar to developed equities in terms of volatility. Most interestingly, credit exhibits the same return as government bonds. This observation is unexpected given that credit is significantly more volatile than government bonds. However, it is important to note that the bulk of credit volatility is related to the credit crunch of 2008 and the subsequent financial crisis. In terms of risk-adjusted returns, the high-risk asset classes rather disappoint given the Sharpe ratios of around 0.2. By this metric, credit ranks second with a figure of 0.69 but is still underperforming compared to bonds that exhibit an impressive Sharpe ratio of 0.95. Further inspecting the assets' dependence structure in Table 4.1, we observe bonds to be hardly correlated to equities. Its correlation to commodities is slightly negative, while the one to credit amounts to 0.53. All of the remaining correlation coefficients range from -0.18 (government bonds vs. commodities) to 0.74 (developed vs. emerging equities). Unsurprisingly, credit is more correlated to both equity indices.

	Return p.a.	Vola p.a.	Sharpe Ratio	Correlation Matrix				
				Bonds	Equities		Commodities	Credit
					Dev.	Emg.		
Bonds	6.6%	3.9%	0.95	1.00				
Developed Equities	7.4%	14.3%	0.31	0.00	1.00			
Emerging Equities	8.3%	23.7%	0.23	-0.04	0.74	1.00		
Commodities	4.8%	15.4%	0.12	-0.18	0.20	0.34	1.00	
Credit	6.6%	5.3%	0.69	0.54	0.25	0.21	0.10	1.00

Table 4.1. *Descriptive statistics*

The table contains descriptive statistics of the multi-asset classes according to the sample period from December 1987 to September 2013 based on monthly local currency returns. On the left-hand side, annualized return and volatility figures are reported and the right-hand side gives the corresponding correlation matrix.

4.2.2. *Extracting and interpreting principal portfolios*

To foster intuition about the uncorrelated risk sources inherent in our multi-asset time series, we investigate the PCA over the whole sample period from December 1987 to September 2013. The economic nature of the principal portfolios is best assessed in terms of the eigenvectors that represent the principal portfolios' weights with respect to the original asset classes. Because eigenvectors are normalized so that

$e_i'e_i = 1$ for all $i = 1, \dots, N$, these weights lie within the $[-1,1]$ -interval by construction. Given that the correlation of the original assets is generally relatively low, the interpretation of the principal portfolios, as collected in Panel A of Table 4.2, is straightforward. The first principal portfolio (PP1) is purely driven by emerging and developed equities with emerging equity having a fairly high weight of 0.86. Therefore, PP1 represents genuine equity risk which is accounting for 69.9% of the overall variance. Conversely, principal portfolio 2 (PP2) reflects the diversification potential of commodities relative to equities as indicated by a commodities weight of 0.96 and has a much smaller variance contributing 19.6% to the overall variance. Principal portfolio 3 (PP3) represents the difference between emerging and developed equities, thus capturing the emerging market spread. Principal portfolio 4 (PP4) mostly loads on credit and government bonds and we interpret it as an interest rate risk factor. Note that PP3 and PP4 explain most of the remaining variance, leaving a minuscule fraction of 0.7% for principal portfolio 5 (PP5). Judging by the weights of PP5, we conceive it to be a mimicking factor of the credit spread. In addition, Panel B of Table 4.2 gives the principal portfolio weights pertaining to a PCA over the last 60 months of the sample period, which is from October 2008 to September 2013. Despite covering the most turbulent of times, these weights prove to be fairly similar to those obtained for the whole sample period. However, the variance of PP1 is elevated by a factor of 2 when compared to the results for the whole sample period.

For estimating the principal portfolios over time, we have to make a choice with regard to the estimation window. The two most common approaches rely either on an expanding window or a rolling window for estimation. The proponents of expanding window estimation appreciate that building on all available data typically gives rise to a quite robust set of components. However, rolling window estimation is believed to be more responsive to potential structural breaks. Therefore, our analysis focuses on the discussion of results arising from rolling window estimation using a 60 months window. In particular, we perform a PCA every month to extract the principal portfolios embedded in the multi-asset classes. Stacking the corresponding principal portfolio variances, Figure 4.1 depicts the variation of the principal portfolios' variances over time.

We observe PP1 to be fairly dominant by accounting for at least 60% of the underlying time series' variation at any given point in time. Given that PP2 and PP3 represent some 20% and 10% of the variation, the remaining principal portfolios PP4 and PP5 do only account for a minor fraction. At the end of the sample period, we find PP1 accounting for 80% of the overall variability which bears testimony of the contagion effects emanating from the financial crisis of 2008.

Asset Class	PP1	PP2	PP3	PP4	PP5
	<i>Equity</i>	<i>Commodities</i>	<i>EM-Spread</i>	<i>Interest Rate</i>	<i>Credit Spread</i>
<i>Panel A: December 1987 to September 2013</i>					
JPM Global Bond	-0.01	-0.05	-0.03	0.53	-0.85
MSCI World	0.43	-0.24	-0.86	-0.11	-0.03
MSCI Emerging Markets	0.86	-0.17	0.47	0.01	-0.01
DJ UBS Commodities	0.26	0.96	-0.13	0.01	-0.05
Barclays U.S. Aggr. Credit	0.05	-0.01	-0.10	0.84	0.53
Variance	7.5%	2.1%	0.7%	0.3%	0.1%
Percent Explained	69.9%	19.6%	6.8%	2.9%	0.7%
Cumulative	69.9%	89.6%	96.4%	99.3%	100.0%
<i>Panel B: October 2008 to September 2013</i>					
JPM Global Bond	-0.03	-0.13	0.28	0.49	-0.81
MSCI World	0.44	-0.28	-0.75	0.41	0.02
MSCI Emerging Markets	0.74	-0.39	0.37	-0.39	-0.07
DJ UBS Commodities	0.50	0.85	0.04	0.14	-0.05
Barclays U.S. Aggr. Credit	0.08	-0.15	0.47	0.65	0.58
Variance	12.7%	1.1%	0.6%	0.3%	0.1%
Percent Explained	86.3%	7.3%	3.9%	2.0%	0.5%
Cumulative	86.3%	93.6%	97.5%	99.5%	100.0%

Table 4.2. *Principal portfolio weights*

The table gives the eigenvectors representing the principal portfolio weights with respect to the underlying asset classes. These eigenvectors either arise from a PCA of the multi-asset class covariance matrix over the whole sample period from December 1987 to September 2013 (Panel A) or from a PCA over the last 60 months of the sample period (Panel B). Weights in excess of 0.4 are in bold, weights in excess of 0.2 are in italics. The principal portfolios' variance is given in absolute terms and relative to the overall data variation. "Cumulative", represents the fraction of variance being explained by a given number of principal portfolios (with the highest variance contributions).

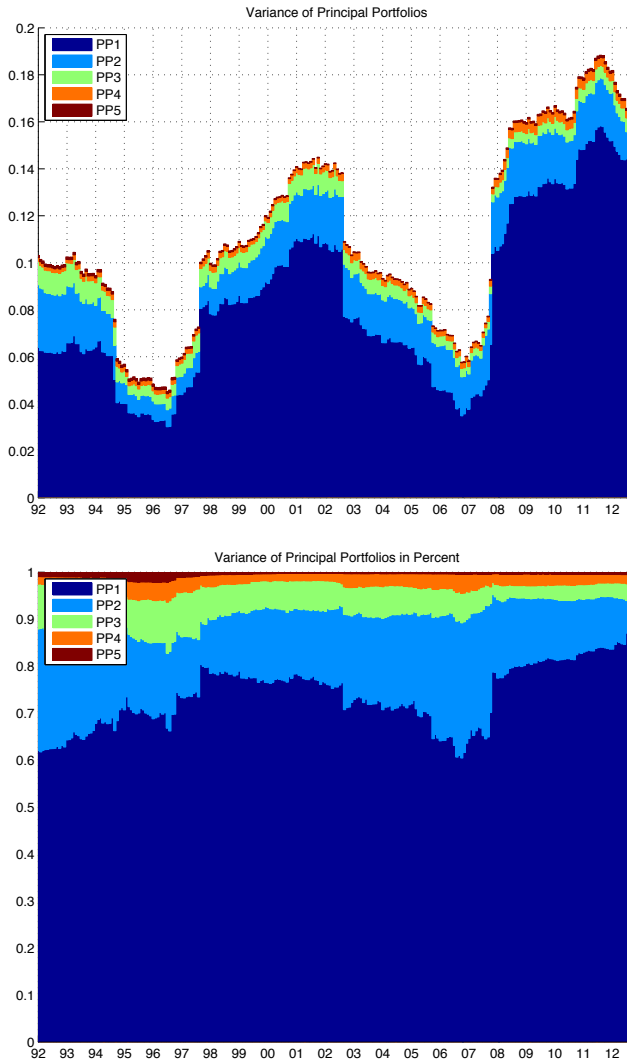


Figure 4.1. *Variances of the principal portfolios. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip*

The figure gives the variance of the principal portfolios in the upper panel and its relative decomposition over time in the lower panel. Each month, a PCA is performed using a 60 months window to extract the principal portfolios embedded in the multi-asset classes and the corresponding principal portfolio variances are stacked in one bar. The results range from January 1993 to September 2013.

A common concern associated with the PCA approach is the stability of the covariance matrix' eigenvectors, especially those pertaining to the smallest eigenvalues. To investigate this issue, we collect the principal portfolio weights throughout time in Figures 4.2 and 4.3. For instance, the left column of Figures 4.2 and 4.3 gives principal portfolio weights when using expanding window estimation. Unsurprisingly, the corresponding weights prove to be very stable throughout the sample period mirroring the general interpretation that we have inferred from the static weights over the whole sample period (see Table 4.2). Obviously, stepping from expanding to rolling window estimation renders the time series of principal portfolio weights more volatile. In this regard, the right column of Figures 4.2 and 4.3 gives results when using rolling estimation windows of 60 months. Note that even though weights are more volatile, there has been no switch in the economic meaning of PP1 (equity risk), PP4 (interest rate risk) and PP5 (credit spread). Only once do we observe a switch between PP2 (commodity risk) and PP3 (EM-spread), which takes place in the middle of the 1990s and lasts for some 3 years.

4.3. Risk-based asset allocation

4.3.1. Risk-based asset allocation schemes

For constructing the diversified risk parity strategy, we first determine the principal portfolios using rolling window estimation. From section 4.2, we know that the optimal DRP strategy is an inverse volatility strategy along the principal portfolios which can be computed analytically given specific sign constraints with respect to the principal portfolios. We choose these constraints such that the sign of each principal portfolio equals the sign of its historical risk premium. We measure the principal portfolios' historical risk premia using an expanding window starting at the beginning of the sample period in 1987. Thus, the weights for the optimal DRP strategy can be computed in principal portfolio space as follows:

$$\tilde{w}_i = \frac{\frac{1}{\lambda_i}}{\sum_{j=1}^N \frac{1}{\lambda_j}} \text{sign}(PP_i) \quad [4.6]$$

The optimal weights in the original assets follow from $\mathbf{w}_{DRP}^{Opt} = \mathbf{E}\tilde{\mathbf{w}}$. Intuitively, the strategy thus aims at capturing long-term risk premia associated with the principal portfolios. More importantly, the strategy's positioning will not establish bets that have not been rewarded historically⁶.

⁶ Note that we standardize the portfolio weights of the optimal DRP strategy to sum to 100%

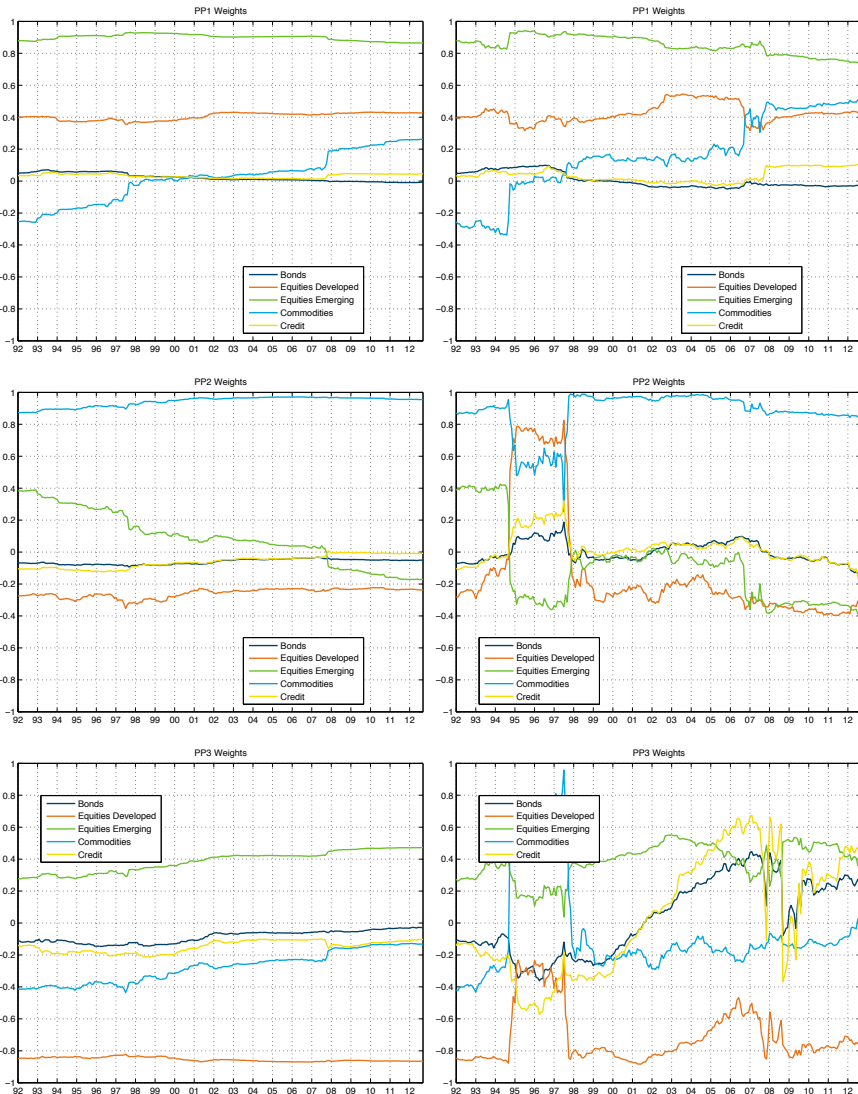


Figure 4.2. Principal portfolio weights: PP1 to PP3. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

The figure gives the principal portfolios' weights over time. The left column gives results when using an expanding window estimation. The right column gives results when using rolling estimation windows of 60 months. The estimation period is from January 1988 to September 2013.

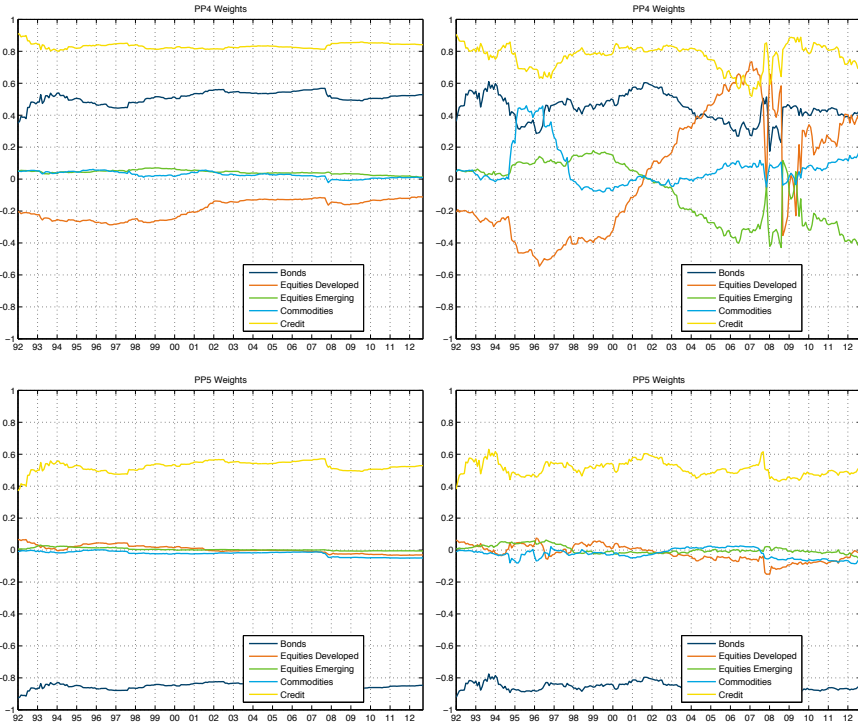


Figure 4.3. Principal portfolio weights: PP4 and PP5. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

The figure gives the principal portfolios' weights over time. The left column gives results when using an expanding window estimation. The right column gives results when using rolling estimation windows of 60 months. The estimation period is from January 1988 to September 2013.

To allow for comparison with alternative risk-based allocation techniques, we also determine a constrained DRP strategy using optimization [4.5] where we enforce full investment and positivity constraints. Rebalancing of all strategies occurs at a monthly frequency. Given that the first PCA estimation consumes 60 months of data, the strategy performance can be assessed from January 1993 to September 2013.

For benchmarking the diversified risk parity strategy, we consider four alternative risk-based asset allocation strategies: $1/N$, minimum-variance, risk parity and the most-diversified portfolio of Choueifaty and Coignard [CHO 08]. First, we implement

the $1/N$ -strategy that rebalances monthly to an equally weighted allocation scheme, hence, the portfolio weights $\mathbf{w}_{1/N}$ are:

$$\mathbf{w}_{1/N} = \frac{\mathbf{1}}{N} \quad [4.7]$$

Second, we compute the minimum-variance (MV) portfolio either building on an expanding or rolling 60 months window for covariance-matrix estimation. The corresponding weights \mathbf{w}_{MV} derive from:

$$\mathbf{w}_{MV} = \underset{\mathbf{w}}{\operatorname{argmin}} \mathbf{w}'\Sigma\mathbf{w} \quad [4.8]$$

subject to the full investment and positivity constraints, $\mathbf{w}'\mathbf{1} = 1$ and $\mathbf{w} \geq \mathbf{0}$.

Third, we construct the original risk parity (RP) strategy by allocating capital such that the asset classes' risk budgets contribute equally to overall portfolio risk⁷. Note that these risk budgets also depend on either expanding or rolling window estimation. Since there are no closed-form solutions available, we follow [MAI 10] to obtain \mathbf{w}_{RP} numerically via:

$$\mathbf{w}_{RP} = \underset{\mathbf{w}}{\operatorname{argmin}} \sum_{i=1}^N \sum_{j=1}^N (w_i(\Sigma\mathbf{w})_i - w_j(\Sigma\mathbf{w})_j)^2 \quad [4.9]$$

which essentially minimizes the variance of the risk contributions⁸. Again, the above full investment and positivity constraints apply.

Fourth, we describe the approach of Coignard [COI 08] for building maximum diversification portfolios. To this end, the authors define a portfolio diversification ratio $D(\mathbf{w})$:

$$D(\mathbf{w}) = \frac{\mathbf{w}' \cdot \boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\Sigma\mathbf{w}}} \quad [4.10]$$

where $\boldsymbol{\sigma}$ is the vector of portfolio asset return volatilities. Thus, the most-diversified portfolio (MDP) simply maximizes the ratio between two distinct definitions of portfolio volatility, i.e. the ratio between the average portfolio assets' volatility and

⁷ Risk parity has been put to the fore by Qian [QIA 06, QIA 11] and Maillard [MAI 10].

⁸ Because the DRP portfolio is equivalent to a risk parity portfolio in principal portfolio space, the approach of Maillard [MAI 10] might also be used to determine the optimal DRP portfolio.

the total portfolio volatility. We obtain MDP's weights vector \mathbf{w}_{MDP} by numerically computing:

$$\mathbf{w}_{MDP} = \underset{\mathbf{w}}{\operatorname{argmax}} D(\mathbf{w}) \quad [4.11]$$

As before, we enforce the full investment and positivity constraints.

The table gives performance and risk statistics of the risk-based asset allocation strategies from January 1993 to September 2013. Annualized average return and volatility figures are reported together with the respective annualized Sharpe Ratio where the risk-free rate is given by the average 3-month U.S. Treasury Rate. Maximum Drawdown (MDD) is computed over 1 month and over the whole sample period. Turnover is the portfolios' mean monthly turnover over the whole sample period. Gini coefficients are reported using portfolios' weights ("Gini Weights") and risk decomposition with respect to the underlying asset classes ("Gini Risk") or with respect to the principal portfolios ("Gini PP Risk"). The # bets is the exponential of the risk decomposition's entropy when measured against the uncorrelated risk sources. D is the diversification ratio.

Statistic	Diversified Risk Parity		Risk-Based Allocations			
	Optimal	Constrained	I/N	MV	RP	MDP
Return p.a.	6.6%	6.5%	7.4%	6.1%	6.6%	5.7%
Volatility p.a.	4.2%	4.0%	9.3%	3.6%	4.6%	4.2%
Sharpe Ratio	0.86	0.89	0.48	0.88	0.80	0.66
MDD 1M	-4.2%	-2.7%	-15.1%	-2.4%	-5.9%	-4.3%
MDD	-6.6%	-7.0%	-33.5%	-5.1%	-13.0%	-10.7%
Turnover	11.7%	6.6%	0.0%	2.6%	1.5%	3.7%
Gini Weights	0.67	0.68	0.00	0.70	0.34	0.52
Gini Risk	0.60	0.60	0.48	0.70	0.00	0.42
Gini PP Risk	0.00	0.13	0.83	0.38	0.50	0.40
# bets	5.00	4.74	1.56	3.26	3.26	3.64
D	1.59	1.69	1.44	1.62	1.78	1.88

Table 4.3. Performance and risk statistics of asset allocation strategies

4.3.2. Performance of risk-based asset allocation schemes

Table 4.3 gives the performance and risk statistics of the two DRP strategies as well as the alternative risk-based asset allocation strategies. The optimal DRP strategy earns 6.6% at 4.2% volatility which is equivalent to a Sharpe ratio of 0.86. This high risk-adjusted return is mostly robust with respect to imposing positive asset weights. The constrained DRP strategy gives 6.5% at 4.0% volatility. Among the

competing strategies, the highest annualized return materializes for the $1/N$ -strategy, but the 7.4% comes at the cost of the highest volatility (9.3%). Moreover, the strategy exhibits the highest drawdown among all alternatives (33.5%). Conversely, the minimum-variance strategy provides a lower return of 6.1%. Given that minimum-variance indeed exhibits the lowest volatility (3.6%), its Sharpe ratio of 0.88 is highly favorable. Also, its drawdown statistics are the least severe amounting to a maximum loss of 5.1% during the whole sample period. Note that the constrained DRP strategy's maximum drawdown is only slightly higher (7.0%).

Paraphrasing [MAI 10], we then find the risk parity strategy to be a middle-ground portfolio between $1/N$ and minimum-variance. Its return is 6.6% at a 4.6% volatility thus giving rise to a Sharpe ratio of 0.80. Also, the maximum drawdown statistics are significantly reduced when compared to the $1/N$ -strategy. The maximum drawdown of the MDP is even smaller (10.7%), however, its return is the lowest among all alternatives (5.7%) which still allows for a fairly adequate Sharpe ratio of 0.66.

To gauge the strategies' evolution over time, we plot their cumulative returns in Figure 4.4. Whereas the $1/N$ -strategy is pursuing a rather rocky path, the remaining strategies exhibit a quite steady evolution of performance. In addition, it seems as if the strategies' resilience with respect to the financial crisis of 2008 is the main driver in explaining the strategies' overall volatility.

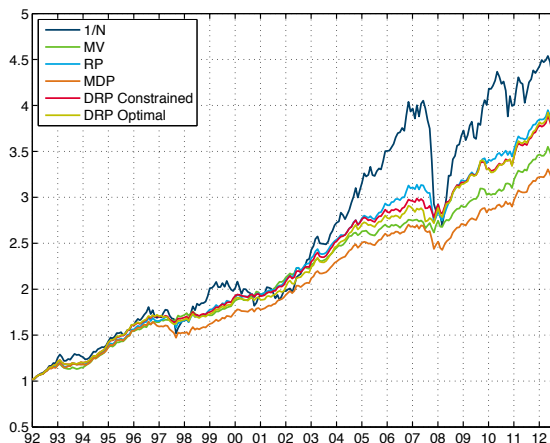


Figure 4.4. Performance of risk-based asset allocation. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

The figure gives the cumulative total return of the risk-based asset allocation strategies when using rolling window estimation of 60 months over the sample period starting January 1993 to September 2013.

While the performance table and figure already give a good grasp of the different strategies, we additionally provide mutual tracking errors and mutual correlation coefficients in Table 4.4. In terms of strategy similarity, we find the constrained diversified risk parity strategy to be very close to the optimal DRP strategy given a correlation of 0.98 and a tracking error of 0.86%. Concerning the alternative risk-based strategies, we find the constrained DRP strategy to be highly correlated to risk parity, or minimum-variance and the MDP. Judging by a tracking error of 1.75%, it is closest to RP. Unsurprisingly, tracking errors are the highest for the $1/N$ -strategy with figures ranging from 5.99% (vs. risk parity) to 8.63% (vs. minimum-variance).

Tracking Error-Correlation-Matrix						
	DRP					
	<i>Optimal</i>	<i>Constrained</i>	<i>1/N</i>	<i>MV</i>	<i>RP</i>	<i>MDP</i>
<i>DRP Optimal</i>	1.00	0.86%	4.21%	7.99%	2.09%	2.34%
<i>DRP Constrained</i>	0.98	1.00	4.00%	7.65%	1.75%	1.83%
<i>1/N</i>	0.53	0.60	1.00	8.63%	5.99%	6.76%
<i>MV</i>	0.87	0.90	0.39	1.00	2.84%	2.29%
<i>RP</i>	0.81	0.88	0.85	0.79	1.00	1.24%
<i>MDP</i>	0.85	0.90	0.76	0.84	0.97	1.00

Table 4.4. Comparison of risk-based asset allocation strategies

The table compares the risk-based asset allocation strategies by reporting mutual tracking errors above the diagonal and mutual correlation figures below the diagonal. All figures refer to a 60-month rolling window estimation over the sample period (January 1993 to September 2013.)

Note that the high risk-adjusted performance of the diversified risk parity strategy is to be taken with a grain of salt, since its monthly turnover is slightly higher than the ones of the other risk-based asset allocation strategies. It amounts to 6.6% on average which compares to 3.7% for the MDP, 2.6% for minimum-variance and 1.5% for risk parity⁹. By construction, the turnover of the $1/N$ -strategy is 0% – disregarding potential rebalancing because of price movement.

4.3.3. How diversified are the risk-based asset allocation schemes?

As argued by Lee [LEE 11], evaluating risk-based portfolio strategies by means of Sharpe ratios is hard to reconcile with the fact that returns are not entering their respective objective function in the first place. In a vein similar to [LEE 11], we rather resort to contrasting the risk characteristics of these portfolios. Thus, we turn to an in-depth discussion of the risk-based strategies' weights and risk allocation in

⁹ Note that we measure turnover based on weight changes of successive models which could understate the actual turnover of the strategies.

Figures 4.5, 4.6 and 4.7. Risk is being decomposed not only by asset class but also by principal portfolios. Hence, the former analysis provides the well-known percentage risk contributions, while the latter analysis performs the very same decomposition with regard to uncorrelated risk sources.

First, we examine both diversified risk parity strategies in Figure 4.5. Only at the beginning of the sample period is the optimal DRP strategy characterized by positive asset weights across all asset classes. In general, the optimal DRP strategy has a large fraction of government bonds and is rather long in emerging markets and commodities. The strategy is often short in credit and sometimes short in developed equities as well. The need to go short is exacerbated in times of crises when correlations spike and the corresponding PPs are less straightforward to implement in a long-only manner. This observation is important when it comes to rationalizing the allocation pattern of the constrained DRP strategy. It allocates two thirds to government bonds and some 20% to commodities. At the beginning of the sample period, the strategy has a 15% credit position that is driven out of the portfolio by an increase in government bonds at the end of the 1990s. Besides, there is a constant exposure to equities with emerging and developed equities being of equal importance. The increase in the equity exposure at the end of 2008 comes at the cost of the commodities position which is completely closed. By construction, the strategy reacts timely to changes in risk structure and thus maintains a quite homogenous risk decomposition by principal portfolios throughout time. Even though facing a long-only and full-investment constraint, the objective of risk parity across principal portfolios is fairly well achieved. Notably, this objective turns out to be harder to realize at the end of the sample period.

Next, we turn to the benchmark strategies as depicted in Figures 4.6 and 4.7. First investigating the $1/N$ -strategy, we find more than 80% of its overall risk to be driven by equities, with the highly volatile emerging equities attracting the highest share of the risk budget. Given that commodities consume most of the remaining risk budget, the other asset classes, namely bonds and credit, are close to being irrelevant. Decomposing the strategy's risk by principal portfolios instead reveals the $1/N$ -strategy to be budgeting risk mostly to PP1, i.e. equity risk. Even more so, as time progresses the $1/N$ -strategy more or less emerges as a single-bet strategy as opposed to an N -bet strategy.

Second, we recover the archetypical weights distribution of minimum-variance that is heavily concentrated in the two low-risk asset classes bonds and credit. While equities hardly enter the minimum-variance portfolio, there is always a diversifying commodities position of some 5–10% in place. This weights decomposition serves as a blueprint for the minimum-variance strategy's traditional risk decomposition. Conversely, the decomposition of risk with respect to the principal portfolios demonstrates minimum-variance to be heavily exposed to a single risk source, PP4, representing interest rate risk. Compared to $1/N$, the minimum-variance strategy appears to be less concentrated because it is also exhibiting a quite marked exposure to PP3 and PP5.

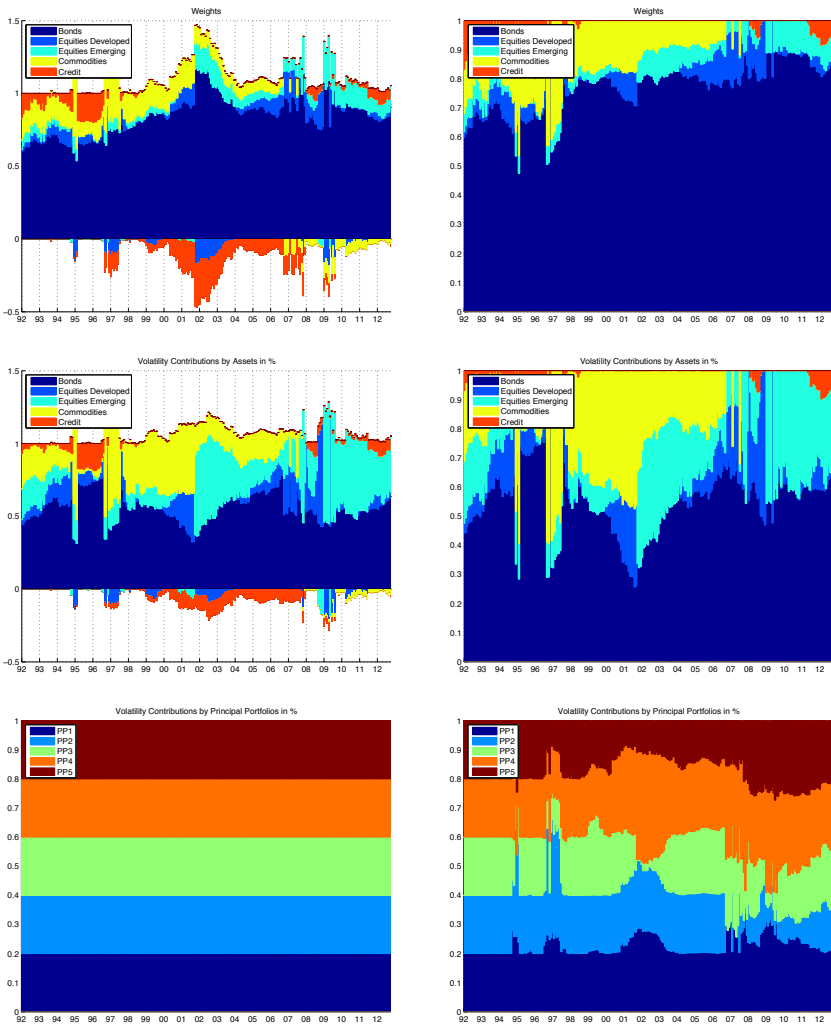


Figure 4.5. *Weights and risk decompositions: DRP optimal vs. constrained. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip*

The figure shows the decomposition of the diversified risk parity strategies in terms of weights and risk. Risk is being decomposed by asset classes and principal portfolios, respectively. The left-hand column gives results for the optimal DRP strategy and the right-hand column gives results for the constrained DRP strategy when using rolling window estimation of 60 months. The sample period is from January 1993 to September 2013.

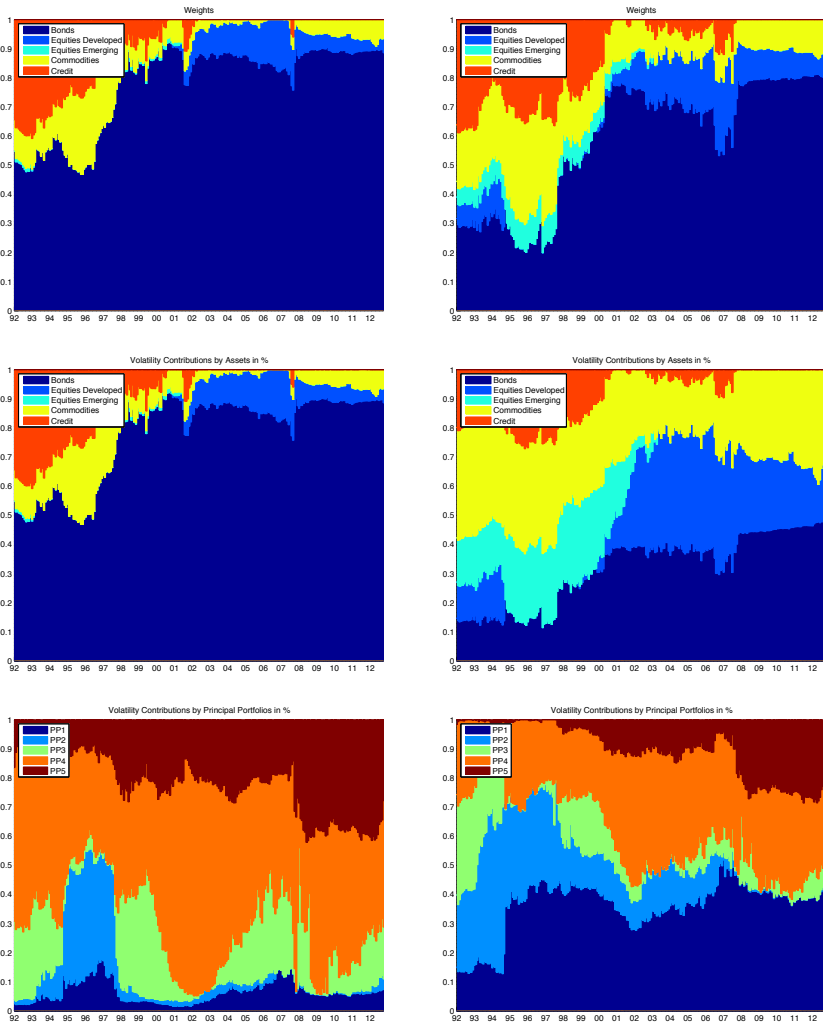


Figure 4.6. *Weights and risk decompositions: minimum-variance and MDP. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip*

The figure gives the decomposition of the risk-based allocation strategies in terms of weights and risk. Risk is being decomposed by asset classes and principal portfolios, respectively. The results build on rolling window estimation over 60 months. The left-hand column gives result for the minimum-variance strategy and the right-hand column gives results for the MDP. The sample period is from January 1993 to September 2013.

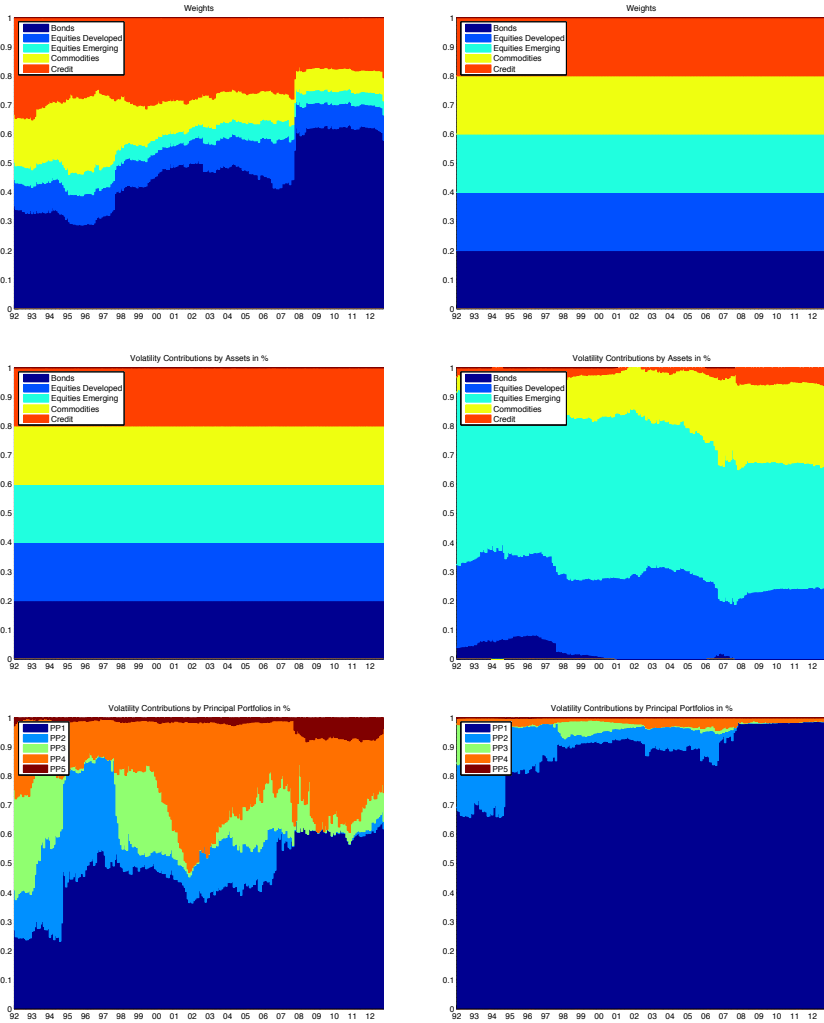


Figure 4.7. *Weights and risk decompositions: risk parity and 1/N. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip*

The figure gives the decomposition of the risk-based allocation strategies in terms of weights and risk. Risk is being decomposed by asset classes and principal portfolios, respectively. The results build on rolling window estimation over 60 months. The left-hand column gives result for the risk parity strategy and the right-hand column gives results for the 1/N-strategy. The sample period is from January 1993 to September 2013.

Third, we examine the risk parity strategy. Its weights decomposition reflects a reasonably smooth allocation over time with global bonds accounting for the highest portfolio fraction; on average, one-third is being allocated to this asset class. While the bond share is increasing over time, we realize that this increase is mainly fueled by a decrease in the credit position. This observation relates to the fact that the rising credit volatility induces the strategy to limit its credit exposure for maintaining risk parity. The remaining asset classes, equities and commodities are characterized by rather constant allocation weights over time that are approximately inversely proportional to their respective time series volatilities. By construction, the traditional risk decomposition exhibits equal weights across asset classes¹⁰. Interestingly, the decomposition of the risk parity strategy with respect to the principal portfolios is significantly less evenly distributed. At the beginning of the sample period PP1 and PP4, each attracts some quarter of the risk budget, while PP2 and PP3 almost completely absorb its remainder. However, PP2 and PP3 are constantly losing share giving rise to a 50% risk contribution of PP1 and some 35% of PP4. Hence, the risk parity strategy is rendered highly concentrated in terms of uncorrelated risk sources at the end of the sample period.

Fourth, we examine the results for the MDP. Overall, its weights decomposition over time is in between the one of risk parity and minimum-variance. Nevertheless, the MDP's reaction to the 2008 crisis is more pronounced with respect to reducing the credit position in favor of global bonds. Given that the MDP's share in commodities is also slightly higher than the one for the risk parity strategy, its traditional risk decomposition is slightly dominated by commodities risk. More interestingly, the risk decomposition with respect to the uncorrelated risk sources is quite evenly distributed. While we also observe a minor degeneration of the profile over time, it is less severe when compared to $1/N$, minimum-variance or risk parity. In a recent paper, Choueifaty *et al.* [CHO 13] show that the MDP is optimal under certain conditions, among which is the homogeneity of the investment universe. This condition is hardly met in an asset allocation context where asset classes are typically characterized by very different risk-return trade-offs. Thus, our results obtained in the MDP case should be taken with a grain of salt. Still, it is interesting to compare diversification ratios across strategies (see Table 4.3). Relative to the MDP (1.88), the traditional risk parity strategy (1.78) and the constrained DRP strategy (1.69) are not too far off, while $1/N$ (1.44) and minimum-variance (1.62) exhibit considerably smaller ratios.

¹⁰ At times, risk parity only holds approximately given that the numerical optimization may be tricky (see [MAI 10])

For directly comparing the degree to which the risk-based asset allocation strategies accomplish the goal of diversifying across uncorrelated risk sources, we plot the number of uncorrelated bets over time in Figure 4.8. Reiterating our above interpretation of the associated risk contributions over time, we find the $1/N$ -strategy to be mostly dominated by the other strategies. Unsurprisingly, the constrained DRP strategy is maintaining the highest number of bets throughout time which often reaches the maximum of five bets. However, there was a short episode in 2000/01 when long-only constraints rendered the constrained DRP strategy with less bets than most of the risk-based alternatives. Intuitively, the latter strategies implement bets that are not deemed attractive by the principal portfolio constraints effective for the constrained DRP strategy. Inbetween $1/N$ and diversified risk parity, we find minimum-variance and risk parity to represent some three bets over time. MDP is close to four bets on average, but these three strategies essentially degenerate in their degree of diversification at the end of the sample period.

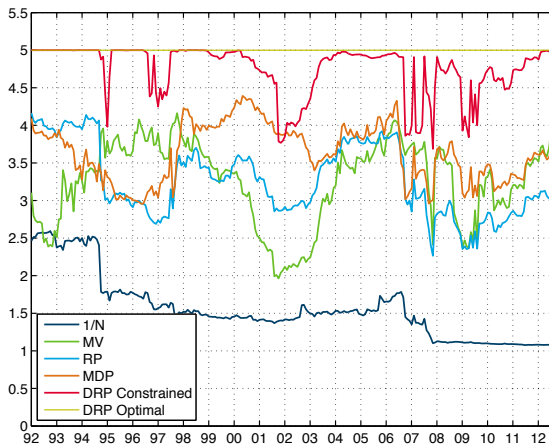


Figure 4.8. *Number of uncorrelated bets. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip*

We plot the number of uncorrelated bets for the risk-based asset allocation strategies when using rolling window estimation of 60 months for the sample period January 1993–September 2013.

4.3.4. In today, out tomorrow? Risk-based strategies in a rising interest rate environment

Most of the risk-based strategies have profited from a high bond exposure in the presence of falling interest rates. The room for further decreases in interest rates is

limited; instead, an increase in rates is more likely. To what extent can we expect risk-based investment strategies to deliver a convincing performance in such an environment? To investigate this question, we simulate the future market development over 60 months for the period 2013–2018 based on a stochastic capital market model. The core of the capital market model forms a series of stochastic processes to reflect a consistent economy building on the main asset classes, money market, global government bonds, global equities, emerging market equities, corporate bonds and commodities. The selection of stochastic processes and their connection ensures that the ensuing 1,000 capital market paths and the underlying distributions display empirically observable effects such as fat tails, volatility clustering, autocorrelation and rising correlations in times of crises¹¹.

In the parameterization of the model, we assume a slow increase in the money market interest rate to 1.4% p.a. at the end of 2018. The average annualized return on government bonds over the 60 months is then 0.95%. For global equities, this value amounts to 6.5% and 8.0% for emerging market equities, respectively. For commodities, we assume 5% p.a., while corporate bonds are expected to give 2.7% p.a. The volatility of government bonds is calibrated to 2.5%, 20% for global equities and 25% for emerging market equities. For corporate bonds, the corresponding figure is 5.5%, and 16% for commodities. The higher moments of the distribution are retrieved from the historical time series.

While Table 4.3 shows the historical performance of the five risk-based allocation techniques in the period from January 1993 to September 2013, Table 4.5 shows the median performance and risk statistics of the risk-based strategies over the various simulated market scenarios for the years 2013–2018. The median annualized return across 1,000 capital market paths for the year 2013–2018 is significantly below the values of the last two decades for all strategies. Their volatilities hardly change and, hence, risk-adjusted returns suffer. Still, strategies striving for optimal diversification, such as DRP and MDP, can provide a relatively convincing performance.

4.4. Conclusion

Within this chapter, we embrace the approach of Meucci [MEU 09] to maximize a portfolio's diversification. His paradigm stipulates rearranging the portfolio assets into uncorrelated risk sources by means of a simple PCA. Maximum diversification is obtained when equally budgeting risk to each of the uncorrelated risk sources, prompting us to label the strategy diversified risk parity, whereas risk-based asset

¹¹ Specifically, the capital market model involves a Cox–Ingersoll–Ross short rate model to span the entire yield curve, autoregressive processes of first order with superimposed jump processes for the spread development, as well as a regime switching model for stocks and commodities

allocation techniques generally yield superior risk-adjusted performance. Judging these strategies by their returns is at odds with the fact that returns are not part of the strategies' underlying objective function. Following [LEE 11], we rather turn to evaluating their *ex ante* risk characteristics, especially with respect to the uncorrelated risk sources. While the diversified risk parity strategy is designed to balance these risk sources, it is reassuring that it is meeting this objective well, even when facing long-only constraints. Unfortunately, the competing alternatives tend to be rather concentrated in a few bets. While the traditional risk parity strategy appeared to be least affected at the outset, we document a decrease in its degree of diversification over time. Also, the traditional risk parity strategy's nature is critically dependent on the choice of assets for contributing equally to portfolio risk. Conversely, diversified risk parity has a built-in mechanism for tracking the prevailing risk structure thus providing a more robust way to achieve maximum diversification throughout time. All of the risk-based strategies have delivered a convincing performance across different market regimes in the past. Going forward, this finding will only continue to hold when striving for a high degree of diversification. Still, investment performance will be more moderate than in the past.

Statistic	Diversified Risk Parity		Risk-Based Allocations			
	<i>Optimal</i>	<i>Constrained</i>	<i>I/N</i>	<i>MV</i>	<i>RP</i>	<i>MDP</i>
Return p.a.	4.1%	3.6%	6.1%	1.6%	2.8%	2.4%
Volatility p.a.	10.6%	4.2%	9.9%	2.1%	3.6%	3.1%
Sharpe Ratio	0.40	0.66	0.54	0.38	0.55	0.49
MDD 1M	-7.9%	-3.2%	-8.4%	-1.4%	-2.9%	-2.5%
MDD	-13.4%	-5.8%	-19.4%	-3.1%	-6.7%	-5.9%

Table 4.5. *Simulated performance and risk statistics: 2013–2018*

The table gives median performance and risk statistics of risk-based asset allocation strategies obtained over the simulated 1,000 capital market scenarios from September 2013 to August 2018. Median annualized return and volatility figures are reported together with the respective annualized Sharpe Ratio where the risk-free rate is given by the simulated average 3-month U.S. Treasury Rate. Maximum Drawdown (MDD) is computed over 1 month and over the simulated period.

4.5. Acknowledgments

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Process Symposium in Monaco, Northfield's 25th Annual Research Conference in San Diego and the 2012 European Quantitative Forum of State Street in London. Note that this chapter expresses the authors' views that do not necessarily coincide with those of Deka Investment GmbH. This chapter is an updated and extended version of our paper which was first published in *The Journal of Risk* [LOH 14]. Reprinted with the kind permission of Incisive Risk Information (IP) Limited.

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Robust Portfolio Allocation with Systematic Risk Contribution Restrictions¹

The standard mean-variance approach can imply extreme weights in some assets in the optimal allocation and a lack of stability of this allocation over time. In order to not only improve the robustness of the portfolio allocation, but also to better control the portfolio turnover and the sensitivity of the portfolio to systematic risk, it is proposed in this chapter to introduce additional constraints on both the total systematic risk contribution of the portfolio and its turnover. Our chapter extends the existing literature on risk parity in three directions: (1) we consider other risk criteria than the variance, such as the value-at-risk (VaR), or the expected shortfall; (2) we manage separately the systematic and idiosyncratic components of the portfolio risk; (3) we introduce a set of portfolio management approaches which control the degree of market neutrality of the portfolio, for the strength of the constraint on systematic risk contribution and for the turnover.

5.1. Introduction

The gap between theory and practice is well illustrated by the example of portfolio management since Markowitz [MAR 52] introduced the mean-variance framework. The resolution of the allocation problem by a simple quadratic optimization is the main advantage of the mean-variance approach. However, in practice, this approach is implemented by replacing the theoretical mean and

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variance by their (unrestricted) historical counterparts, and the associated estimated mean-variance portfolios have several drawbacks: they are very sensitive to errors in the estimates of the mean and variance inputs (see e.g. [CHO 93a, CHO 93b]), the resolution of a large-scale quadratic optimization problem is not straightforward (see e.g. [KON 91]), and dominant factor in the covariance matrix results in extreme weights in optimal portfolios (see e.g. [GRE 92]). Finally, the portfolio allocations are very erratic over time, which implies significant transaction costs or liquidity risks. These drawbacks are even more pronounced when the portfolio is based on a large number of assets.

These difficulties are mainly due to the sensitivity of the mean-variance efficient portfolio allocation to the smallest eigenvalues of the variance matrix and to the poor accuracy of the inverse variance matrix with the standard estimation methods. The literature has proposed different ways to get more robust portfolio allocations, as the potential cost of a loss of efficiency. First, some robust estimation methods have been introduced, following results known in statistics². Typical of such approaches are the shrinkages of the estimated variance matrix, which admit Bayesian interpretation [GAR 07, GOL 03, LED 04], the l_1 - or l_2 -penalizations introduced in the empirical optimization problem (see e.g. [BRO 08, DEM 09a, FAN 12]), or the refresh time subsample approach with far more percentage of data used for any given pair of assets than for all the assets of the portfolio [BAR 08].

Robustness can also be achieved by introducing restrictions in the empirical optimization problem even if these restrictions are not required by Financial Theory. These constraints have often simple interpretations. They can be shortselling restrictions [FRO 88, CHO 93a, JAG 03], gross exposure constraints [FAN 12], at the limit “fully diversified” portfolios in terms of either budget allocations (see [ELT 77, DUC 09, DEM 09b, KRI 10, BEL 12]), or contributions to total risk (see e.g. [MAR 08, CHO 08, MAL 10, BRU 11]).

The idea of imposing additional diversification constraints is now commonly used in the asset management industry, and more enhanced strategies are grouped under the risk parity denomination. Risk parity is a general term for all investment techniques that attempt to take equal risk in the different underlyings of a portfolio. However, risk parity implementations differ considerably: investment universes, risk definitions, risk forecasting methods and risk exposures calculation can be different from one implementation to another implementation. Thus, risk parity is more a conceptual

² It is well-known that the standard OLS estimator in a regression model $y = Xb + u$ is not robust. The expression of the OLS estimator: $\hat{b} = (X'X)^{-1}X'y$ includes the inversion of the design matrix $X'X$, and this inversion is not accurate when the explanatory variables are quasi-colinear. This lack of robustness is solved, either by considering Bayesian estimators, or by introducing l_2 -penalizations, or by constraining the parameters.

approach rather than a specific system, and it is in general difficult to compare the different approaches.

Many questions are raised by risk parity approaches. First, the total risk of a given portfolio is uniquely measured by its volatility, and contributions to total risk by the contribution of each underlying asset to this volatility. However, in a risk parity allocation, it is more natural to define the total risk as the potential loss at the portfolio level and the contribution to total risk as the amount of initial wealth measured in risk unit on each portfolio underlying. These amounts are called risk budget in the literature (see e.g. [CHO 01, LEE 01]). By defining risk budget through volatility contributions, Gaussian returns are implicitly assumed (see e.g. [INK 10]). Once the potential loss of capital for each portfolio underlying has been estimated, the portfolio can be determined. Second, the optimality of the standard risk parity portfolio, which imposes equal risk budgets on the underlying, can be discussed. This risk parity approach does not ensure that the total risk of the portfolio is optimized. Third, the definition of the investment universe has a significant impact on the risk parity portfolio. In particular, the risk parity portfolio allocation changes when we duplicate one asset. Thus it does not satisfy the duplication invariance property in the terminology of Choueifaty *et al.* [CHO 13]. Finally, risk parity approaches decrease portfolio concentration by construction in increasing the small cap weights. Then they create liquidity issues, since we have to dynamically rebalance an equity portfolio with a bigger liquidity exposure on small caps.

We develop in this paper a new implementation of the risk parity principle that circumvents the usual limitations of the current implemented ones. Our contribution to the literature is threefold. Firstly, we use a more appropriate risk measure than the variance to account for extreme risks and give a reserve interpretation of the risk contributions in a general non-Gaussian framework. Secondly, we introduce the risk contribution restriction on the total contribution of the portfolio to systematic risk and do not impose equal contributions to the systematic and unsystematic components of the portfolio risk. Thirdly, we discuss the interest of such a restriction in terms of portfolio turnover and transaction costs.

This chapter is organized as follows. In section 5.2, we focus on the difference between the standard optimal portfolios and the associated risk parity portfolios. Section 5.3 considers asset returns with systematic and idiosyncratic components. Then we construct and compare different risk parity portfolios, when the parity is written on both types of components. Section 5.4 derives and compares optimal portfolios for different risk measures, especially the volatility, the VaR and the expected shortfall. Section 5.5 presents empirical applications on portfolio of futures on commodities and section 5.6 concludes. Some extensions and proofs are given in the appendices.

5.2. Portfolio allocation with risk contribution restrictions

We review in this section basic results on portfolio and risk allocations to highlight the difference between the standard optimal portfolios and the portfolios with risk contribution restrictions. We denote by y_1, \dots, y_n the returns of n risky assets, Y the corresponding vector of returns, μ the vector of expected returns, Ω the associated volatility matrix and w the portfolio allocation, satisfying the standardized budget constraint $w'e = 1$, with e being an n -dimensional vector of 1. We denote by $R(w)$ the scalar risk measure associated with allocation w . The risk measure depends on allocation w through the distribution of the portfolio return $w'Y$.

5.2.1. Minimum risk portfolios

Let us focus first on the risk minimization problem. We obtain the minimum risk allocation by solving the program:

$$w^* = \underset{w'e=1}{\operatorname{argmin}} R(w).$$

The optimization problem above is written under the standardized budget constraint $w'e = 1$. This possibility to standardize the budget constraint exists if the risk measure is homogeneous of degree 1, that is, if $R(cw) = cR(w)$, for any positive scalar³ c .

5.2.2. Portfolios with risk contribution restrictions

The recent literature on risk measures focuses on the risk contribution of each asset to the total portfolio risk. In this respect, the risk contributions differ from the weights in portfolio allocations, since they also account for the effect of each individual asset on the total risk. Let us consider a global portfolio risk measured by $R(w)$. This total risk can be assigned to the different assets as:

$$R(w) = \sum_{i=1}^n R_i(w), \quad [5.1]$$

where $R_i(w)$ denotes the risk contribution of asset i to the risk of the whole portfolio. If the risk measure is homogenous of degree 1, we get the Euler formula:

$$R(w) = \sum_{i=1}^n w_i \frac{\partial R(w)}{\partial w_i}.$$

³ Indeed, the solution of an optimization problem such as $w^*(c) = \min_w R(w)$, *s.t.* $w'e = 1/c$ is equal to $w^*(c) = cw^*$. Thus, the solution with another budget restriction is deduced from the solution of the standardized optimization problem by an appropriate scaling.

The Euler formula has an interpretation in terms of marginal contribution to global risk w.r.t. a change of scale in the portfolio allocation⁴.

This explains why it is often proposed in the literature to choose:

$$R_i(w) = w_i \frac{\partial R(w)}{\partial w_i}, \quad [5.2]$$

called the Euler allocation [LIT 96, p.28; GAR 97, footnote 2; QIA 06]. The difference between the portfolio allocation and the risk contribution is captured by the marginal risk $\partial R(w)/\partial w_i$ (see equation [5.2]).

The Euler decomposition can be used to construct portfolios with constraints on the risk contributions. For instance, equally weighted risk contribution portfolios have been considered in the literature (see e.g. [SCH 07, MAI 10, ASN 12]), and are gaining in popularity among practitioners [ASN 10, SUL 10, DOR 11].

This practice can be generalized by imposing the risk contributions to be proportional to some benchmarks $\pi_i, i = 1, \dots, n$, which are not necessarily equal to:

$$\frac{\partial R(w)}{\partial w} = \lambda(w) \text{diag}(\pi_i) \text{vec}(1/w), \quad [5.3]$$

where $\text{diag}(\pi_i)$ is the diagonal matrix with $\pi_i, i = 1, \dots, n$ as diagonal elements. That is, we consider risk parity portfolios after an appropriate adjustment for the notion of parity.

5.2.3. Risk contribution restrictions and portfolio turnover

The introduction of restrictions [5.3] can be justified by the effect of trading costs. Let us assume that the investor's portfolio allocation at the beginning of the period is: $w_0 = (w_{0,1}, \dots, w_{0,n})'$, and that the investor updates his portfolio to get the new allocation $w = (w_1, \dots, w_n)'$. He will account for the risk $R(w)$ of the new allocation and for the trading costs when reallocating the portfolio from w_0 to w . Under no short sale constraints: $w_{0,i} \geq 0, w_i \geq 0, \forall i$, the trading cost (turnover) may be measured by:

$$T(w, w_0) = c \sum_{i=1}^n w_{0,i} \ln \left(\frac{w_{0,i}}{w_i} \right). \quad [5.4]$$

⁴ The Euler formula is obtained by differentiating the homogeneity condition $R(cw) = cR(w)$, with respect to c and setting $c = 1$.

Indeed, when the allocation adjustments are small, we have:

$$T(w, w_0) \approx -c \left[\sum_{i=1}^n w_{0,i} \frac{w_i - w_{0,i}}{w_{0,i}} - \frac{1}{2} \sum_{i=1}^n w_{0,i} \frac{(w_i - w_{0,i})^2}{w_{0,i}^2} \right]$$

$$\approx \frac{c}{2} \sum_{i=1}^n \frac{(w_i - w_{0,i})^2}{w_{0,i}},$$

since the two portfolios satisfy the budget constraint: $e'w = e'w_0 = 1$.

This approximation has a direct interpretation in terms of transaction costs, in which the cost for trading asset i is proportional to $1/w_{0,i}$. This assumption on trading costs can find a justification if the initial allocation corresponds to a market portfolio. Assets with the highest market weights $w_{0,i}$ are also the most liquid ones, and their trading is associated with low transaction cost. At the opposite, assets with the lowest market weights are less liquid and then trading these assets is expensive in terms of transaction costs. This cost for trading asset i is proportional to $(w_i - w_{0,i})^2$. Thus, the implied market impact function for trading asset i is strictly convex.

The investor has to balance risk reduction and trading cost in his portfolio management. Thus, he/she can minimize a combination of both criteria, and choose:

$$w = \operatorname{argmin}_w R(w) + \lambda c \sum_{i=1}^n w_{0,i} \ln \left(\frac{w_{0,i}}{w_i} \right), \quad [5.5]$$

where $\lambda > 0$ is a smoothing parameter introduced to control the portfolio turnover. With $\lambda = 0$, the investment objective focuses on risk control. For high λ , the control is on the portfolio turnover, and the investment objective is to enhance the initial portfolio allocation in terms of risk control, but with a limited turnover.

The associated first-order condition is:

$$\frac{\partial R(w)}{\partial w_i} - \lambda c \frac{w_{0,i}}{w_i} = 0 \Leftrightarrow w_i \frac{\partial R(w)}{\partial w_i} = \lambda c w_{0,i}. \quad [5.6]$$

The risk contributions are proportional to the initial portfolio allocations: $\pi_i \propto w_{0,i}$. In particular, the benchmark levels of risk contributions π_i , $i = 1, \dots, n$ depend on the current investor's portfolio. This approach is clearly suitable for advising investors that do not want to enhance their risk management without generating a high portfolio turnover.

This solution is especially appealing in a multiperiod framework. Indeed, in a myopic dynamic portfolio management, the sequence of optimization problems is:

$$w_t^* = \underset{w_t}{\operatorname{argmin}} R_t(w_t) + \lambda c_t \sum_{j=1}^n w_{t-1,j}^* \ln \left(\frac{w_{t-1,j}^*}{w_{t,j}} \right), \quad [5.7]$$

where the conditional risk measure $R_t(w_t)$ and the trading cost c_t depend on time. Then the risk contribution restrictions are proportional to $w_{t-1,i}^*$ and path dependent. In this dynamic framework, the λ parameter controls the speed of convergence of the current portfolio toward the time dependent minimum risk portfolio. In a stable risk environment, that is, if R_t and c_t do not depend on time, the optimal dynamic reallocation approaches the minimum risk portfolio in several steps instead of doing the reallocation at a single date. This point is especially appealing for big institutional investors that want to reallocate huge portfolios without destabilizing the markets⁵. This multiperiod optimal reallocation approach is also appealing when managing portfolios of illiquid assets.

5.3. Portfolio allocation with systematic risk contribution restrictions

In this section, we consider portfolio allocations constructed to monitor the systematic and idiosyncratic components of the portfolio return. This is done by imposing the risk contribution restrictions on these two components of the total risk. We consider factor models to discuss the effects of the systematic and idiosyncratic components of the risk.

5.3.1. Systematic and idiosyncratic risks

Let us assume that the asset returns follow the one-factor model:

$$y_i = \beta_i f + \sigma_i u_i, \quad i = 1, \dots, n, \quad [5.8]$$

where f is the common (or systematic) factor, β_i is the factor loading of asset i w.r.t. factor f , and u_i is the idiosyncratic (or specific) component, independent of the factor. We assume that the idiosyncratic terms are mutually independent⁶, with unconditional zero mean and unit variance. We get the following decomposition of the return covariance matrix:

$$\Omega = \beta \beta' \sigma_f^2 + \Sigma, \quad [5.9]$$

⁵ This is an important criterion for food commodity markets, when the commodity is also traded for consumption by households, for instance.

⁶ Any residual dependence might be captured by introducing additional common factors. This would lead to a multifactor model. We consider the one-factor model for expository purpose.

where $\Sigma = Vu = \text{diag}(\sigma^2)$, σ_f^2 is the variance of the common factor and β is the vector of factor loadings. The portfolio return can be decomposed accordingly into a systematic and an idiosyncratic component as:

$$w'Y = \left(\sum_{i=1}^n w_i \beta_i \right) f + \sum_{i=1}^n w_i \sigma_i u_i. \quad [5.10]$$

The effects of the systematic and idiosyncratic components can be analyzed for both risk contributions and portfolio allocations.

5.3.2. Systematic and idiosyncratic risk contributions

The decomposition principle [5.8] can be applied to disentangle the systematic and idiosyncratic components of the risk as follows:

$$R_i(w) = R_{is}(w) + R_{iu}(w), i = 1, \dots, n,$$

where $R_{is}(w)$ [respectively $R_{iu}(w)$] denotes the systematic (respectively idiosyncratic) risk contribution of asset i to the total systematic (respectively idiosyncratic) component of the risk. The risk decompositions above can be aggregated to get a decomposition of the total risk as:

$$R(w) = R_s(w) + R_u(w),$$

where $R_s(w) = \sum_{i=1}^n R_{is}(w)$ and $R_u(w) = \sum_{i=1}^n R_{iu}(w)$. These decompositions are summarized in Table 5.1. This table shows how to pass from the assets $i = 1, \dots, n$ tradable on the market, to the virtual assets f and $(u_1, \dots, u_n) = u$, which are not directly tradable, that is, how to transform the decomposition of the total risk with respect to basic assets $i = 1, \dots, n$ to a decomposition with respect to virtual assets. This is done by constructing an appropriate two entries table, and summing up per column instead of summing up per row (see [GOU 13]).

Assets	Systematic factor	Idiosyncratic error terms	Total
1	$R_{is}(w)$	$R_{iu}(w)$	$R_1(w)$
\vdots			\vdots
i			$R_i(w)$
\vdots			\vdots
n			$R_n(w)$
Total	$R_s(w)$	$R_u(w)$	$R(w)$

Table 5.1. Decomposition of the global risk measure

How to derive this thinner risk decomposition in practice, while keeping an interpretation in terms of Euler decomposition? Let us consider the virtual portfolio with allocation $w_{i,s}$ in the systematic component and $w_{i,u}$ in the idiosyncratic one. Thus, the associated portfolio return becomes:

$$\left(\sum_{i=1}^n w_{i,s} \beta_i \right) f + \sum_{i=1}^n w_{i,u} \sigma_i u_i.$$

This portfolio invests $w_{i,s}$ in $\beta_i f$, and $w_{i,u}$ in $\sigma_i u_i$, $i = 1, \dots, n$. If we denote \tilde{w} the components $w_{i,s}, w_{i,u}$, the risk measure of this virtual portfolio can be written as: $\tilde{R}(w_s, w_u)$, where $\tilde{R}(w, w) = R(w)$. The extended risk measure \tilde{R} is also homogenous of degree 1. Thus, we can apply the Euler formula to \tilde{R} and obtain:

$$\tilde{R}(w_s, w_u) = \sum_{i=1}^n w_{i,s} \frac{\partial \tilde{R}}{\partial w_{i,s}}(w_s, w_u) + \sum_{i=1}^n w_{i,u} \frac{\partial \tilde{R}}{\partial w_{i,u}}(w_s, w_u).$$

Then, for $w_s = w_u = w$, we deduce the thinner decomposition:

$$\tilde{R}(w) = \sum_{i=1}^n w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w, w) + \sum_{i=1}^n w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w, w),$$

and can define: $R_{i,s}(w) = w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w, w)$, $R_{i,u}(w) = w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w, w)$. Finally, the risk measure $R(w)$ is also function of parameters β_i, σ_i , $i = 1, \dots, n$, involved in the factor model and we get:

$$w_i \frac{\partial \tilde{R}}{\partial w_{i,s}}(w, w) = \frac{\partial R}{\partial \beta_i}(w), \quad w_i \frac{\partial \tilde{R}}{\partial w_{i,u}}(w, w) = \frac{\partial R}{\partial \sigma_i}(w), \quad [5.11]$$

in which the dependence of R with respect to β_i, σ_i is not explicitly written for expository purpose. We get a decomposition, which highlights the effects on the total portfolio risk of shocks either on the factor, or on the idiosyncratic term.

5.3.3. Portfolios with systematic risk contribution restrictions

In the standard portfolios with risk contribution restrictions, the constraints are written on the basic assets. The approach can be extended by considering risk contributions written on the systematic and unsystematic components of the portfolio. Let us consider the following optimization problem:

$$w(\delta, \pi) = \underset{w' e=1}{\operatorname{argmin}} R^2(w) + \delta [(1 - \pi)R_s(w) - \pi R_u(w)]^2, \quad [5.12]$$

where $\delta \in (0, \infty)$ is a smoothing parameter. In the limiting case $\delta = \infty$, we get the optimization with a strict constraint on the contribution to systematic risk⁷: $R_s(w) = \pi R(w)$. When $\delta = 0$, we get the minimum risk portfolio.

As in section 5.2.3, we can justify the introduction of this risk contribution restriction by the effect of trading costs, both on individual assets and on the factors, when derivative instruments allows investors to directly trade on the virtual factor asset. This is the case for equity investing, where the factor is usually the market portfolio.

5.4. Illustrations with different risk measures

This section provides the closed form expressions of the minimum risk portfolios and the risk contributions for three risk measures, that are the volatility, the VaR, and the distortion risk measures, including the expected shortfall.

5.4.1. The volatility risk measure

When the risk is measured by the volatility, we get: $R(w) = (w' \Omega w)^{1/2}$.

5.4.1.1. Minimum variance portfolio

Let us assume that the set of assets does not include the riskfree asset, or equivalently that the volatility matrix Ω is invertible. For the volatility risk measure, we get the minimum-variance portfolio (see [MAR 52]), whose optimal allocation has the closed form expression:

$$w^* = \frac{\Omega^{-1} e}{e' \Omega^{-1} e}.$$

5.4.1.2. Risk contributions

We have:

$$\frac{\partial R(w)}{\partial w} = \frac{\Omega w}{(w' \Omega w)^{1/2}},$$

and the risk contributions are:

$$R_i(w) = \frac{w_i}{(w' \Omega w)^{1/2}} \sum_{j=1}^n \Omega_{i,j} w_j = \frac{\text{Cov}(w_i y_i, w' Y)}{V(w' Y)},$$

⁷ If the level π belongs to the domain of admissible values of $R_s(w)/R(w)$, when w vary. Otherwise, we get the portfolio allocation with a systematic risk budget the closest to π .

where $\Omega_{i,j}$ is the generic element of Ω , and $Cov(\cdot)$ and $V(\cdot)$ denote, respectively, the covariance and the variance. Thus, the contribution $R_i(w)$ is the beta coefficient of the part of the portfolio invested in asset i with respect to the total portfolio.

5.4.1.3. Systematic and idiosyncratic risk contributions

In a single factor model, we have: $R(w) = [w'(\beta\beta'\sigma_f^2 + \Sigma)w]^{1/2}$ and the Euler risk contributions can be written as:

$$R_i(w) = \frac{w_i}{(w'\Omega w)^{1/2}} [\beta_i w' \beta \sigma_f^2 + w_i \sigma_i^2] = R_{is}(w) + R_{iu}(w),$$

where $R_{is}(w)$ is the systematic risk contribution i and $R_{iu}(w)$ is the idiosyncratic risk contribution of asset i :

$$R_{is}(w) = w_i \beta_i \frac{w' \beta \sigma_f^2}{(w'\Omega w)^{1/2}}, \quad R_{iu}(w) = w_i \frac{w_i \sigma_i^2}{(w'\Omega w)^{1/2}}.$$

The expression of component $R_{is}(w)$ shows the quantity $w_i \beta_i$ invested in the systematic factor f , and the risk contribution $\frac{w' \beta \sigma_f^2}{(w'\Omega w)^{1/2}}$ of a unit invested in f . By adding the decompositions per asset, we get the decomposition of the total portfolio risk as:

$$R(w) = R_s(w) + R_u(w), \quad \text{with } R_s(w) = (w' \beta)^2 \frac{\sigma_f^2}{(w'\Omega w)^{1/2}} \quad \text{and}$$

$$R_u(w) = \frac{w' \Sigma w}{(w'\Omega w)^{1/2}},$$

that is the standard variance decomposition equation.

5.4.2. The α -VaR risk measure

The introduction of the VaR corresponds to the safety first criterion initially introduced by Roy [ROY 52]. The α -VaR risk measure is defined by:

$$R(w) = -q_\alpha(w'Y),$$

where q_α is the α -quantile of the distribution of the portfolio return. More precisely, the α -VaR is defined by the condition: $P[w'Y < q_\alpha(w'Y)] = \alpha$.

5.4.2.1. Minimum α -VaR portfolio

Let us first consider the Gaussian case before discussing the general framework:

– let us assume that the set of basic assets does not include the riskfree asset and consider the allocation minimizing the α -VaR in a Gaussian framework. When the vector of returns is Gaussian with mean μ and variance-covariance Ω , the optimal allocation minimizes:

$$-q_\alpha(w'Y) = -w'\mu - q_\alpha w'\Omega w,$$

where q_α denotes the α -quantile⁸ of the standard Gaussian distribution under the budget restriction $w'e = 1$. The minimum α -VaR portfolio allocation is then given by:

$$w^* = \frac{\Omega^{-1}e}{e'\Omega^{-1}e} + \frac{1}{2q_\alpha}\Omega^{-1}\left[\mu - \frac{e'\Omega^{-1}\mu}{e'\Omega^{-1}e}e\right].$$

This formula highlights the key role of the minimum variance portfolio as the benchmark portfolio for a very risk averse investor (when $\alpha \rightarrow 0$ and $q_\alpha \rightarrow \infty$), but also the importance of the excess expected returns;

– in the general framework, the returns are not necessarily Gaussian and the minimum α -VaR portfolio is the solution of the system of equations:

$$\frac{\partial q_\alpha(w'Y)}{\partial w_i} = \lambda(w), i = 1, \dots, n, \quad [5.13]$$

where the Lagrange multiplier $\lambda(w)$ is fixed by the budget restriction $w'e = 1$. The derivative of the α -VaR is equal to [GOU 00, HAL 03]:

$$\frac{\partial q_\alpha}{\partial w_i}(w'Y) = E[y_i | w'Y = q_\alpha(w'Y)], i = 1, \dots, n. \quad [5.14]$$

This derivative has no closed form expression in general and the minimum α -VaR allocation has to be computed numerically.

⁸ Since α is small, q_α is negative. Thus, the α -VaR is an increasing function of the variance of the portfolio return and a decreasing function of its expected return.

5.4.2.2. Risk contributions

When the risk is measured by the α -VaR, we get the following decomposition formula of the global conditional quantile (see [GOU 13]):

$$q_\alpha(w'Y) = w' \frac{\partial q_\alpha(w'Y)}{\partial w} = w' E[Y | w'Y = q_\alpha(w'Y)], \quad [5.15]$$

and

$$R_i(w) = E[w_i y_i | w'Y = q_\alpha(w'Y)]. \quad [5.16]$$

It measures the part of the expected loss due to asset i when the total portfolio is in distress.

5.4.2.3. Systematic and idiosyncratic risk components

In the α -VaR case, the marginal effect of a change of weight of asset i can be decomposed by equation [5.11] as:

$$w_i \frac{\partial q_\alpha(w'Y)}{\partial w_i} = \beta_i \frac{\partial q_\alpha(w'Y)}{\partial \beta_i} + \sigma_i \frac{\partial q_\alpha(w'Y)}{\partial \sigma_i}. \quad [5.17]$$

The Euler components associated with systematic and idiosyncratic risks can be explicitated as follows:

$$\frac{\partial q_\alpha(w'Y)}{\partial \beta} = E[f | w'Y = q_\alpha(w'Y)], \quad \frac{\partial q_\alpha(w'Y)}{\partial \sigma_i} = E[u_i | w'Y = q_\alpha(w'Y)].$$

In the linear factor model, the general decomposition [5.11] becomes:

$$R_{is}(w) = \beta_i E[f | w'Y = q_\alpha(w'Y)], \quad R_{iu}(w) = \sigma_i E[u_i | w'Y = q_\alpha(w'Y)], \quad [5.18]$$

and the decomposition of the total portfolio risk is:

$$R(w) = R_s(w) + R_u(w),$$

with

$$R_s(w) = \beta' E[f | w'Y = q_\alpha(w'Y)], \quad R_u(w) = \sum_{i=1}^n \sigma_i E[u_i | w'Y = q_\alpha(w'Y)]. \quad [5.19]$$

5.4.3. Distorsion risk measures

A distortion risk measure is a weighted function of the VaRs associated with the different risk levels. It can be written as:

$$R(w) = \int VaR_\alpha(w) dH(\alpha) = - \int q_\alpha(w) dH(\alpha),$$

where H is a given distortion measure on $(0, 1)$, that is, an increasing concave function. The expected shortfall is obtained when H is the cumulative distribution function of the uniform distribution on the interval $[0, \alpha]$ (see e.g. [WAN 00, ACE 02a, ACE 02b]).

5.4.3.1. Minimum DRM portfolio

The optimal allocations have no closed form expression and have to be derived numerically. The minimum DRM portfolios solve the first-order condition (see [GOU 00]):

$$(1 - q_\alpha(w'Y)) E[Y|w'Y = q_\alpha(w'Y)] = 0.$$

5.4.3.2. Risk contributions

Let us, for instance, consider the expected shortfall ES_α . By definition we have:

$$ES_\alpha(w'Y) = w'E[Y|w'Y > q_\alpha(w'Y)], \quad [5.20]$$

with risk contribution [TAS 00]: $R_i(w) = E[w_i y_i | w'Y > q_\alpha(w'Y)]$. Thus, the risk decompositions for VaR and ES differ by their conditioning set. These conditioning sets correspond to different definitions of portfolio distress.

5.5. Application

We apply in this section the different portfolio management solutions of section 5.3 above to futures on commodities.

5.5.1. The investment universe

We consider futures contracts on physical commodities. These assets are split into five sectors as described in Table 5.2.

The prices are daily closing prices from 14 May, 1990 up to 24 September, 2012, and are all denominated in US\$, even for metals traded in London. The physical commodity prices include the storage and transportation costs. The returns are

adjusted by rolling the futures' positions in order to avoid the delivery process and to get a stable time-to-maturity over time. We obtain rather symmetric distributions, except for commodity "brent crudeoil", which is left skewed, and for "cotton", which is right skewed⁹. All distributions feature tails fatter than Gaussian tail with kurtosis up to 30–40 for "brent crudeoil" and "cotton". The historical betas of each asset returns with respect to the Dow Jones-UBS (DJUBS) commodity index are all non-negative and some returns are very sensitive to changes in the index such as "brent crudeoil" and "soybeans". These large values do not reflect the composition of the DJUBS index only. Indeed this index includes currently 20 physical commodities for 7 sectors. Thus, several commodities in Table 5.2 are not included in the index. The commodities included in the index are marked with a "*" in Table 5.2. Moreover, if the weights of included assets are fixed according to their global economic significance and market liquidity, they are capped. No commodity can compose more than 15% of the index and no sector more than 33%. For instance, cocoa, coffee and cotton have similar weights in the index, but cotton has a much higher beta than the two other commodities.

<i>Energy</i>	<i>Grains & Seeds</i>	<i>Softs</i>	<i>Live Stock</i>	<i>Metals</i>
brent crudeoil*	corn*	cocoa	lean hogs*	copper*
heating oil*	rice	coffee*	live cattle*	gold*
light crudeoil*	soybean oil*	cotton*		palladium
natural gas*	soybeans*	orange juice		platinum
	wheat*	sugar*		silver*

Table 5.2. *The commodities*

For expository purposes, we focus in this section on the "Grains & Seeds" sector¹⁰. Figure 5.1 plots the return evolutions of the five corresponding assets. These evolutions can be very different in such a sector, which is clearly not homogenous. Even if we observe common volatility clustering, there is a switching trend in both mean and volatility for commodity "soybeans" and partly for the commodity "soybean oil" positively correlated with it. It is this change of regime in 2004, which explains the double regime dependence mentioned earlier.

5.5.2. Portfolio management with total risk contribution restrictions

Let us now consider four portfolio allocations for the "Grains & Seeds" sector: an equally-weighted portfolio, a minimum-variance portfolio and two risk parity

⁹ Even if we do not focus on portfolio performances in this chapter, note that positive historical skewness of individual asset returns might explain some good performance properties of the equally weighted portfolio [BEL 12]. The small observed skewness show that this argument will not apply to commodities.

¹⁰ The analysis for the other commodity sectors are available from the authors upon request.

portfolios using either the volatility or the VaR at 5%, respectively. The three first portfolios are frequently considered in the H.F. literature (see e.g. [DEM 09b]), and can be used as benchmarks for comparison. The fourth portfolio allocation focuses on extreme risks. The VaR and VaR contributions are estimated by kernel methods¹¹, the means and variances by their historical counterparts based on the 252 previous observations. For each portfolio, we provide the evolutions of the portfolio weights (Figure 5.2) and of the contributions to VaR (Figure 5.3), computed under shortselling restrictions.

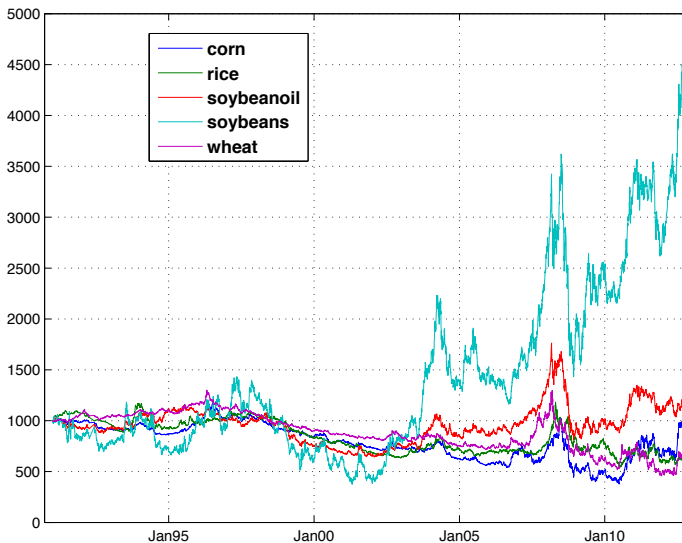


Figure 5.1. *Returns for the Sector Grains & Seeds. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip*

The main expected effect is to diminish the weights of highly risky assets for all strategies controlling the risk (see Figure 5.2). At the extreme, the commodity “soybeans” is not introduced in the min-variance allocation, whereas it appears underweighted for strategies based on risk contributions, which are using the “substitutability” with the less risky “soybean oil”. We also observe the instability over time of the weights for the min-variance portfolio largely mentioned in the literature. However, the two risk parity portfolios exhibit stable weights with a lower turnover. However, the final allocation depends on the risk measure selected to write the risk contribution restrictions.

¹¹ The standard Nadaraya–Watson estimator has to be adjusted to ensure that the estimated VaR and VaR contributions are compatible, that is satisfy exactly the Euler restriction.

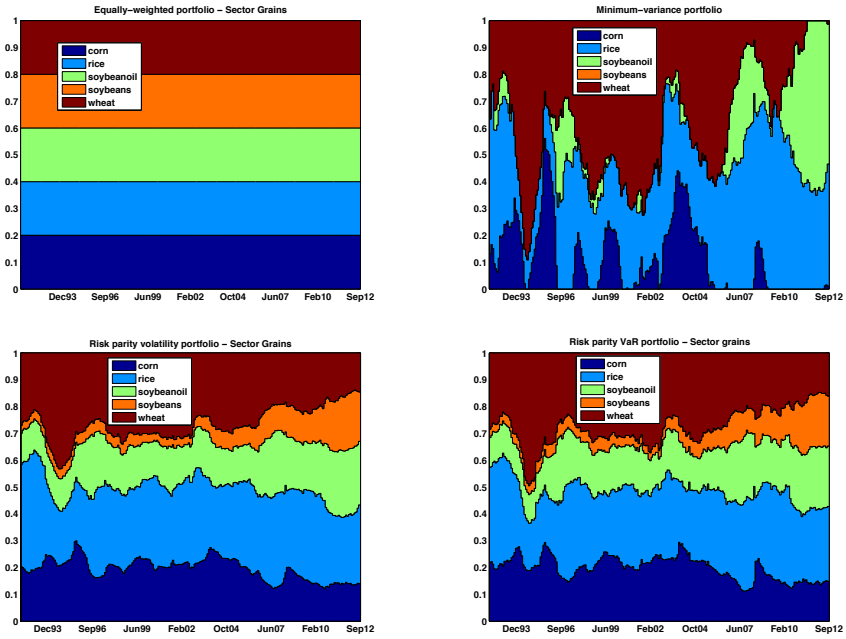


Figure 5.2. Evolution of Portfolio Weights for the Sector Grains & Seeds. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

The risk parity portfolios have rather stable risk contributions for the VaR risk measure (see Figure 5.3), especially when we compare their contributions to the VaR with the contribution of the equally weighted and min-variance portfolios. Whereas the min-variance portfolio shows very erratic contributions to total risk, we observe a highly risky trend in the evolution of the risk contribution for the portfolio with naive $1/n$ diversification. Finally, the VaR contributions of the two last portfolios are almost the same, even if the portfolio VaR and portfolio volatility differ significantly.

5.5.3. Portfolio management with systematic risk contribution restrictions

Let us now consider the constrained optimization problem introduced in section 5.3.3 with a control for turnover:

$$\begin{aligned}
 \min_{w_t} & VaR_t(w_t) + \delta [(1 - \pi)VaR_{s,t}(w_t) - \pi VaR_{u,t}(w_t)]^2 \\
 & + \lambda \sum_{j=1}^n w_{t-1,j}^* \ln \left(\frac{w_{t-1,j}^*}{w_{t,j}} \right), \\
 \text{s.t.} & w_t' e = 1, w_{it} \geq 0, i = 1, \dots, n,
 \end{aligned} \tag{5.21}$$

which corresponds to a mix between the minimization of the total VaR, the constraint on the risk contribution for systematic risk and the turnover. The risk measure is the VaR at 5%, and the systematic component is driven by a single factor chosen equal to the DJUBS index return.

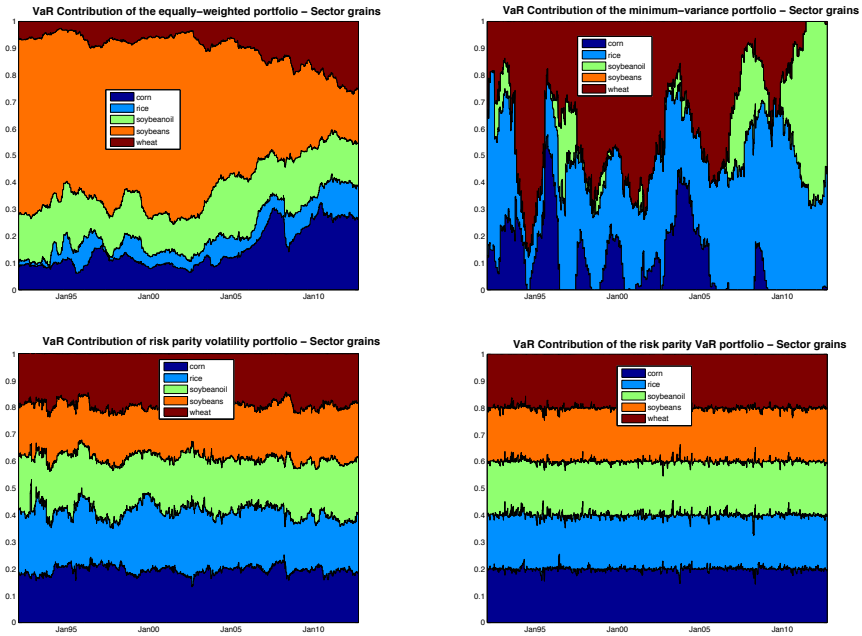


Figure 5.3. Evolution of VaR Contributions in the Sector Grains & Seeds. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

The optimal allocation depends on control parameters δ , π and λ :

- δ is a smoothing parameter: we get the min-VaR portfolio when $\delta = 0$, and the min-VaR portfolio with strict restriction on the systematic risk contribution when $\delta \rightarrow \infty$;

- the benchmark systematic risk contribution π takes values in $(0,1)$. When the factor is a market index, π measures the degree of market neutrality of the portfolio for extreme risks. When $\pi = 0$, we are looking for a portfolio with no market influence on extreme risks;

- when used, the control for turnover takes two different values: $\lambda = 0.01$ and $\lambda = 1$.

We provide in Figures 5.4–5.5 the dynamic evolution of the weights for the portfolios obtained with equation [5.21] for different values of the couple (π, δ) and in the two cases $\lambda = 0.01$ and $\lambda = 1$.

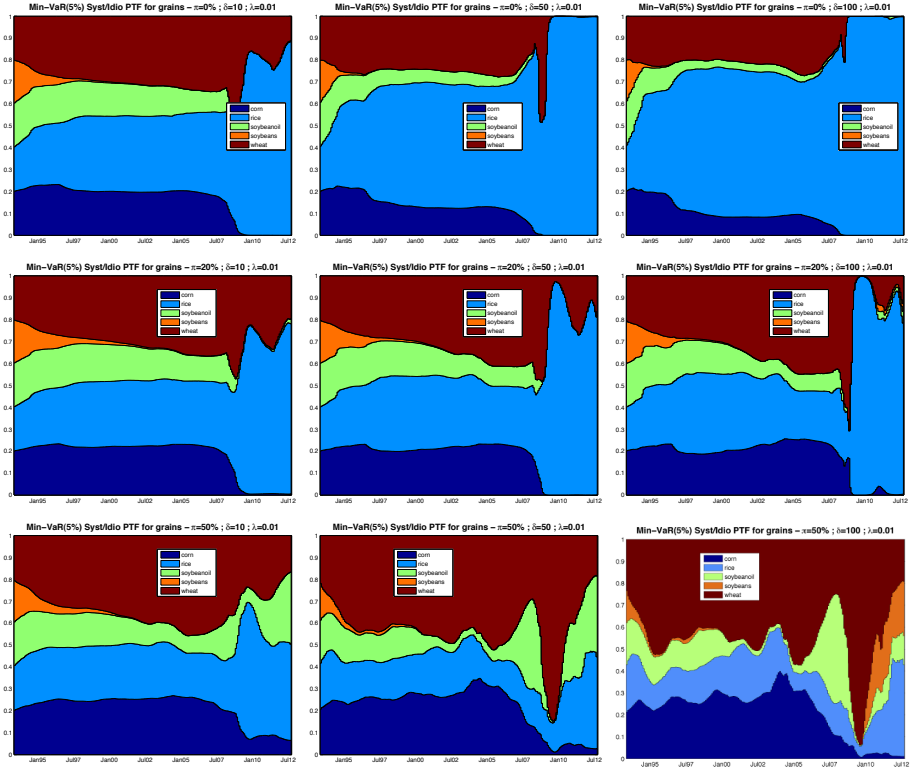


Figure 5.4. Allocations of the 5%-VaR portfolio in sector Grains & Seeds including the transaction costs balanced with a quite low parameter $\lambda = 0.01$ and for different couples of parameters (π, δ) . Each row is for a fixed value of $\pi \in \{0; 0.2; 0.5\}$, and each column is for a fixed value of $\delta \in \{10; 50; 100\}$. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

For the small value of $\lambda = 0.01$, it seems quite possible to control the market neutrality of the final portfolio by changing the values of π and δ . The higher π and δ values are, the higher the allocations of the assets with a higher β (this is the case for rice and wheat in Figure 5.4). However, when the manager focuses on reducing the turnover of its portfolio (e.g. when $\lambda = 1$), then it seems very difficult to balance this condition whatever are the values of π and δ . Note also that a portfolio management which controls for extreme risk does not necessarily imply a “diversification” in terms of portfolio allocation. It may be less risky to allocate the budget in a small number of

assets. This phenomena is clearly seen on the top right panels of Figure 5.4 at the end of the period. Table 5.3 reports the decomposition of total risk for a min-VaR (5%) portfolio with a turnover control parameter $\lambda = 0.01$. We observe that, if rice is the worst asset in terms of total risk contribution, its contribution to the systematic risk is the lowest one.

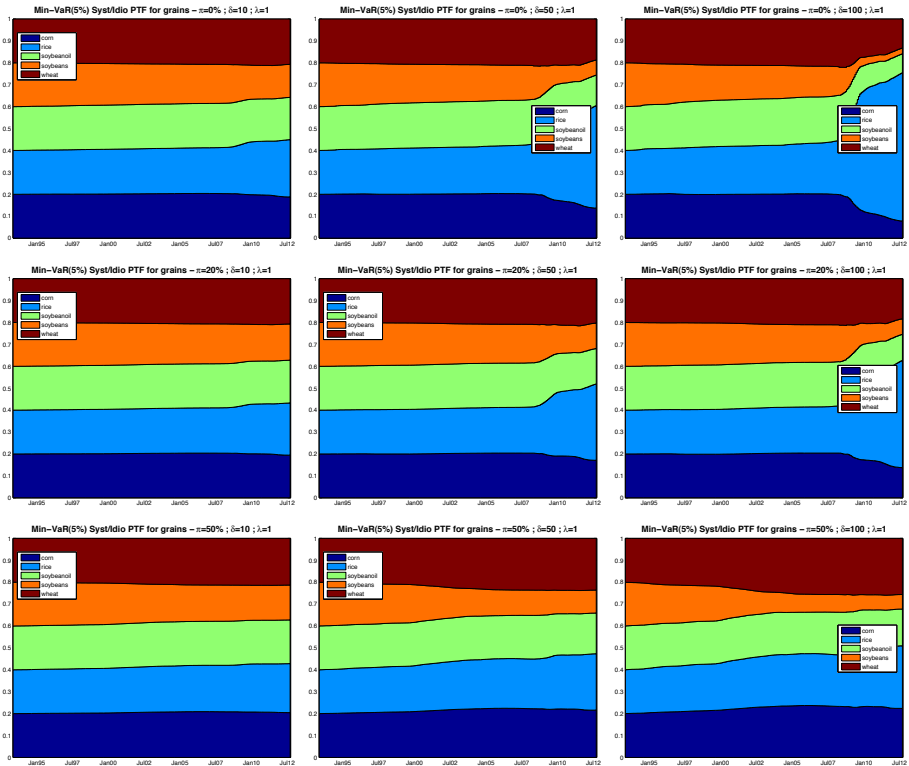


Figure 5.5. Allocations of the 5%-VaR portfolio in sector Grains & Seeds including the transaction costs balanced with a quite high parameter $\lambda = 1$ and for different couples of parameters (π, δ) . Each row is for a fixed value of $\pi \in \{0; 0.2; 0.5\}$, and each column is for a fixed value of $\delta \in \{10; 50; 100\}$. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

When the solicited level of systematic risk contribution is high (Figure 5.5, $\lambda = 1$), the tradeoff between the control on the systematic risk contributions and the turnover is clearly in favor of this latter and we get then stuck with the initial portfolio, which is, in the case of Figure 5.5, set as being the equally-weighted portfolio. This example

shows that the calibration of the different control parameters is essential and yields to quite different portfolio profiles.

<i>Assets</i>	<i>Beta</i>	<i>Risk parity VaR weight (%)</i>	<i>Systematic factor (%)</i>	<i>Idiosyncratic error (%)</i>	<i>Total (%)</i>
<i>Corn</i>	1.09	11.9	9.3	6.9	16.2
<i>Rice</i>	0.35	34.4	8.7	18.2	26.8
<i>Soybeanoil</i>	0.76	24.4	13.2	8.8	22.1
<i>Soybeans</i>	0.87	16.3	10.2	8	18.2
<i>Wheat</i>	1.12	13	10.4	6.3	16.7
<i>Total</i>		100	51.8	48.2	100

Table 5.3. *Decomposition of total risk for a min-VaR (5%) portfolio with a turnover control parameter $\lambda = 0.01$ in the sector Grains and seeds as of 24 September 2012*

5.6. Concluding remarks

We have introduced in this chapter a unified optimization framework for asset allocation, which provides a mix between risk minimization, weakened risk contribution restrictions and turnover. These allocation techniques include the most well-known allocation procedures, such as the mean-variance and the minimum-variance allocation as well as the equally weighted and risk parity portfolios. There exist at least four reasons for considering such a mix focusing on the systematic component of the risk. The first reason is to account for transaction costs, when looking for the portfolio adjustment. In this respect, the introduction of constraints on the risk contributions can have such an interpretation. The second reason is to account for the regulation for financial stability, that is, for the introduction of constraints on the budgets allocated to the different types of assets, according to their individual risk, but also to the capital required for systematic risk, which is based on the risk contribution. This justifies a restriction written on the systematic component of the portfolio. The third reason is the possibility to manage the degree of market neutrality of the portfolio. Finally, the standard mean-variance approach applied to a large number of assets is very sensitive to small changes in the inputs, especially to the estimate of the volatility–covolatility matrix of asset returns. The introduction of budget and/or risk contributions, on either asset classes or types of risks (systematic vs unsystematic), will robustify such an approach as well as the accounting for turnover.

However, if such a mix is needed, there is no general method for selecting an optimal mix, which might depend not only on the preference of the investor, but also on the liquidity features and on the potential regulation. In this framework, the best approach consists of considering different mix, to apply them empirically for portfolio

allocation and compare the properties of the associated portfolios in terms of stability over time of budget allocations, risk contributions and performances.

Our approach is easily extended to other type and number of factors. At the limit, these factors on which the risk budgeting constraints are written might be at the disposal of the portfolio manager and be selected to create oriented portfolio managements (see e.g. [MEC 07]).

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Risk-Based Investing but What Risk(s)?

In this chapter, we propose a generalized risk-based investing framework, which allows us to deal in a simple and flexible way with various risks beyond volatility and correlation, namely valuation, asymmetry, tail and illiquidity risks. We empirically illustrate the methodology by proposing a risk-based strategic allocation for a multi-asset portfolio made of bonds, equities, commodities, real estate, hedge funds and private equity over the period 1990–2013.

6.1. Introduction

Risk-based investment strategies, such as minimum variance, maximum diversification and risk parity, have attracted considerable attention from the investor and academia community since the advent of the 2008 crisis. In most applications, risk is measured through volatility, i.e. the standard deviation of a portfolio's returns. To many investors, associating risk with volatility is highly disputable.

Asness [ASN 14] illustrates this debate through the opposition between two schools, with the “quant/geeks” using volatility as preferred to risk measure, while some other investors argue valuation is the only true risk¹. To some extent, this opposition is coming from differences in investment horizons, and in the short term a valuation approach can itself lead to huge losses – for instance, when assets are entering into a “deep-value” phase. The story from Asness [ASN 14] ends up on a

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¹ As a recent example, Montier [MON 14] writes: “Putting volatility at the heart of your investment approach seems very odd to me as, for example, it would have had you increasing exposure in 2007 as volatility was low, and decreasing exposure in 2009 since volatility was high – the exact opposite of the value-driven approach”.

more positive note by emphasizing that both approaches can be reconciled noting that quants can incorporate valuation risks in their models through a positive (negative) expected return hypothesis for the cheap (expensive) assets. More globally, risk is a multi-facet concept as shown by many examples. Corporate bonds are characterized by an intrinsic asymmetry of risk associated with default risk. Some prominent alternative assets such as private equity, real estate and distressed securities are affected by structural liquidity risks.

The purpose of our chapter is to propose a simple and flexible analytical solution to deal with this diversity of risks. While not pretending to be exhaustive, we develop a risk-based asset allocation framework, based on variations around expected shortfall (ES) measures, where volatility, correlation, valuation, asymmetry, tail and illiquidity risks can be fairly easily incorporated. We provide an illustration of the methodology with a strategic asset allocation problem involving traditional and alternative assets. To keep the text as readable as possible, most mathematical details are relegated to the mathematical appendix, section 6.6. The data appendix, section 6.7, describes the construction of the dataset used in the empirical application.

6.2. Expected shortfall as risk measure

6.2.1. *Expected shortfall: definition and properties*

Our modeling framework is built around ES, also known as conditional value at risk or expected tail loss. For a given confidence level of $(1 - \alpha)$, ES corresponds to the expected loss of the portfolio during the worst α -proportion of times for a predefined investment horizon. In practice, the risk level α is typically chosen as being low, e.g. $\alpha = 5\%$. This makes ES a more sensible risk measure than volatility as it does concentrate on the left-tail of the return distributions corresponding to the largest losses for the investors. Furthermore, ES is known to have properties superior to other tail loss estimators such as value-at-risk (VaR). In particular, contrary to VaR, ES is a subadditive risk measure [ART 99]. This means it is fulfilling the usual diversification property which merely says that a portfolio can never be more risky than the sum of risk of its individual components, which is not guaranteed for VaR. Mathematically, this comes from the fact that ES is convex in portfolio's weights, a property of utmost importance to obtain a risk-based allocation solution, as we discuss below.

To formally compute ES, let us consider a portfolio $\mathbf{w} = (w_1, \dots, w_n)$ made up of n assets. Let μ_i and σ_i be the expected return and volatility of the i -th asset's returns, R_i . The associated portfolio returns are then given by $R_p = \sum_{i=1}^n w_i R_i$, with mean $\mu_p = \sum_{i=1}^n w_i \mu_i$ and volatility $\sigma_p = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}}$, where σ_{ij} denotes the

covariance between the returns of assets i and j . It is convenient to re-express the portfolio return through a location-scale representation

$$R_p = \mu_p + \sigma_p Z, \quad [6.1]$$

where $Z = (R_p - \mu_p) / \sigma_p$ is a zero-mean unit-variance random variable, with distribution function $G(\cdot)$. The ES of the portfolio returns then equals:

$$\text{ES}_\alpha(\mathbf{w}) = -\mu_p - \sigma_p E[Z | Z \leq G^{-1}(\alpha)], \quad [6.2]$$

with

$$E[Z | Z \leq G^{-1}(\alpha)] = \frac{1}{\alpha} \int_0^\alpha G^{-1}(s) ds,$$

where $G^{-1}(\cdot)$ is the quantile function associated with the portfolio's return distribution. Because $E[Z | Z \leq G^{-1}(\alpha)]$ is a negative number in general for low α , ES is expected to be a positive term conforming to the usual convention. The higher the ES, the higher the loss. Higher expected return μ_p reduces ES. Higher volatility σ_p increases ES.

The measurement of [6.2] requires us to estimate the quantities $G^{-1}(\alpha)$ and $E[Z | Z \leq G^{-1}(\alpha)]$.

This can be done empirically from the time-series of innovations Z , either non-parametrically or parametrically by postulating a specific distribution. The simplest choice is to assume that Z follows a standard normal distribution, in which case [6.2] becomes:

$$\text{ES}_\alpha^N(\mathbf{w}) = -\mu_p + \sigma_p \lambda_\alpha^N \quad [6.3]$$

with

$$\lambda_\alpha^N = \frac{\phi(z_\alpha)}{\alpha}, \quad [6.4]$$

where $\phi(\cdot)$, $\Phi(\cdot)$ and $z_\alpha = \Phi^{-1}(\alpha)$ are the standard Gaussian density, cumulative distribution and quantile functions, respectively. λ_α^N is a constant depending only on α ; it is higher for lower α , i.e. larger losses. For instance, we have $\lambda_{0.01}^N = 2.6652$, $\lambda_{0.05}^N = 2.0627$ or $\lambda_{0.10}^N = 1.7550$. While simple, the approach might be insufficient as normality has well-known limitations to describe financial return distributions. Later in the chapter, we discuss the Gaussian hypothesis.

6.2.2. Contribution to expected shortfall and risk-based portfolios

Risk-budgeting principle is based on the decomposition of a risk measure in contributions coming from the different components of a portfolio. For instance, portfolio's volatility can be decomposed as $\sigma_p = \sum_{i=1}^n \text{CVOL}_i$, where $\text{CVOL}_i = w_i \partial \sigma_p / \partial w_i$ is the contribution of the i -th asset to the volatility of the portfolio and $\partial \sigma_p / \partial w_i = \sigma_{ip} / \sigma_p$ is the marginal contribution to volatility with σ_{ip} the covariance of asset i returns with the ones of the portfolio. More generally, the linear risk decomposition can be transposed to any risk measure which is homogenous of degree 1 in the weights, since in this case we can apply Euler's decomposition [TAS 04]. ES is an example of a linear homogenous risk measure, with $\text{ES}_\alpha(\mathbf{w}) = \sum_{i=1}^n \text{CES}_{\alpha(i)}$, where $\text{CES}_{\alpha(i)} = w_i \partial \text{ES}_\alpha(\mathbf{w}) / \partial w_i$ is the contribution of asset i to the ES of the portfolio. The percentage contributions are given by $\% \text{CES}_{\alpha(i)} = \text{CES}_{\alpha(i)} / \text{ES}_\alpha(\mathbf{w})$, with $\sum_{i=1}^n \% \text{CES}_{\alpha(i)} = 1$.

Risk-based investing is an application of risk-budgeting principles, which consists of determining portfolio weights such that risk contributions match predefined risk-budgeting policy. If we consider ES as the risk measure, then following Bruder and Roncalli [BRU 12], any specific risk-based portfolio can be numerically obtained as:

$$\begin{aligned} \mathbf{w}^* &= \arg \min \sum_{i=1}^n (\% \text{CES}_{\alpha(i)} - b_i)^2, \\ \text{s.t. } &\begin{cases} \sum_i^{\mathbf{w}} b_i = 1 \\ \sum_i w_i = 1 \end{cases} \quad \forall i, \end{aligned} \tag{6.5}$$

where b_i corresponds to some arbitrary set of risk budgets. A simple example is the risk parity strategy where we give the same risk budget to each asset in the portfolio, i.e. $b_i = n^{-1}$ [MAI 10]; a key difference being that risk is here measured by ES and not by volatility only. Other risk-budgeting policies can be considered as well. For instance, the minimum expected shortfall portfolio is such that the risk budgets are equal to capital weights, i.e. $b_i = w_i$.

Two additional remarks can be made. First, for long-only portfolios, program [6.5] leads to a unique solution due to the convex property of ES [BER 04, RON 13]. Second, the "traditional" risk-based portfolio solutions are obtained as special cases of [6.5] where all assets have zero expected returns and the distribution of return innovations is Gaussian². Departing from these hypotheses is what we propose in the next section.

² From Gaussian ES specification [6.3], we immediately deduce that $\text{CES}_{\alpha(i)}^N = -w_i \mu_i + \lambda_\alpha^N \text{CVOL}_i$. It is straightforward to see that total ES contributions $\text{CES}_{\alpha(i)}^N$ become equivalent to total risk contributions to volatility CVOL_i when $\mu_i = 0 \forall i$.

6.3. Broadening risk measures

We now review how to incorporate different types of risks within the ES framework.

6.3.1. *Volatility and correlation risks*

The usual risk-based investment strategies determine their capital allocations such that all components of the portfolio contribute to the same extent to its volatility. Contributions to volatility are both dependent on individual volatilities and correlation between assets.

In general terms, individual volatilities mainly measure the amplitude of returns; the higher the volatility, the higher the range of potential returns. A common criticism is that this measure is symmetric and hence does not differentiate between gains and losses, while for investment purposes, the left-tail of the distribution (losses) is probably more critical than the right-tail (gains) from a risk perspective. Hence, it is often recommended to look at downside risk measures concentrated on losses, and in particular on extreme losses. As the purpose of this chapter is to claim that risk-based portfolios should be built on broader risk measures, we cannot totally disagree. This being said, the criticism on volatility is a bit exaggerated. First, volatility is potentially a more robust parameter as its estimation is based on the full portfolio return distribution, and hence on larger samples. Second, volatility remains a key component of extreme losses measure as clearly shown by equation [6.2] in the case of ES.

Correlation between assets is often disregarded in risk-based solutions [CHA 11]. This has the advantage of solving the issue of estimation of these parameters but has disadvantages as well. As an example, ignoring correlation renders risk parity portfolios totally exposed to the duplication risk [CHO 13]. Applied to typical multi-asset applications, ignoring correlation might turn into the same mistake that was made some years ago by allocating to alternative assets as a way to diversify the portfolio, while ignoring the common economic risk factors that affect both traditional and alternative assets.

6.3.2. *Valuation risk*

Valuation is one of the major risks mentioned by investors. A natural way to include valuation risk within the ES framework is to consider that the expected returns μ_i are reflecting valuation misalignment. More specifically, we suggest using carry as a measure of expected returns.

Carry corresponds to the expected return of an asset if its price does not move in the future [KOI 13]. As first-order approximation, carry will typically be measured through bond or earning yields. More details are given in the data appendix (section 6.7). Carry is intrinsically related to valuation since it is obtained as the ratio between the expected income associated with the holding of an asset and its current price. For the same income stream, carry is larger (smaller) when asset prices are lower (higher). A more general valuation model would imply to specifically measure the difference between the current price and the fundamental value of the asset, but the latter is not observable and is thus highly subjective. We do believe that the model-free dimension of carry make it a more proper candidate for measuring valuation in a risk-based context. All in all, our approach leads to the following mechanism: lower (higher) carry implies lower (higher) expected returns, which translates into higher (lower) ES through [6.2].

A related discussion is pursued by Roncalli [RON 14] who suggests using a Gaussian ES to incorporate tactical views in a risk-budgeting framework. The author draws a parallel with traditional Markowitz allocation problem, which does consist of the minimization of the quantity $-\mu_p + \frac{\gamma}{2} \sigma_p^2$, where γ is the risk-aversion parameter. Comparing this expression by [6.3], we deduce the implied risk-aversion parameter in ES as being equal to $\gamma^{\text{imp}} = 2 (\lambda_\alpha^N / \sigma_p)$. It is interesting to note that the incorporation of expected returns into the ES is made at very specific conditions, and notably at very high level of risk aversion. For instance, if $\alpha = 0.05$ and $\sigma_p = 10\%$, we have $\gamma^{\text{imp}} = 41.2542$, which is extremely high as typical risk-aversion estimates are below 10 [MEH 85]. Hence, even if we allow the incorporation of expected returns in a risk-based framework, the program remains fairly different from Markowitz's one in practice³.

6.3.3. *Asymmetry and tail risks*

It is widely acknowledged that financial return distributions frequently deviate from a Gaussian distribution, as they present both asymmetrical and fat-tailness characteristics. Asymmetry of risk describes a situation where potential gains and losses are uneven. This risk is typical of asset classes such as credit, where issuers are subject to default risk, implying a fundamentally negatively-skewed distribution of returns, where the buy-and-hold bondholder can gain as a maximum the bond yield to maturity (say, 5%) but can lose all its investment if the issuer goes bankrupt and the recovery rate is zero. Tail risk expresses the fact that extreme events occur more frequently than expected from a normal distribution, leading to a higher probability of big losses or big gains. Different methodologies are available to model non-Gaussian distributions.

³ See [JUR 15b] for an analytical framework allowing the combination of active views with a risk-based portfolio.

Following Zangari [ZAN 96], Cornish–Fisher (CF) approximations have gained popularity due to their convenience and flexibility. The advantage of this approach in the present context is to offer the opportunity of obtaining closed-form formulas for the risk contributions, while alternative approaches would necessitate simulations at each iteration used to solve the program [6.5]⁴. Under a second-order CF approximation, with $s_p^2 \approx 0$, ES is equal to (see the mathematical appendix, section 6.6)⁵:

$$\text{ES}_\alpha^{CF}(\mathbf{w}) = -\mu_p + \lambda_\alpha^N (1 + \lambda_\alpha^{CF}) \sigma_p, \quad [6.6]$$

where $\lambda_\alpha^{CF} = [\frac{1}{6}z_\alpha s_p + \frac{1}{24}(z_\alpha^2 - 1)(k_p - 3)]$ is the adjustment to ES for the non-Gaussian features of the distribution of returns; and s_p and k_p are the skewness and kurtosis of the portfolio, respectively. For Gaussian returns, we have $s_p = 0$, $k_p = 3$, $\lambda_\alpha^{CF} = 0$, implying that we are back to [6.3]. Non-Gaussian distributions lead ES to be impacted by skewness and kurtosis⁶. $\text{ES}_\alpha^{CF}(\mathbf{w})$ is linearly decreasing in skewness s_p and linearly increasing in kurtosis k_p . These are desirable features because they imply that an asset with negative skewness or high kurtosis will tend to receive a lower weight in a risk-based portfolio based on ES, all other things being equal.

To determine the exact risk-based portfolio composition, the next step is to get individual ES contributions as inputs to program [6.5], which, under a CF hypothesis, can be shown to be equal to (see Appendix 1)⁷:

$$\text{CES}_{\alpha(i)}^{CF} = \text{CMEAN}_i + \text{CVOL}_i + \text{CSKEW}_i + \text{CKURT}_i, \quad [6.7]$$

where CMEAN_i , CVOL_i , CSKEW_i and CKURT_i are asset i mean, volatility, skewness and kurtosis contributions to portfolio ES. Hence, the contributions of each asset to ES can be broken down into its distributional characteristic risk contributions. As we will illustrate in the empirical section, this decomposition facilitates the identification of the various sources of risk at the level of individual

4 See [BRO 11] for an extensive study of ES under different distributional characteristics. While in many cases, closed-form formulas can be obtained for ES itself, this is not the case for the individual contributions to risk $\text{CES}_{\alpha(i)}$. Technically, this is due to the fact that the quantile of the distribution $G^{-1}(\alpha)$ is a complex function of portfolio's weights. The absence of closed-form formulas renders the optimization of the program [6.5] cumbersome. Our approach is much more efficient in that perspective.

5 See [CHR 05] for a similar semi-parametric specification of the ES in which the term in s_p^2 is ignored.

6 In practice, we need to consider that under a CF approximation, the set of skewness-kurtosis pairs must be restricted in order to yield monotonic quantile functions, with the skewness varying between -3 and 3 and the excess kurtosis ranging between 0 and 8 [MAI 14].

7 A related formula is proposed by Boudt *et al.* [BOU 08]. Formula [6.7] has the advantage of showing a clear decomposition of contributions to ES as a sum of contributions to the different sources of risks. Furthermore, the expression is more tractable, making the optimization program [6.5] easier to solve.

assets, as well as at an aggregated portfolio level as summing over all ES contributions [6.7] gives back the total ES risk itself, i.e. $\sum_i \text{CES}_{\alpha(i)}^{CF} = \text{ES}_{\alpha}^{CF}(\mathbf{w})$.

The calculation of [6.7] necessitates computing all comoment matrices up to an order of four. In practice, this can lead to a very large number of parameters. To cope with this issue, we introduce in the mathematical appendix (section 6.6) a linear factor model where the dependence between assets is modeled through a predefined set of Gaussian factors, while the individual return innovations are potentially non-Gaussian. This leads to a huge reduction in the number of parameters to be estimated.

6.3.4. Illiquidity risk

Illiquidity induces that transactions cannot be observed frequently, leading to some form of stale pricing and smoothing return patterns. A common way to correct for the associated biases in empirical risk measures is to unsmooth the time-series by assuming that current returns form a moving-average of true returns; see, among others, [GEL 93, ASN 01, GET 04, CAO 07]. In Appendix 1, we expand these results to higher moments and comoments of the distribution. In particular, we give expressions to correct covariance, coskewness and cokurtosis empirical estimates and we derive a new expression for individual contributions to ES isolating illiquidity risk, as follows:

$$\text{CES}_{\alpha(i)}^{CF} = \text{CES}_{\alpha(i)}^o + \text{ILLIQUID}_i, \quad [6.8]$$

where $\text{CES}_{\alpha(i)}^{CF}$ and $\text{CES}_{\alpha(i)}^o$ are defined as in equation [6.7] and obtained through the application of this formula to illiquidity-corrected and raw moments, respectively. ILLIQUID_i corresponds to the contribution of asset i to total ES coming from the impact of illiquidity on the various moments of the return distribution, with $\text{ILLIQUID}_i = 0$ if the asset is liquid.

6.4. Empirical results

In this section, we empirically illustrate the ES decomposition methodology by determining a risk-based strategic allocation for a multi-asset portfolio spanning a range of traditional and alternative assets.

6.4.1. Data overview and preliminary analysis

Our database is made up of the quarterly returns and carry indicators for eight traditional and alternative asset classes over the period spanning from the first quarter of 1990 to the third quarter of 2013. Table 6.1 gives the list of assets and details sources and carry definitions (more details are available in the data appendix, section 6.7).

Code	Asset class (index)	Source	Carry definition
EQ	Equities Large Cap (S&P 500)	Bloomberg	Earning yield plus inflation expect
SC	Equities Small Cap (Russell 2000)	Bloomberg	Earning yield plus inflation expect
IG	U.S. Investment Grade (Barclays)	Barclays' Website	Yield to worst plus roll-down
HY	U.S. Corp. High Yield (Barclays)	Barclays' Website	Yield to worst plus roll-down
CO	Commodities (basket of 14 comm.)	Bloomberg	1-year roll-yield
HF	Hedge Funds (HFR)	Bloomberg	Regression-based implied carry
PE	Private Equity (Cambridge)	Cambridge's Website	Regression-based implied carry
RE	Real Estate (NCREIF)	NCREIF's Website	Rental income returns

Table 6.1. *Data description*

Table 6.2 displays descriptive statistics for the asset returns. The average (arithmetic) returns spread from 1.8% per quarter (or 7.2% per year) for investment grade bonds to 3.6% per quarter (or 14.4% per year) for private equity. All assets have delivered substantial premia over the risk-free rate which was roughly 1% per quarter over the same time period. Volatilities range from roughly 2.5% per quarter for investment grade bonds and real estate to more than 10% per quarter for small cap equities. Associated Sharpe ratios gravitate around 0.25 for most assets with the noticeable difference of private equity and real estate. However, as we will see later, the latter asset classes are characterized by substantial illiquidity that downward bias the volatility measure and leads to overestimate the Sharpe ratio. Deviations from normality are substantial and frequently significant as shown by Jarque–Bera statistics. Most asset classes are indeed characterized by a negative skewness and an excess kurtosis of the return distribution⁸. Incorporating this dimension of the asset returns is critical when it comes to risk-based portfolio construction.

	EQ	SC	IG	HY	CO	HF	PE	RE
Average return	2.53%	2.90%	1.80%	2.30%	2.49%	1.84%	3.61%	1.83%
Volatility	8.15%	10.61%	2.73%	5.24%	9.22%	3.63%	5.14%	2.44%
Sharpe ratio	0.19	0.18	0.29	0.25	0.16	0.23	0.51	0.34
Skewness	-0.55**	-0.43**	-0.11	0.29	0.03	-0.76***	-0.50**	-1.94***
Excess Kurtosis	0.48	0.62	1.45**	5.34***	3.48***	3.07***	2.00***	4.95***
Jarque–Bera	5.70	4.45	8.53**	114.08***	47.90***	46.48**	19.77***	156.82***

Notes. *, ** and *** indicate significance at the 10, 5 and 1% level, respectively, (two-tail tests for skewness and kurtosis). Sharpe ratios are not annualized and assume a 1% per quarter risk-free rate assumption.

Table 6.2. *Descriptive statistics of quarterly returns, 1990Q1–2013Q3*

⁸ The sole exceptions to this deviation from normality are for public equities (large cap and small cap), which can be deemed as a surprising result. However, the frequency of data seems to play an important role here. Indeed, normality assumption is strongly rejected at a higher frequency. This is mainly explained by a reduction in kurtosis at lower frequency.

Table 6.3 displays the average carry (per quarter) and average correlation with other assets. We refer the reader to the data appendix (section 6.7) for more details on carry calculations. Carry estimates vary from almost zero for commodities to more than 2.5% per quarter for high yield. For some assets such as fixed income, or even real estate, the carry constitutes the bulk of the return in the long run. At the opposite, for commodities or hedge funds, the portion due to carry is very low. Equity (public and private) is intermediary between these extremes. As far as correlation is concerned, we can isolate two groups of assets. On the one hand, public and private equities, and high yield bonds, all have average correlation to the rest of the universe of assets which are roughly 0.4. On the other hand, investment grade bonds, commodities or real estate appear as the most diversifying assets as their correlation with the rest of the universe is close to 0.

EQ	SC	IG	HY	CO	HF	PE	RE
Panel A: average carry (per quarter)							
2.11%	1.49%	1.66%	2.59%	0.10%	0.36%	1.81%	2.00%
Panel B: average correlation							
0.47	0.44	0.08	0.39	0.15	0.41	0.44	0.07

Notes. See Appendix 2 for more details on carry estimates. Average correlation is based on the full sample matrix.

Table 6.3. Average carry and correlation

Table 6.4 summarizes the estimations performed to correct the asset comoments for illiquidity. We follow the steps described in section C of the mathematical appendix (section 6.6). We first estimate moving-average models and then infer corrections for the different comoment matrices. In Panel A, we first report Bayesian information criterion (BIC) for the different moving-average models. We highlight in bold the lag k that we retain for each asset as it leads to minimize the BIC statistic. With the exception of high yield, traditional assets and commodities are all characterized by a zero-lag, meaning they are deemed liquid through this criterion. The exception for high yield is not surprising as it has the reputation of offering one of the poorest liquidity among fixed income buckets. Alternative assets are also characterized by estimated poor liquidity, particularly for private equity and even more for real estate. These results are not surprising due to the characteristics of these privately-exchanged assets, but our models provide a quantification which is based on a consistent methodology across all assets. Panel B displays the associated smoothing coefficients, while in Panel C we report the implied scaling coefficients for the moments⁹. Cases where $\theta_{i,0} = 1$ indicate situations where there is no estimated illiquidity-bias and no need for scaling up the moments. At the opposite, cases where $\theta_{i,0} < 1$ indicate cases where some illiquidity bias is estimated.

⁹ Comoments will also be corrected through appropriate products of the smoothing coefficients of the involved assets. See Appendix C of Appendix 1 for more details.

Correcting for this leads to inflate moments. For instance, the volatility of real estate is estimated to be understated by a factor of more than 2 due to the illiquid nature of the asset. This notably implies that the impressive Sharpe ratios of Table 6.2 for the less liquid assets will be questioned. Corrections for skewness and kurtosis are also significant for some other assets.

	EQ	SC	IG	HY	CO	HF	PE	RE
Panel A: BIC criterion for MA(k)								
$k = 0$	-207.76	-157.59	-415.86	-291.85	-184.29	-361.57	-295.29	-436.73
$k = 1$	-203.47	-153.99	-412.13	-301.97	-180.19	-362.98	-301.13	-480.00
$k = 2$	-199.04	-150.54	-409.60	-301.49	-175.81	-359.01	-302.40	-519.48
$k = 3$	-194.49	-146.91	-405.07	-297.04	-172.92	-354.91	-298.55	-515.39
$k = 4$	-190.05	-143.03	-404.37	-293.65	-168.66	-350.58	-295.47	-533.61
Panel B: smoothing coefficients								
$\theta_{i,0}$	1.000	1.000	1.000	0.658	1.000	0.808	0.650	0.305
$\theta_{i,1}$	0.000	0.000	0.000	0.342	0.000	0.192	0.205	0.204
$\theta_{i,2}$	0.000	0.000	0.000	0.000	0.000	0.000	0.145	0.203
$\theta_{i,3}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.131
$\theta_{i,4}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.157
Panel C: implied scaling coefficients for moments								
Volatility	1.000	1.000	1.000	1.348	1.000	1.204	1.435	2.143
Skewness	1.000	1.000	1.000	1.255	1.000	1.072	1.182	1.976
Kurtosis	1.000	1.000	1.000	1.502	1.000	1.113	1.304	3.641

Table 6.4. *Illiquidity corrections*

To shed additional light on these issues, we report in Table 6.5 different risk-adjusted performance measures. On top of the traditional Sharpe ratios, we report illiquidity-corrected Sharpe ratio where volatility is incorporating the illiquidity correction displayed in Table 6.4. We also report so-called conditional Sharpe ratios, which are obtained as the ratio between asset excess returns and the 95% ES, where ES is estimated alternatively through three different models of increased completeness: a Gaussian approximation (see equation [6.3]), a skewness-kurtosis expanded version (see equation [6.6]) or the most complete model incorporating expansions for non-normal behavior and illiquidity-bias. The risk measure can have a significant implication on the assets ranks, notably when we consider conditional (ES-based) Sharpe ratios involving the different risk dimensions. Most spectacularly, while real estate is the best asset as far as uncorrected conditional Sharpe ratio is concerned, it becomes the worst when we incorporate correction for non-normality and illiquidity. These differences will also get out in the risk-based allocations as we will see in the next section.

Measure	Corrections		Assets							
	Skew-Kurt	Illiquidity	EQ	SC	IG	HY	CO	HF	PE	RE
SR (Sharpe ratio)	No	No	0.19	0.18	0.29	0.25	0.16	0.23	0.51	0.34
SR	No	Yes	0.19	0.18	0.29	0.18	0.16	0.19	0.35	0.16
Conditional SR	No	No	0.18	0.15	0.47	0.27	0.15	0.33	0.52	0.57
Conditional SR	Yes	No	0.15	0.13	0.39	0.20	0.12	0.21	0.36	0.24
Conditional SR	Yes	Yes	0.15	0.13	0.39	0.11	0.12	0.16	0.20	0.05

Table 6.5. Comparison of risk-adjusted performance measures

6.4.2. Equal risk contribution under different risk models

Let us assume that an investor wants to build a fully invested portfolio where each asset will contribute to the same extent to the global risk of the portfolio. Investors would recognize “risk parity” in this portfolio construction seeking to maximize risk diversification on an *ex-ante* basis. But, risk parity is in general associated with volatility alone and might involve some leverage. Accordingly, let us call “Equal Risk Contribution” [MAI 10] this risk-budgeting policy.

Our purpose is to illustrate how different will be the portfolios when the investor uses different risk models. More specifically, we consider five different models of increasing realism:

- naive risk parity (NRP): risk measure is volatility, ignoring correlation;
- risk parity (RP): risk measure is volatility, incorporating correlation;
- Gaussian expected shortfall (GES): risk measure is ES, assuming a Gaussian distribution;
- modified expected shortfall (MES): risk measure is ES, where ES is modified to incorporate skewness and kurtosis;
- liquidity adjusted modified expected shortfall (LAMES): risk measure is ES, where ES is modified to incorporate skewness and kurtosis, and comoment matrices are corrected for illiquidity.

Associated portfolio weights are given in Panel A of Table 6.6. We observe a substantial variation across models. NRP is naturally allocating more massively to assets with low volatility such as investment grade bonds and real estate, and will give small weights to high-volatility assets such as equities and commodities. RP is introducing correlation and will mitigate the NRP allocation by increasing the allocation to low-correlation assets such as investment grade bonds, commodities and real estate, and will reduce the allocation to the other ones. The next column marks the entry into ES risk measures with first the Gaussian approximation. Relatively to

RP, the key change in GES is to introduce carry, or valuation risk in our framework¹⁰. The allocation is increased to assets presenting a high carry-to-volatility ratio, namely investment grade bonds and real estate. Portfolio weights change significantly when we correct ES for non-normal behavior, with notably a significant drop in real estate weight, an asset characterized by high kurtosis and fairly negative skewness, to the benefit of public equities and investment grade bonds. An interesting result is the fact that the allocation to high yield bonds barely changes as its high kurtosis is compensated by a (modest) positive skewness. Finally, the last column presents the portfolio weights associated with the LAMES risk model, i.e. based on ES incorporating both non-normal and illiquidity adjustments. As an asset characterized by a poor liquidity, real estate sees its portfolio weight reduced further. Private equity is also affected but to a lesser extent as its weight is already small due to its high volatility, high correlation and non-normal features. In Panel B of Table 6.6, we finally report the portfolio metrics. More realistic risk models lead to build portfolios with better carry to volatility profiles. But, the most significant improvements are appearing for higher moments such as skewness and kurtosis, as well as for the modified ES, while normal-Gaussian ES barely changes.

	NRP	RP	GES	MES	LAMES
	Panel A: portfolio weights				
EQ	6.96%	4.55%	2.87%	7.02%	8.67%
SC	5.34%	3.80%	2.29%	7.19%	8.36%
IG	20.81%	27.85%	31.47%	52.10%	52.01%
HY	10.83%	7.89%	6.54%	7.22%	8.13%
CO	6.15%	6.76%	3.85%	3.21%	3.88%
HF	15.64%	11.05%	6.66%	7.58%	8.33%
PE	11.03%	7.44%	5.02%	6.32%	5.39%
RE	23.23%	30.66%	41.30%	9.38%	5.23%
	Panel B: portfolio metrics				
Average carry	1.58%	1.61%	1.73%	1.64%	1.62%
Volatility	3.61%	3.09%	2.88%	3.03%	3.26%
Skewness	-2.64	-3.52	-4.19	-1.00	-0.62
Excess Kurtosis	12.15	17.74	18.69	3.92	2.91
Gaussian expected shortfall	5.87%	4.77%	4.17%	4.61%	5.11%
Modified expected shortfall	16.99%	18.15%	17.97%	7.83%	7.46%

Notes. NRP is naive risk parity, i.e. an equalization of contributions to portfolio volatility ignoring correlation. RP is risk parity, i.e. an equalization of contributions to portfolio volatility taking into consideration correlation. GES is the allocation equalizing ES contributions assuming a normal-Gaussian distribution. MES is the allocation equalizing ES contributions where ES is the modified expected shortfall incorporating skewness and kurtosis. LAMES is the allocation equalizing ES contributions where ES is the modified expected shortfall incorporating skewness and kurtosis, and corrected for illiquidity bias. Portfolio statistics are computed using illiquidity-corrected comoments matrices. Modified expected shortfall is incorporating skewness, kurtosis and illiquidity corrections.

Table 6.6. *Equal risk contribution portfolio: differences across models*

¹⁰ The valuation interpretation of the carry will be even more sensible in dynamic allocation exercises. We leave this to future research.

In Table 6.7, we provide more details on LAMES portfolio. In particular, we give a breakdown of the portfolio ES (7.46%; see Table 6.6) along two axes, splitting contributions by asset class and moments. For instance, the large cap equities are contributing to 1.22% out of the 7.46% ES through the volatility of the portfolio. The last column gives the contribution of each asset to total ES. All contributions are equal as this is the objective of the equal risk contribution portfolio construction. Figure 6.1 reports, for each asset class and the LAMES portfolio, the breakdown of the ES contribution per moment. For liquid asset classes, the largest contributions come from volatility. Most assets are contributing positively through the skewness with the exception of bonds. Alternative asset classes are mainly contributing through illiquidity, with the exception of commodities contributing everywhere to some extent. Finally, we notice that even liquid assets such as investment grade can have an impact through illiquidity, but this is mainly coming from products in comoment matrices with other (illiquid) asset classes.

	Moments attribution					Total
	Mean	Volatility	Skewness	Kurtosis	Illiquidity	
EQ	-0.18%	1.22%	0.15%	-0.21%	-0.05%	0.93%
SC	-0.12%	1.45%	0.00%	-0.33%	-0.05%	0.93%
IG	-0.86%	1.78%	-0.75%	0.39%	0.37%	0.93%
HY	-0.21%	0.72%	-0.07%	0.38%	0.11%	0.93%
CO	0.00%	0.16%	0.29%	0.35%	0.13%	0.93%
HF	-0.03%	0.41%	0.29%	0.10%	0.16%	0.93%
PE	-0.10%	0.36%	0.29%	0.04%	0.34%	0.93%
RE	-0.10%	0.01%	0.21%	0.13%	0.68%	0.93%
Portfolio	-1.62%	6.10%	0.41%	0.86%	1.70%	7.46%

Table 6.7. *LAMES portfolio: contributions to expected shortfall*

6.5. Conclusion

Risk-based investing strategies are experiencing a growing popularity among investors. Yet, in most applications, risk is usually measured through volatility, while other risk characteristics are also important, especially when considering investing in alternative asset classes. In this chapter, we propose a generalized risk-based investing framework based on a semi-parametric ES risk measure that allows investors to deal with various source of risks beyond volatility and correlation, such as valuation, asymmetry, tail and illiquidity risks.

We obtain closed-form formulas for the moment contributions of each asset to the portfolio ES, e.g mean, volatility, skewness and kurtosis risk contributions. We also show how to isolate the individual illiquidity risk contributions through a simple adjustment of the asset comoments. This decomposition facilitates the identification

of the various source of risks both at the individual and portfolio level through the aggregation of the individual risk contributions.

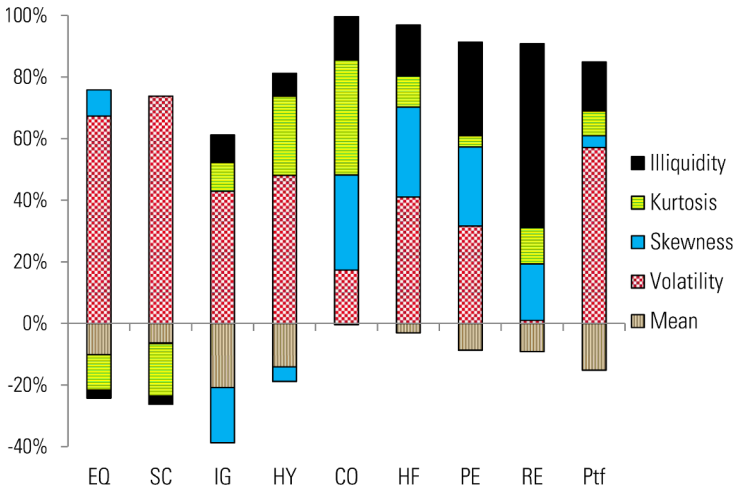


Figure 6.1. *LAMES portfolio: contributions to expected shortfall by asset*

We illustrate the usefulness of our methodology with an in-sample analysis of a typical risk-based strategic allocation across a range of traditional and alternative assets. We show how the non-normality of returns and illiquidity biases can affect significantly the capital allocation. We also find that the breakdown of the individual risk contributions can be very different across assets, with the largest contributions coming from volatility for traditional asset classes, while non-normality and illiquidity are more important for alternative ones.

A natural extension of our research would consist of comparing the out-of-sample performance of our generalized risk-based approach with alternative risk-based or maximum Sharpe ratio portfolios. Another interesting extension would consist of assessing the empirical performance of our robust factor-based moment estimators.

6.6. Mathematical Appendix

In this appendix, we detail the calculations used for computing and estimating ES and contributions to ES. We first recall how to compute the sensitivity of the first four moments of the portfolio's return distribution to portfolio weights (Appendix A). We then apply those results to compute contributions to ES, first under the hypothesis of non-Gaussian distribution (Appendix B), and of illiquidity of assets (Appendix C).

We finally provide structured multifactor estimators of the covariance, coskewness and cokurtosis matrices (Appendix D).

Appendix A: Sensitivity of distribution centered moments to portfolio's weights

Let \mathbf{R}_t and \mathbf{w} be the $(N \times 1)$ vectors of the t -period returns and portfolio's weights for the different assets, respectively. The portfolio returns are denoted by $R_{p,t} = \mathbf{w}^T \mathbf{R}_t$ with mean $\mu_p = E(R_{p,t})$. $m_{(k)} = E[(R_{p,t} - \mu_p)^k]$ corresponds to the portfolio's k -th centered moment. By definition, the first centered moment $m_{(1)}$ is equal to zero. Following the approach of Athayde and Flores [ATH 04] and Jondeau and Rockinger [JON 06], it is convenient to express the centered moments of order 2, 3 and 4 as follows:

$$\begin{aligned} m_{(2)} &= \mathbf{w}^T \boldsymbol{\Omega} \mathbf{w}, \\ m_{(3)} &= \mathbf{w}^T \boldsymbol{\Xi} (\mathbf{w} \otimes \mathbf{w}), \\ m_{(4)} &= \mathbf{w}^T \boldsymbol{\Gamma} (\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w}), \end{aligned} \quad [\text{A.1}]$$

where $\boldsymbol{\Omega} = E[(\mathbf{R}_t - \boldsymbol{\mu})(\mathbf{R}_t - \boldsymbol{\mu})^T]$ is the $(N \times N)$ covariance matrix, with generic term $\sigma_{ij} = E[(R_{i,t} - \mu_i)(R_{j,t} - \mu_j)]$, $\boldsymbol{\Xi} = E[(\mathbf{R}_t - \boldsymbol{\mu})(\mathbf{R}_t - \boldsymbol{\mu})^T \otimes (\mathbf{R}_t - \boldsymbol{\mu})^T]$ is the $(N \times N^2)$ coskewness matrix, with generic term $s_{ijl} = E[(R_{i,t} - \mu_i)(R_{j,t} - \mu_j)(R_{l,t} - \mu_l)]$, $\boldsymbol{\Gamma} = E[(\mathbf{R}_t - \boldsymbol{\mu})(\mathbf{R}_t - \boldsymbol{\mu})^T \otimes (\mathbf{R}_t - \boldsymbol{\mu})^T \otimes (\mathbf{R}_t - \boldsymbol{\mu})^T]$ is the $(N \times N^3)$ cokurtosis matrix, with generic term $k_{ijklm} = E[(R_{i,t} - \mu_i)(R_{j,t} - \mu_j)(R_{l,t} - \mu_l)(R_{m,t} - \mu_m)]$ and \otimes is the Kronecker product symbol.

In what follows, we need to estimate the sensitivity of these moments to changes in portfolio's weights w_i , denoted by $\partial_i m_{(k)}$. From [A.1], we have:

$$\begin{aligned} \partial_i m_{(2)} &= 2 [\boldsymbol{\Omega} \mathbf{w}]_i = 2 \sigma_{ip}, \\ \partial_i m_{(3)} &= 3 [\boldsymbol{\Xi} (\mathbf{w} \otimes \mathbf{w})]_i = 3 s_{ip}, \\ \partial_i m_{(4)} &= 4 [\boldsymbol{\Gamma} (\mathbf{w} \otimes \mathbf{w} \otimes \mathbf{w})]_i = 4 k_{iip}, \end{aligned} \quad [\text{A.2}]$$

where $[\mathbf{A}]_i$ is the i -th row of matrix \mathbf{A} , $\sigma_{ip} = E[(R_{i,t} - \mu_i)(R_{p,t} - \mu_p)]$, $s_{ip} = E[(R_{i,t} - \mu_i)(R_{p,t} - \mu_p)^2]$ and $k_{iip} = E[(R_{i,t} - \mu_i)(R_{p,t} - \mu_p)^3]$ correspond to the covariance, coskewness and cokurtosis between the i -th asset and the portfolio p .

Appendix B: Cornish–Fisher expected shortfall approximation and risk contributions

The intuition of the Cornish–Fisher (CF) expansion is to approximate the quantile function of a standardized non-Gaussian random variable Z by the quantile of a

standard normal variable z_α augmented by terms capturing the non-normal characteristics through the direct introduction of skewness and kurtosis [ZAN 96]. Following Maillard [MAI 14], the CF quantile function can be expressed as:

$$G_{CF}^{-1}(\alpha) = a_0 + a_1 z_\alpha + a_2 z_\alpha^2 + a_3 z_\alpha^3 \quad [\text{B.1}]$$

with

$$\begin{cases} a_0 = -\frac{1}{6}s_p, \\ a_1 = 1 - \frac{1}{8}(k_p - 3), \\ a_2 = \frac{1}{6}s_p, \\ a_3 = \frac{1}{24}(k_p - 3), \end{cases}$$

where $G_{CF}(\cdot)$ is the distribution function under a second-order CF approximation, assuming $s_p^2 = 0$, and $(1 - \alpha)$ is the confidence level. $s_p = m_{(3)}/m_{(2)}^{3/2}$ and $k_p = m_{(4)}/m_{(2)}^2$ are the portfolio's skewness and kurtosis, respectively. The volatility is defined as $\sigma_p = m_{(2)}^{1/2}$. The ES of the standardized return Z can then be obtained by integrating the approximate CF quantile function [B.1]:

$$\begin{aligned} E[Z | Z \leq G_{CF}^{-1}(\alpha)] &= \frac{1}{\alpha} \int_0^\alpha G_{CF}^{-1}(s) ds \quad [\text{B.2}] \\ &= \frac{1}{\alpha} \int_{-\infty}^{z_\alpha} a_0 \phi(z) dz + \frac{1}{\alpha} \int_{-\infty}^{z_\alpha} a_1 z \phi(z) dz \\ &\quad + \frac{1}{\alpha} \int_{-\infty}^{z_\alpha} a_2 z^2 \phi(z) dz + \frac{1}{\alpha} \int_{-\infty}^{z_\alpha} a_3 z^3 \phi(z) dz, \end{aligned}$$

where $G_{CF}^{-1}(\cdot)$ is given in [B.1] and $\phi(\cdot)$ is the standard normal density function. Using the fact that $\int_{-\infty}^{z_\alpha} z^0 \phi(z) dz = \alpha$ and $\int_{-\infty}^{z_\alpha} z^k \phi(z) dz = -z_\alpha^{k-1} \phi(z_\alpha) + (k-1) \int_{-\infty}^{z_\alpha} z^{k-2} \phi(z) dz$, for $k > 0$, we deduce:

$$E_{CF}[Z | Z \leq G_{CF}^{-1}(\alpha)] = -\lambda_\alpha^N \left[1 + \frac{1}{6} z_\alpha s_p + \frac{1}{24} (z_\alpha^2 - 1) (k_p - 3) \right], \quad [\text{B.3}]$$

where $\lambda_\alpha^N = \phi(z_\alpha) / \alpha$. Substituting [B.3] in the ES portfolio formula [6.2] yields:

$$\text{ES}_\alpha^{CF}(\mathbf{w}) = -\mu_p + \lambda_\alpha^N (1 + \lambda_\alpha^{CF}) \sigma_p, \quad [\text{B.4}]$$

where $\lambda_\alpha^{CF} = [\frac{1}{6} z_\alpha s_p + \frac{1}{24} (z_\alpha^2 - 1) (k_p - 3)]$ is the adjustment to ES for the non-Gaussian features of the distribution of returns.

The estimation of a risk-based portfolio necessitates to obtain the individual contributions to ES, $\text{CES}_{\alpha(i)}$. Computing the first derivative of [B.4] relatively to w_i , we get:

$$\begin{aligned} \partial_i \text{ES}_\alpha^{CF}(\mathbf{w}) &= -\mu_i + \lambda_\alpha^N (1 + \lambda_\alpha^{CF}) \partial_i \sigma_p \\ &\quad + \lambda_\alpha^N \sigma_p \left[\frac{1}{6} z_\alpha \partial_i s_p + \frac{1}{24} (z_\alpha^2 - 1) \partial_i k_p \right], \end{aligned} \quad [\text{B.5}]$$

where ∂_i denotes the partial derivative relatively to w_i . From the centered moments derivatives given in Appendix A [A.2], we infer:

$$\begin{aligned}\partial_i \sigma_p &= \frac{\partial_i m_{(2)}}{2 m_{(2)}^{1/2}} = \frac{\sigma_{ip}}{\sigma_p}, \\ \partial_i s_p &= \frac{m_{(2)}^{3/2} \partial_i m_{(3)} - \frac{3}{2} m_{(2)}^{1/2} m_{(3)} \partial_i m_{(2)}}{m_{(2)}^3} = 3 \left(\frac{s_{ip}}{\sigma_p^3} - \frac{\sigma_{ip}}{\sigma_p^2} s_p \right), \\ \partial_i k_p &= \frac{m_{(2)}^2 \partial_i m_{(4)} - 2 m_{(4)} m_{(2)} \partial_i m_{(2)}}{m_{(2)}^4} = 4 \left(\frac{k_{ip}}{\sigma_p^4} - \frac{\sigma_{ip}}{\sigma_p^2} k_p \right),\end{aligned}\tag{B.6}$$

Inserting these values into [B.5] and rearranging terms leads to:

$$\begin{aligned}\partial_i \text{ES}_\alpha^{CF}(\mathbf{w}) &= -\mu_i + \lambda_\alpha^N \frac{\sigma_{ip}}{\sigma_p} + \lambda_\alpha^N \sigma_p \frac{z_\alpha}{6} \left(\partial_i s_p + \partial_i \sigma_p \frac{s_p}{\sigma_p} \right) \\ &\quad + \lambda_\alpha^N \sigma_p \frac{1}{24} (z_\alpha^2 - 1) \left[\partial_i k_p + \partial_i \sigma_p \frac{k_p - 3}{\sigma_p} \right].\end{aligned}\tag{B.7}$$

Premultiplying by portfolio's weights w_i yields the contribution of each asset to the total ES, i.e. $\text{CES}_{\alpha(i)}^{CF}$, which can finally be rewritten as a weighted sum of contributions to the first four moments of the portfolio as follows;

$$\text{CES}_{\alpha(i)}^{CF} = \text{CMEAN}_i + \text{CVOL}_i + \text{CSKEW}_i + \text{CKURT}_i,\tag{B.8}$$

with

$$\left\{ \begin{array}{l} \text{CMEAN}_i = c_1 w_i \mu_i, \quad \sum_{i=1}^n \text{CMEAN}_i = c_1 \mu_p, \\ \text{CVOL}_i = c_2 \frac{w_i \sigma_{ip}}{\sigma_p}, \quad \sum_{i=1}^n \text{CVOL}_i = c_2 \sigma_p, \\ \text{CSKEW}_i = c_3 w_i \left(\partial_i s_p + \partial_i \sigma_p \frac{s_p}{\sigma_p} \right), \quad \sum_{i=1}^n \text{CSKEW}_i = c_3 s_p, \\ \text{CKURT}_i = c_4 w_i \left(\partial_i k_p + \partial_i \sigma_p \frac{k_p - 3}{\sigma_p} \right), \quad \sum_{i=1}^n \text{CKURT}_i = c_4 (k_p - 3), \end{array} \right.$$

where CMEAN_i , CVOL_i , CSKEW_i and CKURT_i denote the contribution from the i -th asset mean, volatility, skewness and kurtosis to the portfolio risk. c_1 , c_2 , c_3 and c_4 are constants with $c_1 = -1$, $c_2 = \lambda_\alpha^N$, $c_3 = \lambda_\alpha^N \sigma_p \frac{z_\alpha}{6}$ and $c_4 = \lambda_\alpha^N \sigma_p \left(\frac{z_\alpha^2 - 1}{24} \right)$. Summing over all assets, we retrieve ES as the sum of individual contributions $\text{ES}_\alpha^{CF}(\mathbf{w}) = \sum_{i=1}^n \text{CES}_{\alpha(i)}^{CF}$.

Appendix C: Modeling illiquidity through smoothing

Following Getmanski *et al.* [GET 04], we assume that, due to illiquidity, the actual returns $R_{i,t}$ cannot be observed directly and that the reported returns of the illiquid asset, denoted by $R_{i,t}^o$, are governed by a MA(K) process:

$$R_{i,t}^o = \sum_{k=0}^K \theta_{i,k} R_{i,t-k} \quad \text{for } i = (1, \dots, n),\tag{C.1}$$

where $\theta_{i,k} \in [0, 1]$ for $k = (0, \dots, K)$ and $\sum_{k=0}^K \theta_{i,k} = 1$. Getmanski *et al.* [GET 04] have shown that the smoothing mechanism [C.1] leads to a bias in the volatility, while expected returns are unaffected, i.e.:

$$\begin{aligned} \mu_i^o &= \mu_i, \\ \sigma_i^o &= \sigma_i \times \left(\sum_{k=0}^K \theta_{i,k}^2 \right)^{1/2} \leq \sigma_i. \end{aligned} \tag{C.2}$$

Cao and Teiletche [CAO 07] expand these results to the bivariate case, from which we infer for covariance:

$$\sigma_{ij}^o = \sigma_{ij} \left(\sum_{k=0}^K \theta_{i,k} \theta_{j,k} \right).$$

Covariances are thus understated, $|\sigma_{ij}^o| \leq |\sigma_{ij}|$, as $\sum_{k=0}^K \theta_{i,k} \theta_{j,k} \leq 1$. Following the same type of reasoning, we deduce similar results for skewness s_i , coskewness s_{ijl} , kurtosis k_i and cokurtosis k_{ijklm} , that is:

$$\begin{aligned} s_i^o &= s_i \frac{\sum_{k=0}^K \theta_{i,k}^3}{\left(\sum_{k=0}^K \theta_{i,k}^2 \right)^{3/2}}, \\ s_{ijl}^o &= s_{ijl} \left(\sum_{k=0}^K \theta_{i,k} \theta_{j,k} \theta_{l,k} \right), \\ k_i^o &= k_i \frac{\sum_{k=0}^K \theta_{i,k}^4}{\left(\sum_{k=0}^K \theta_{i,k}^2 \right)^2}, \\ k_{ijklm}^o &= k_{ijklm} \left(\sum_{k=0}^K \theta_{i,k} \theta_{j,k} \theta_{l,k} \theta_{m,k} \right). \end{aligned} \tag{C.3}$$

In all cases, we also see that the smoothing process [C.1] leads to an underestimation (in absolute terms) of comoments. To cope with these biases, we correct the covariance, coskewness and cokurtosis matrices before applying the risk contribution calculations [6.7] and the program [6.5]. More specifically, we correct the comoments in the following way:

$$\begin{aligned} \sigma_{ij}^{\text{corrected}} &= \frac{\sigma_{ij}^o}{\left(\sum_{k=0}^K \hat{\theta}_{i,k} \hat{\theta}_{j,k} \right)}, \\ s_{ijl}^{\text{corrected}} &= \frac{s_{ijl}^o}{\left(\sum_{k=0}^K \hat{\theta}_{i,k} \hat{\theta}_{j,k} \hat{\theta}_{l,k} \right)}, \\ k_{ijklm}^{\text{corrected}} &= \frac{k_{ijklm}^o}{\left(\sum_{k=0}^K \hat{\theta}_{i,k} \hat{\theta}_{j,k} \hat{\theta}_{l,k} \hat{\theta}_{m,k} \right)}, \end{aligned} \tag{C.4}$$

where the terms $\hat{\theta}_{i,k}$ are inferred from the estimation of moving-average models. We first estimate the MA(K) process, $R_{i,t}^o = \mu_i + \varepsilon_{i,t} + \vartheta_{i,1} \varepsilon_{i,t-1} + \dots + \vartheta_{i,K} \varepsilon_{i,t-K}$, and then simply deduce the smoothing parameters as $\hat{\theta}_{i,0} = 1 / \left(1 + \hat{\vartheta}_{i,1} + \dots + \hat{\vartheta}_{i,K} \right)$ and $\hat{\theta}_{i,k} = \hat{\vartheta}_{i,k} / \left(1 + \hat{\vartheta}_{i,1} + \dots + \hat{\vartheta}_{i,K} \right)$ for $k > 1$.

For each asset i , the implication of the illiquidity can then be judged by contrasting the risk allocation of a considered portfolio on the basis of the corrected statistics versus the ones based on the raw statistics with:

$$\text{ILLIQUID}_i = \text{CES}_{\alpha(i)}^{CF} - \text{CES}_{\alpha(i)}^o, \quad [\text{C.5}]$$

where $\text{CES}_{\alpha(i)}^{CF}$ and $\text{CES}_{\alpha(i)}^o$ correspond to the smoothed and unsmoothed individual ES risk contributions, respectively. It is straightforward to see that ILLIQUID_i is equal to 0 if all assets are liquid, i.e. $\theta_{i,k} = 1$ for $k = 1$ and $\theta_{i,k} = 0$ for $k > 0$.

Appendix D: Multifactor estimators for higher order moment matrices

The implementation of ES risk decomposition analysis requires an estimate of the higher order moments of the return distribution. To reduce the dimensionality problem typically involved in the estimation of the coskewness and cokurtosis matrices, we assume that asset returns can be represented by the following linear Q -factor model:

$$\mathbf{R}_t = \boldsymbol{\mu} + \mathbf{B} \mathbf{f}_t + \boldsymbol{\varepsilon}_t, \quad [\text{D.1}]$$

where $\mathbf{f}_t = (f_{1,t}, \dots, f_{Q,t})^T$ is the $(Q \times 1)$ vector of risk factors assumed to be jointly normally distributed, with $E[\mathbf{f}_t] = 0$ and $E[\mathbf{f}_t \mathbf{f}_t^T \otimes \mathbf{f}_t^T] = 0$; $\boldsymbol{\varepsilon}_t = (\varepsilon_{1,t}, \dots, \varepsilon_{N,t})^T$ is the $(N \times 1)$ vector of error terms which are supposed to be independent with each of the risk factors and also cross-sectionally independent, i.e. for any powers r and s and $i \neq j$, $E[(\mathbf{f}_t)^r \varepsilon_{i,t}^s] = E[(\mathbf{f}_t)^r] E[\varepsilon_{i,t}^s]$ and $E[\varepsilon_{i,t}^r \varepsilon_{j,t}^s] = E[\varepsilon_{i,t}^r] E[\varepsilon_{j,t}^s]$; and \mathbf{B} is the $(N \times Q)$ matrix of factor loadings with row vectors $\mathbf{b}_i^T = (\beta_{i1}, \dots, \beta_{iQ})$. The common risk factors can correspond either to some set of observable macrofinancial variables, fundamental-based mimicking portfolios or to latent statistical factors.

Under the simplifying assumption of the multifactor model [D.1], the comoment matrices $\boldsymbol{\Omega}$, $\boldsymbol{\Xi}$ and $\boldsymbol{\Gamma}$ can then be decomposed as:

$$\begin{aligned} \boldsymbol{\Omega} &= \mathbf{B} \boldsymbol{\Omega}_f \mathbf{B}^T + \boldsymbol{\Delta}_\varepsilon, \\ \boldsymbol{\Xi} &= \boldsymbol{\Psi}_\varepsilon, \\ \boldsymbol{\Gamma} &= \mathbf{B} \boldsymbol{\Gamma}_f (\mathbf{B}^T \otimes \mathbf{B}^T \otimes \mathbf{B}^T) + \boldsymbol{\Phi}_\varepsilon, \end{aligned} \quad [\text{D.2}]$$

where $\boldsymbol{\Omega}_f = E[\mathbf{f}_t \mathbf{f}_t^T]$ is the $(Q \times Q)$ factor covariance matrix with generic term $\sigma_{ij}^f = E[f_{i,t} f_{j,t}]$ and $\boldsymbol{\Delta}_\varepsilon$ is the $(N \times N)$ diagonal residual covariance matrix, with elements $\delta_{ij} = E[\varepsilon_{i,t}^2]$ when $i = j$ and zero otherwise; $\boldsymbol{\Psi}_\varepsilon$ is the $(N \times N^2)$ sparse residual matrix of coskewness, with elements $\psi_{ijk} = E[\varepsilon_{i,t}^3]$ when $i = j = k$

and zero otherwise; $\Gamma_f = E \left[\mathbf{f}_t \mathbf{f}_t^T \otimes \mathbf{f}_t^T \otimes \mathbf{f}_t^T \right]$ is the $(Q \times Q^3)$ factor cokurtosis matrix, with generic term $k_{ijlm}^f = \sigma_{ij}^f \sigma_{lm}^f + \sigma_{il}^f \sigma_{jm}^f + \sigma_{im}^f \sigma_{jl}^f$ and Φ_ε is the $(N \times N^3)$ residual matrix of cokurtosis, with $\phi_{ijlm} = 6 \mathbf{b}_i^T \Omega_f \mathbf{b}_i E [\varepsilon_{i,t}^2] + E [\varepsilon_{i,t}^4]$ when $i = j = l = m$, $\phi_{ijlm} = 3 \mathbf{b}_i^T \Omega_f \mathbf{b}_m E [\varepsilon_{i,t}^2]$ when $i = j = l$ and $m \neq i$ (and similarly for $i = j = m$ and $l \neq i$, $i = l = m$ and $j \neq i$, or $j = l = m$ and $i \neq j$), $\phi_{ijlm} = \mathbf{b}_i^T \Omega_f \mathbf{b}_i E [\varepsilon_{i,t}^2] + \mathbf{b}_l^T \Omega_f \mathbf{b}_l E [\varepsilon_{l,t}^2] + E [\varepsilon_{i,t}^2] E [\varepsilon_{l,t}^2]$ when $(i = j) \neq (l = m)$ (and analogously for $(i = m) \neq (j = l)$ or $(i = l) \neq (j = m)$), $\phi_{ijlm} = 3 \mathbf{b}_j^T \Omega_f \mathbf{b}_l E [\varepsilon_{i,t}^2]$ when $i = j$, $l \neq i$, $l \neq m$ and $i \neq m$, and zero if $i \neq j \neq l \neq m$.

As shown in the table below, the resulting number of parameters required to estimate the comoment matrices is dramatically reduced under [D.1]. For universes of medium sizes, say 50 assets and 4 factors, there are only 360 parameters to estimate as opposed to 316200 in the general sample case. This translates in 878-fold increase in the degrees of freedom if we use 5 years of daily data.

Moments	Unconstrained	Q-factor model			
Ω	$N(N+1)/2$	$Q(Q+1)/2 + N(Q+1)$			
Ξ	$N(N+1)(N+2)/6$	N			
Γ	$N(N+1)(N+2)(N+3)/24$	$Q(Q+1)/2 + N(Q+2)$			
Total	$N(N+1)/2$ $+N(N+1)(N+2)/6$ $+N(N+1)(N+2)(N+3)/24$	$Q(Q+1)/2 + N(Q+3)$			
Number of assets	Total	Total			
		Q = 1	Q = 2	Q = 3	Q = 4
N = 5	120	21	28	36	45
N = 50	316200	201	253	306	360
N = 100	4598025	401	503	606	710

Notes. The table represents the number of parameters to estimate for a universe of N assets, under a general unconstrained set-up or through a Q -factor model.

6.7. Data Appendix

Our dataset covers eight traditional and alternative asset classes, over the period spanning from the first quarter of 1990 to the third quarter of 2013. Due to limitations related to private equity and real estate, we use quarterly data. Data are downloaded from Bloomberg, and from Cambridge associates and National Council of Real Estate investment Fiduciaries (NCREIF) Websites.

Our carry measures are inspired from [ILM 11] and [KOI 13], but with some key methodological choices that we here detail. Furthermore, we extend the approach of these authors to alternative assets for which carry estimates are less easily developed.

For equities, we prefer to use earning yields rather than dividend yields, as the latter forecasting abilities have notoriously decreased as companies have preferably used shares buy-backs or accumulating cash rather than distributing it. In this regard, earnings are giving a broader overview of the available income of the company. More globally, earning yield, i.e. the ratio of earnings with current price, is a core measure of equity valuation. Finally, we add inflation expectations to earning yields to get the final carry measure as carry is reputed being a real variable. Inflation expectations are based on University of Michigan Survey of Consumers 1 year ahead expectation, as it is available at sufficient high frequency (monthly) and over a long history, and is model-free as a direct survey, while inflation-linked bonds were not available before 1997 and require assumptions to derive inflation expectations. The source for earning yields is Bloomberg where we use the reciprocal of the ratio between the 12 months trailing weighted diluted earnings per share with the latest available price. For Russell 2000, price-earning ratios are not available before 1995. We fill the gap from 1990 to 1994, by considering that small cap stocks post earning yields half of the ones of large cap stocks, as it does roughly correspond to the ratio observed between 1995 and 2013.

For fixed income securities, carry is given by the sum of current yield and the roll-down. We use Barclays indices for U.S. Investment Grade and High Yield. The yield measure is the yield to worst, which adjusts yield to maturity for prepayment, call or other features characteristic of some bonds. To compute the roll-down, we make use of informations we draw from different Barclays indices. In particular, we contrast the “Global” all-maturity index we use as basis with the “Intermediate” versions, which are based on shorter maturity bonds. More specifically, the roll-down is based on the following formula:

$$\text{Roll return}_t = -D \times \frac{y_t^{\text{Intermediate}} - y_t^{\text{Global}}}{M_t^{\text{Global}} - M_t^{\text{Intermediate}}},$$

where y_t and M_t denote the yield-to-worst and maturity, respectively, and D is the duration of the “Global” index.

Commodities portfolio is based on a basket of 14 primary commodities, which offer available data back to 1990: sugar, coffee, cotton, cocoa, live cattle, lean hogs, gold, silver, copper, WTI crude oil, Brent, heating oil, gasoil and unleaded gasoline. The basket is equally weighted in capital and rebalanced every end of quarter. For each commodity, we take all available contracts over the next 12 months, and compute a roll yield as follows:

$$\frac{12}{K-1} \sum_{k=2}^K \frac{F_t^{(k-1)} - F_t^{(k)}}{(T_k - T_1) F_t^{(k)}}, \quad [6.9]$$

where k denotes the various available contracts with $k = 1$ the front contract and K is the number of available contracts, while T_k is the maturity in months of the k -th

contract and $F_t^{(k)}$ is its price. Similarly to earnings, aggregating yields over a full year helps dealing with seasonality issues that affect many commodities. Carry for commodities is finally the sum of the average roll yield and the cash returns earned by investing every 3 months at USD Libor 3 months.

Hedge funds returns are measured through the Hedge Fund Research (HFR) fund-of-hedge funds index. We select funds of hedge funds rather than single hedge funds as the associated performance indices are much less subject to the typical biases affecting single fund indices, such as survivorship and backfill biases. There is no direct measure of carry for hedge funds and the idea could even sound strange as hedge funds are not assets per se and they frequently benchmark themselves against cash. However, a large academic literature indicates that a significant portion of their returns can be linked to exposures to asset classes or risk factors either contemporaneously or with a lag (see, among others, [HAS 07, ASN 01]). On this basis, we suggest estimating the implied carry for hedge funds through a regression-based aggregation of individual asset classes carry. More specifically, we regress hedge funds (HF) excess returns on contemporaneous and lagged excess returns of S&P 500 (EQ), Russell 2000 (SC), Barclays U.S. Corporate Investment Grade (IG) and High Yield (HY), and commodities (CO) indices. We propose up to four quarterly lags and retain significant lags through a stepwise regression at the 10% significance level over the period 1990Q1–2013Q3. The retained model is as follows:

$$HF_t = 0.2373 SC_t + 0.1136 CO_t, R^2 = 0.4908.$$

Carry for hedge funds is then obtained as the beta-weighted average of individual carry for the retained factors. From data in Table 6.3 for small cap equities and commodities, we obtain an estimate of carry equal to 0.36% per quarter.

Private equity carry is not readily available as well. It has been shown that PE returns are directly linked to traditional asset returns and, particularly, to the public equity market contemporaneous and lagged returns. Using the same methodology as for hedge funds, we have the following model:

$$PE_t = 0.3102 EQ_t + 0.1482 EQ_{t-1} + 0.1582 EQ_{t-2} + 0.1414 EQ_{t-4} \\ + 0.1348 SC_t + 0.119 CO_t, R^2 = 0.2569.$$

Carry is then finally computed similarly to hedge funds on the relevant PE factors. From data in Table 6.3, we obtain an estimate of carry equal to 1.81% per quarter.

Contrary to other alternative assets, sensible measures for real estate carry are readily available. In particular, real estate returns can be decomposed into two components: an income return coming from gross rental income minus operating expenses, and a capital return coming from the change in market value of the

property. We retain the former measure as carry, because this is the return a real estate investor captures if the property price does not move, hence being consistent with the carry definition used across asset classes. In practice, income returns computed by NCREIF have been historically very stable around 2% per quarter since 1978 (see, for example, http://www.ncreif.org/documents/event_docs/NCREIF_Academy/NCREIF-Database-Query-Tools.pdf, p. 38), and this is the estimate we retain in our empirical section.

6.8. Bibliography

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Target Volatility

7.1. Introduction

It is a common practice among investment practitioners to scale (leverage) their portfolio positions in risky assets (or trading strategies) according to forecasted volatility. There is a cross-sectional and a time series aspect of this procedure. In the cross-sectional dimension, we scale all assets to a common target volatility. After scaling, all assets will display similar realized volatility, i.e. all assets will be equally important for overall portfolio risk and performance¹. The objective here is better *asset diversification*. If all assets display equal Sharpe ratio and equal correlation, the resulting portfolio is also mean variance efficient. In the time series dimension, we want to achieve better *time diversification*, i.e. we want to make every period equally important for overall portfolio risk and performance. Both aspects of diversification can be found to different degrees in risk parity (predominantly cross sectional volatility scaling) as well as target volatility (predominantly time series volatility scaling) products. This contribution will focus on the theoretical and empirical foundations of target volatility. If target volatility offers better time diversification (even without larger expected returns), target volatility products will outperform buy and hold products by offering lower risk per unit of return. This will then lead to higher Sharpe ratios and mean variance utility. Section 7.2 will prove this conjecture via an old argument made by Paul Samuelson on the proper definition of time diversification. Section 7.3 will look at the empirical evidence, while section 7.4 relates target volatility to systematic beta variation. In section 7.5, we will ask whether target volatility is also a tail hedge. If volatility is kept targeted,

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¹ For this to be strictly true, we also need equal correlation among assets. Given that correlation varies widely across time, i.e. is difficult to estimate, most practitioners see this as an appropriate regularization constraint.

it seems plausible that large blowouts in times of risen volatility can be avoided. Section 7.6 analyzes the impact of asymmetric leverage constraints in target volatility products. Section 7.7 extends our analysis away from US stocks to global asset classes. Section 7.8 concludes the whole chapter.

7.2. Better leverage and the Samuelson puzzle

Why should a strategy – as simple as scaling volatility to a target level – lead to outperformance versus a buy and hold strategy? In the author’s view, the answer to this question has already been given by Paul Samuelson [SAM 92] in a related context². Samuelson once famously asked on the consequences of uninformative (random) market timing: “What is riskier, 100% in a 50%/50% stock/bond portfolio all the time or 100% either in a pure stock or pure bond portfolio half the time? Samuelson found that non-informative and random market timing leaves expected returns unchanged but increases portfolio variance.

How does target volatility relate to random market timing? Suppose we want to target a 10% constant volatility for a given market that displays on average a 20% volatility ($\bar{\sigma}$). This requires a nominal weight of a given assets evolves according to:

$$\bar{w}_{nom.} = \frac{\sigma_{target}}{\bar{\sigma}} = \frac{0.10}{0.20} = 0.5 \quad [7.1]$$

Unfortunately – and in line with reality – we are going to assume that our asset also displays large variations in volatility across time. If we decided, as suggested above, to give that asset a constant weight of 50% (i.e. a fixed weight of 50% cash and 50% market exposure), we would experience large variations in realized volatility. Alternatively, we can also view this as variations in the effective weight (random market timing) with respect to a constant volatility asset. What would that mean for our hypothetical example? Suppose realized volatility for a given period is 30%. What would have been the synthetic weight in a 20% volatility asset? In other words, by how much does the weight in a 20% assumed volatility asset randomly vary with changing volatility? The answer is given below:

$$w_{eff,t} = \bar{w}_{nom} \frac{\sigma_{realized,t}}{\bar{\sigma}} = 0.50 \frac{0.30}{0.20} = 0.75 \quad [7.2]$$

² Hallerbach [HAL 12] provides an interesting mathematical proof.

We find a 50% fixed weight in a 30% volatility asset equals a (now increased) 75% weight in a 20% volatility asset. Fixing nominal exposure (\bar{w}_{nom}) in a world of changing volatility is equivalent to randomly changing effective weights in a world of fixed volatility. This amounts to random (non-informative) market timing with the negative effects on Sharpe-ratios as established by Samuelson [SAM 92], Krizman [KRI 00] and Ilmanen [ILM 14]³. Removing this element of random market timing is the value proposition of volatility scaling. Under constant volatility scaling, the nominal weight varies now according to:

$$w_{nom,t} = \frac{\sigma_{target}}{\sigma_{forecast,t}} \quad [7.3]$$

while the effective weight becomes:

$$w_{eff,t} = \left(\frac{\sigma_{target}}{\sigma_{forecast,t}} \right) \left(\frac{\sigma_{realized,t}}{\bar{\sigma}} \right) = \bar{w}_{nom} \left(\frac{\sigma_{realized,t}}{\sigma_{forecast,t}} \right) \quad [7.4]$$

For perfect volatility forecasts, i.e. $\sigma_{forecast} = \sigma_{realized}$, the effective weight is constant across time and equal to \bar{w}_{nom} . This would effectively remove random market timing with its negative effects on portfolio risk.

We can now calculate $Var(r_{portfolio,t}) = Var(w_{eff,t}r_t^*)$ under alternative hypothesis on the distribution of $\left(\frac{\sigma_{realized,t}}{\sigma_{forecast,t}} \right)$. In order to compute the effect of changing volatility on the risk of a (volatility unscaled) portfolio (excess) returns, we need to work out:

$$Var(r_{portfolio,t}) = Var(w_{eff,t}r_t^*) \quad [7.5]$$

where r_t^* denotes the (unobservable) return of constant volatility asset. As an example, we assume random weights to be uniformly distributed according to:

$$w_{eff,t} \sim U(\bar{w}_{nom} - \Delta, \bar{w}_{nom} + \Delta) \quad [7.6]$$

³ This statement rests on the assumption that volatility does not represent an economic state variable, i.e. volatility is not correlated with future expected performance.

Without apology, we also assume market excess returns to be normally distributed as well as effective weights and returns to be independently distributed. This allows us to write down the joint distribution of random effective weights and random returns in a constant volatility assets as:

$$f(w_{eff}r^*) = f(w_{eff})f(r^*) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{r^* - \mu}{\sigma_{long-run}}\right)^2} \frac{1}{(\bar{w}_{nom} + \Delta) - (\bar{w}_{nom} - \Delta)} \quad [7.7]$$

Note that:

$$Var(w_{eff,t}r_t^*) = E\left[(w_{eff}r^*)^2\right] - E[w_{eff}r^*]^2 \quad [7.8]$$

Given the joint probability density [7.7], we start integrating over the joint probability density to calculate:

$$\begin{aligned} E\left[(w_{eff}r^*)^2\right] &= \int_{-\infty}^{\infty} \int_{\bar{w}_{nom}-\Delta}^{\bar{w}_{nom}+\Delta} (w_{eff}r^*)^2 f(w_{eff}, r^*) dw_{eff} dr^* \\ &= \frac{1}{3}(\Delta^2 + 3\bar{w}_{nom}^2)\left(\mu^2 + \bar{\sigma} \lim_{x \rightarrow \infty}^2\right) \end{aligned} \quad [7.9]$$

We can now write:

$$\begin{aligned} Var(w_{eff}r^*) &= \frac{1}{3}(\Delta^2 + 3\bar{w}_{nom}^2)\left(\mu^2 + \bar{\sigma}^2\right) - \mu^2\bar{w}_{nom}^2 \\ &= \bar{w}_{nom}^2\bar{\sigma}^2 + \Delta^2\frac{\mu^2 + \bar{\sigma}^2}{3} \\ &= \sigma_{target}^2 + \Delta^2\frac{\mu^2 + \bar{\sigma}^2}{3} \end{aligned} \quad [7.10]$$

The reader will notice that for $\Delta \rightarrow 0$ (forecasted volatility gets close to realized volatility), we converge to the standard expression for portfolio risk with constant volatility. Realized risk equals targeted risk. What does [7.10] imply? As long as we have time varying volatility – that will implicitly vary our effective exposures – realized portfolio volatility will be higher than what we expect, even if our view on average volatility view was correct. If we could adopt our nominal weights (e.g. by forecasting realized volatility, instead of relying on average volatility) such that effective weights remain constant, we could reach target volatility.

7.3. Target volatility and Sharpe ratio improvement

We want to evaluate the statistical properties of target volatility over the longest possible time period, i.e. with the highest statistical confidence. This section focuses on long term, high quality US equity data and in particular on returns for the Fama and French three factor model complemented by momentum as well as US industry returns (14 sectors). Our data source is French [FRE 15]. First, we calculate realized volatility for each month t , $\hat{\sigma}_t$, using daily intra month excess returns (returns minus risk free rate), i.e.:

$$\hat{\sigma}_t = \sqrt{\frac{1}{\#i \in month_t} \sum_{i \in month_t} (r_i - \bar{r})^2} \sqrt{250} \quad [7.11]$$

where r_i (with $i \in month_t$) are daily returns belonging to month t . Secondly, we calculate the after transaction costs (tc) monthly returns for buy and hold ($r_{B\&H,t}$), target volatility with lagged volatility ($r_{TV,t}$) and perfect (i.e. the same month realized volatility) volatility forecast ($r_{TV^*,t}$) as given below⁴:

$$r_{TV,t} = \left(n^{-1} \sum \hat{\sigma}_t \right) \left[\left(\frac{1}{\hat{\sigma}_{t-1}} \right) r_{B\&H,t} - \left| \left(\frac{1}{\hat{\sigma}_{t-1}} \right) - \left(\frac{1}{\hat{\sigma}_{t-2}} \right) \right| tc \right] \quad [7.12]$$

$$r_{TV^*,t} = \left(n^{-1} \sum \hat{\sigma}_t \right) \left[\left(\frac{1}{\hat{\sigma}_t} \right) r_{B\&H,t} - \left| \left(\frac{1}{\hat{\sigma}_t} \right) - \left(\frac{1}{\hat{\sigma}_{t-1}} \right) \right| tc \right] \quad [7.13]$$

Note that we scale weights to generate the average (full sample) realized volatility $\left(n^{-1} \sum \hat{\sigma}_t \right)$. Tables 7.1 (for factor returns) and 7.2 (for industry returns) report Sharpe-ratios as well as Sharpe-ratio differences together with their respective t -values – according to the Leung/Wong [LEU 08] test. Our calculations are based on monthly returns from November 1926 to December 2014. Assumed transaction costs (as a fraction of the traded underlying) amount to 30bps ($tc = 0.003$). With the exception of momentum (UMD) and size (SMB), none of the realistic target volatility strategies display statistically significant Sharpe differences to a buy and hold strategy. Only perfect volatility forecasts deliver statistically significant Sharpe-ratio differences across factors and industries.

⁴ We do not consider performance related drift up to the end of month in this formulation.

	$SR_{B\&H}$	SR_{TV}	SR_{TV^*}	$SR_{TV} - SR_{B\&H}$	$SR_{TV^*} - SR_{TV}$
<i>MKT</i>	0,41	0,46	0,99	0,05	0,53
	3,93***	4,32***	9,28***	0,76	11,59***
<i>SMB</i>	0,12	-0,01	0,18	-0,13	0,18
	1,27	-0,06	1,66	-2,66**	5,00***
<i>HML</i>	0,40	0,34	0,26	-0,06	-0,08
	3,72***	3,18***	2,44**	-0,98	-2,08*
<i>UMD</i>	0,51	0,80	1,04	0,29	0,24
	4,72***	7,55***	9,81***	4,31***	5,08***

Table 7.1. Target Volatility for Fama/French factors (*MKT*, *SMB*, *HML*) and momentum (*UMD*)

Table 7.1 displays individual Sharpe-ratios as well Sharpe-ratio differences together with their respective t -values – according to the Leung/Wong [LEU 08] test – for buy and hold ($B \& H$) as well as target volatility portfolios across four market factor returns. Realistic target volatility portfolios (TV) are formed at the end of each month using realized volatility based on daily intra month returns. Perfect foresight target volatility (TV^*) portfolios are constructed using next month realized volatility. We use daily data from November 1926 to December 2014. Assumed transaction costs (as a fraction of the traded underlying) are 30 bps. Data source: French [FRE 15].

What drives these results? Do assets differ in their volatility persistence? Does realized volatility work as a state variable, i.e. does low (high) volatility forecast high (low) future returns? In order to answer this question, we run two separate regressions. First, we investigate how persistent realized volatility is. High persistence allows better forecasts, which in turn improves returns from target volatility. For this purpose, we estimate:

$$\hat{\sigma}_t = a_0 + a_1 \hat{\sigma}_{t-1} + e_t \quad [7.14]$$

for November 1926 to December 2014 (1058 observations). Tables 7.3 and 7.5 show our results. While persistence is high (correlation between this month and last month realized volatility hovers around 0.7) there are no significant differences between assets where target volatility had significantly improved Sharpe-ratios and where it had not improved Sharper-ratios. We acknowledge, this might be due to our simplistic forecast generation model or our infrequent rebalancing frequency. Second, we investigate whether past realized volatility is a good predictor for future

returns. Given that target volatility increases weights in the risky asset in times of falling volatility, we would expect that low volatility, predicts high future returns. For this purpose we estimate:

$$r_{B\&H,t} = b_0 + b_1 \hat{\sigma}_{t-1} + \nu_t \quad [7.15]$$

Tables 7.4 and 7.6 summarize the results for factors and industries. Most assets display insignificant and wrong signed (positive instead of negative coefficients) for b_1 . Exceptions are the momentum factor (UMD) and the telecom industry. Predictability is concentrated in those assets that displayed improved Sharpe-ratios after volatility targeting. It seems that the information content of volatility scaling (realized volatility as state variable) is more important than time diversification.

	$SR_{B\&H}$	SR_{TV}	SR_{TV^*}	$SR_{TV} - SR_{B\&H}$	$SR_{TV^*} - SR_{TV}$
Non-durables	0,467	0,468	0,985	0,001	0,517
	4,943***	4,393***	9,250***	-0,887	8,412***
Durables	0,368	0,367	0,601	-0,001	0,234
	3,390***	3,448***	5,644***	0,109	5,379***
Manufacturing	0,391	0,416	0,838	0,025	0,422
	3,646***	3,900***	7,864***	0,523	10,259***
Energy	0,405	0,455	0,704	0,049	0,250
	4,212***	4,268***	6,615***	0,098	6,239***
Chemicals	0,439	0,445	0,775	0,006	0,330
	4,230***	4,173***	7,273***	-0,066	7,147***
Business Eq.	0,377	0,424	0,767	0,046	0,343
	3,569***	3,977***	7,202***	0,815	8,487***
Telecom	0,384	0,528	0,725	0,143	0,197
	4,078***	4,954***	6,804***	1,448	3,569***
Utilities	0,311	0,396	0,754	0,085	0,358
	3,505***	3,718***	7,079***	0,331	7,204***
Shops	0,407	0,470	0,880	0,063	0,410
	4,082***	4,415***	8,265***	0,670	7,844***
Health	0,459	0,465	0,847	0,006	0,381
	4,666***	4,368***	7,951***	-0,511	7,503***
Finance	0,352	0,421	0,832	0,069	0,411
	3,415***	3,951***	7,815***	0,927	8,409***
Other	0,287	0,314	0,775	0,026	0,462
	2,748**	2,943***	7,280***	0,386	10,705***

Table 7.2. Target volatility for industry portfolios

Table 7.2 displays individual Sharpe-ratios as well as Sharpe-ratio differences together with their respective t-values—according to the Leung/Wong [LEU 08] test

– for buy and hold ($B \& H$) as well as target volatility portfolios across 14 US industry portfolios. Realistic target volatility portfolios (TV) are formed at the end of each month using realized volatility based on daily intra month returns. Perfect foresight target volatility (TV^*) portfolios are constructed using next month realized volatility. We use daily data from November 1926 to December 2014. Assumed transaction costs (as a fraction of the traded underlying) are 30bps. Data source: French [FRE 15].

Table 7.3 shows regression coefficients from regressing last month realized volatility against this month realized volatility, i.e. from $\hat{\sigma}_t = a_0 + a_1 \hat{\sigma}_{t-1} + e_t$. All 1058 data points are calculated from intramonth daily data. Our sample stretches from November 1926 to December 2014. Data source: French [FRE 15].

	a_0	a_1	\bar{R}^2
<i>MKT</i>	0,039	0,715	0,511
	10,965***	33,231***	
<i>SMB</i>	0,019	0,732	0,537
	10,315***	34,946***	
<i>HML</i>	0,016	0,781	0,610
	9,251***	40,624***	
<i>UMD</i>	0,025	0,725	0,526
	10,240***	34,190***	

Table 7.3. Volatility persistence – factors

Table 7.4 shows regression coefficients from regressing last month realized volatility $\hat{\sigma}_{t-1}$ against this month returns, i.e. from $r_{B\&H,t} = b_0 + b_1 \hat{\sigma}_{t-1} + \nu_t$. All 1058 data points are calculated from intramonth daily data. Our sample stretches from November 1926 to December 2014. Data source: French [FRE 15].

	b_0	b_1	\bar{R}^2
<i>MKT</i>	0,007	-0,004	0,000
	2,379**	-0,204	
<i>SMB</i>	0,001	0,009	0,000
	0,334	0,523	
<i>HML</i>	0,000	0,048	0,006
	0,204	2,439**	
<i>UMD</i>	0,014	-0,089	0,020
	6,593***	-4,651***	

Table 7.4. Return predictability – factors

Table 7.5 shows regression coefficients from regressing last month realized volatility against this month realized volatility, i.e. from $\hat{\sigma}_t = a_0 + a_1\hat{\sigma}_{t-1} + e_t$. All 1058 data points are calculated from intramonth daily data. Our sample stretches from November 1926 to December 2014. Data source: French [FRE 15].

	a_0	a_1	\bar{R}^2
Non-Durables	0,041	0,643	0,414
	12,728***	27,281***	
Durables	0,054	0,732	0,536
	10,849***	34,894***	
Manufacturing	0,044	0,730	0,533
	10,546***	34,692***	
Energy	0,051	0,707	0,499
	11,536***	32,395***	
Chemicals	0,043	0,714	0,509
	11,054***	33,085***	
Business Eq.	0,051	0,748	0,560
	10,395***	36,610***	
Telecom	0,040	0,706	0,499
	11,114***	32,427***	
Utilities	0,029	0,769	0,591
	8,775***	39,018***	
Shops	0,042	0,709	0,503
	11,230***	32,674***	
Health	0,051	0,65	0,425
	12,765***	27,916***	
Finance	0,037	0,767	0,588
	9,127***	38,806***	
Other	0,046	0,712	0,507
	11,079***	32,963***	

Table 7.5. Volatility persistence – industries

Table 7.6 shows regression coefficients from regressing last month realized volatility against this month returns, i.e. from $r_{B\&H,t} = b_0 + b_1\hat{\sigma}_{t-1} + \nu_t$. All 1058 data points are calculated from intramonth daily data. Our sample stretches from November 1926 to December 2014. Data source: French [FRE 15].

	b_0	b_1	\bar{R}^2
Non-durables	0,004	0,026	0,002
	1,556	1,332	
Durables	0,000	0,041	0,004
	-0,014	2,148*	
Manufacturing	0,003	0,030	0,003
	0,732	1,643	
Energy	0,008	-0,001	0,000
	2,239*	-0,052	
Chemicals	0,005	0,018	0,001
	1,489	1,023	
Business Eq.	0,008	0,000	0,000
	1,923	-0,009	
Telecom	0,010	-0,031	0,004
	4,027***	-2,073*	
Utilities	0,005	0,006	0,000
	2,021	0,413	
Shops	0,006	0,010	0,000
	1,771	0,524	
Health	0,007	0,010	0,000
	1,968	0,514	
Finance	0,009	-0,008	0,000
	2,560**	-0,506	
Other	0,005	0,007	0,000
	1,248	0,343	

Table 7.6. Return predictability – industries

7.4. Informative or uninformative leverage

Target volatility adds time varying exposures (leverage) to a buy and hold strategy. It is hence natural to ask how much of these excess returns (versus a buy and hold strategy) can be explained by time varying exposures created by variations in economic state variables⁵. For this, we use the dynamic beta model as introduced by Ferson and Schadt [FER 96]. Let z_{t-1} denote the demeaned series (surprises

⁵ Anderson et al. [AND 12, AND 14] also document that time varying leverage can significantly increase or decrease realized Sharpe-ratios.

from the unconditional mean) of a set of instruments (predictors). We run the following regression:

$$\begin{aligned} (r_{iv,t} - r_{bh,t}) &= \alpha + \beta(z_{t-1})' r_{bh,t} + \varepsilon_t \\ \beta(z_{t-1}) &= \beta_0 + \sum_{i=1}^n \beta_i z_{t-1} \end{aligned} \quad [7.16]$$

to focus explicitly on the % variation explained by economic state variables. Tables 7.7 and 7.8 display our regression results. We use dividend yields, term structure slope and real rates as state variables. Monthly inputs for state variables (dividend yields, long term rates, inflation) are sourced from Shiller [SHI 15].

Table 7.7 displays regression coefficients for target volatility factor returns

$$\begin{aligned} (r_{iv,t} - r_{bh,t}) &= \alpha + \beta(z_{t-1})' r_{bh,t} + \varepsilon_t \\ \beta(z_{t-1}) &= \beta_0 + \sum_{i=1}^n \beta_i z_{t-1} \end{aligned}$$

We use dividend yields, yield curve slope and inflation as economic state variables. All regressions use monthly data from November 1926 to December 2014. Data source: French [FRE 15], Shiller [SHI 15].

	α	β_0	β_{slope}	β_{div}	β_{real}	\bar{R}^2
<i>MKT</i>	0,001	-0,033	-0,069	-0,076	0,089	0,190
-	1,563	-1,866	-4,049***	-6,404***	4,652***	
<i>SMB</i>	-0,001	-0,013	-0,094	0,029	0,065	0,067
	-1,530	-0,741	-4,717	1,702	2,855**	
<i>HML</i>	0,000	-0,107	0,037	-0,085	0,152	0,213
	-0,497	-6,138***	1,842	-6,349***	7,688***	
<i>UMD</i>	0,004	-0,169	0,042	-0,039	0,179	0,212
	5,584***	-9,737***	1,927	-3,434***	7,995***	

Table 7.7. Dynamic factor timing using economic state variables – factors

Table 7.8 displays regression coefficients for target volatility industry returns according to

$$(r_{iv,t} - r_{bh,t}) = \alpha + \beta(z_{t-1})' r_{bh,t} + \varepsilon_t$$

$$\beta(z_{t-1}) = \beta_0 + \sum_{i=1}^n \beta_i z_{t-1}$$

We use dividend yields, yield curve slope and real interest rates as economic state variables. All regressions employ monthly data from November 1926 to December 2014. Data source: French [FRE 15], Shiller [SHI 15].

	α	β_0	β_{slope}	β_{div}	β_{real}	\bar{R}^2
Non-Durables	0,000	0,018	-0,161	-0,050	-0,077	0,090
	0,194	0,956	-8,544***	-3,733***	-3,704***	
Durables	0,002	-0,001	-0,100	-0,055	0,228	0,363
	2,179*	-0,083	-6,142	-5,002***	11,613	
Manufacturing	0,002	0,006	-0,071	-0,089	0,216	0,368
	1,571	0,367	-4,280***	-7,787***	11,668	
Energy	0,001	0,007	-0,053	-0,084	-0,025	0,050
	0,724	0,406	-2,979**	-6,200***	-1,456	
Chemicals	0,000	-0,044	-0,016	-0,096	0,095	0,167
	0,510	-2,477*	-1,043	-7,524***	4,808	
Business Eq.	0,002	-0,098	-0,032	-0,041	0,101	0,144
	1,970	-5,853***	-1,838	-3,492***	4,751	
Telecom	0,003	-0,078	-0,084	0,032	0,021	0,042
	3,149***	-3,808***	-4,128***	2,075***	0,855	
Utilities	0,001	0,108	0,029	-0,109	0,323	0,211
	0,878	4,184***	1,133	-6,659***	9,991	
Shops	0,002	-0,022	-0,087	-0,059	0,044	0,119
	1,723	-1,247	-5,259***	-4,819***	2,324	
Health	0,000	0,021	-0,137	-0,085	-0,057	0,150
	-0,521	1,211	-7,899***	-7,926***	-3,028	
Finance	0,002	-0,022	-0,035	-0,055	0,221	0,251
	1,951	-1,148	-1,790	-4,447***	9,630	
Other	0,001	0,032	-0,081	-0,119	0,144	0,367
	0,736	1,916	-4,967***	-11,720***	8,432	

Table 7.8. *Dynamic factor timing using economic state variables – industries*

The success of this model is modest. A time varying beta model can explain about 20% of excess return variation for most factors but leaves significant alphas unchanged. If anything, additional alphas become significant or increase in t-values. This suggests that beta timing using realized volatility catches different information than beta timing using economic state variables.

7.5. Target volatility and tail hedging

Let us casually define tail risk hedging as a strategy to avoid extreme outcomes. One reason for the “large sigma” losses investors experienced in 2008 and 2009 was the calculation of sigma. A 40% loss in a month is a 6.93 sigma event for 20% per annum volatility, but only a 2.31 sigma loss for 60% volatility. In this view, tail losses arise mainly from investors failure to target the portfolio volatility they desire. Instead, they let portfolio volatility considerably increase in crisis periods. Volatility targeting provides a systematic approach to avoid a repeat. It uses cash/leverage as a way to reduce tail risk by keeping volatility at a predefined level, i.e. raising cash in times of rising volatilities, but equally leveraging existing assets in quiet times. It therefore attempts to avoid blowouts. So far the narrative. How can we provide evidence on our conjecture? Target volatility increases leverage in times of falling volatility in up markets (some few target volatility hence as close relative to trend following). It is unrealistic to expect smaller drawdowns after a trend reversal. Larger leverage must lead to sharper drawdowns⁶. However, given that drawdowns occur in times of rising volatility (and hence quickly falling leverage) we could reasonably expect lower maximal cumulative drawdowns if volatility was persistent. We therefore focus on maximum cumulative drawdown to evaluate tail risk.

Maximum cumulative drawdown (MDD) is defined as the worst case holding period return, i.e. the return for the unluckiest investor that bought at the worst moment (past high) and equally sold it at the worst moment (lowest low after past high). Some investors view MDD as a better risk measure as it incorporates the volatility of returns as well their tendency to exhibit large losses or to cluster in time. It is however also a noisier measure of (downside) risk as, unlike in volatility, not all data points have the same impact on its calculation. Observations in the drawdown period get a larger weight. The practical appeal of MDD for many investors arises from its direct interpretation as an actually experienced historical loss (unlike the

⁶ This contrast with simulation studies based on artificial data that found limited drawdowns for volatility targeting. The problem here are of course the artificial data. Simulating long-term stock market returns using a Heston stochastic volatility model ignores jumps (occurring most of the time on the downside). Continuous and reactive adjustments will lead to lower drawdowns if the market moves smoothly. However, for a process with jumps it will often be too late to react. Lower drawdowns look then not realistic.

abstract notion of variance). MDD is not only useful for clients but also for hedge fund owners as it resembles the distance from a hedge funds high watermark, i.e. it

Table 7.9 displays maximum cumulative drawdowns relative to volatility. We use monthly data from November 1926 to December 2014. Data source: French [FRE 15], Shiller [SHI 15]

	$MMD_{B\&H}$	MDD_{TV}	MDD_{TV^*}
<i>MKT</i>	-2,892	-2,209	-1,936
<i>SMB</i>	-5,725	-7,097	-5,461
<i>HML</i>	-3,661	-2,535	-3,360
<i>UMD</i>	-3,709	-1,541	-1,427

Table 7.9. *Maximum drawdown to volatility – factors*

Table 7.10 displays maximum cumulative drawdowns relative to volatility according to [7.18]. We use monthly data from November 1926 to December 2014. Data source: French [FRE 15], Shiller [SHI 15].

	$MMD_{B\&H}$	MDD_{TV}	MDD_{TV^*}
Non-durables	-2,175	-1,741	-0,921
Durables	-2,845	-3,076	-1,826
Manufacturing	-2,571	-1,689	-1,200
Energy	-2,378	-0,936	-0,836
Chemicals	-2,209	-1,313	-0,988
Business Eq	-3,108	-2,176	-2,141
Telecoms	-4,817	-2,767	-3,236
Utilities	-2,025	-3,918	-0,844
Shops	-1,980	-3,160	-0,406
Health	-2,222	-1,994	-1,419
Finance	-3,079	-1,763	-1,731
Other	-2,673	-2,445	-1,506

Table 7.10. *Maximum drawdown to volatility – industries*

represents the moneyness of the performance fee option. Given a random process x on $[0, T]$ the historic maximum drawdown (MDD) at T is defined as:

$$MDD(x) = \sup_{t \in [0, T]} \left[\sup_{s \in [0, t]} x_s - x_t \right] \quad [7.17]$$

Equation [7.17] looks for each t for the worst possible entry point in the past starting from zero up to t . The worst of these drawdowns is called the maximum drawdown or sometimes also called the maximum cumulative drawdown. In practice, MDD is often scaled with respect to volatility to make it comparable across strategies.

$$\frac{MDD(x)}{\sigma} = \frac{\sup_{t \in [0, T]} \left[\sup_{s \in [0, t]} x_s - x_t \right]}{\sigma} \quad [7.18]$$

The intuition is that we would expect a more volatile investment to show larger MDD. Relating MDD to volatility also allows comparisons across investments with different risks. MDD to volatility is among practitioners also used as a characteristic attributable to investment strategies.

Tables 7.9 and 7.10 display the results of our calculations. As conjectured, the maximum drawdown to volatility is in most cases considerably smaller than for a buy and hold investment. In the absence of improved Sharpe ratios, lower cumulative drawdowns could explain the appeal of target volatility to investors.

7.6. Asymmetric leverage

Many real-world investors dislike leverage. For a volatility targeting investor, leverage aversion might turn out to be a costly disorder. Target volatility is likely to underperform the unleveraged portfolios in the long run, as it will display a beta < 1 due to the (self) imposed leverage constraint.

In Figure 7.1, we display beta, Sharpe-ratio and turnover for target volatility funds with varying target volatility investing into the S&P 500. We use daily data for the period January 2003 to December 2013. The fund is rebalanced daily. Under a leverage constraint, nominal weights must not exceed 100%.

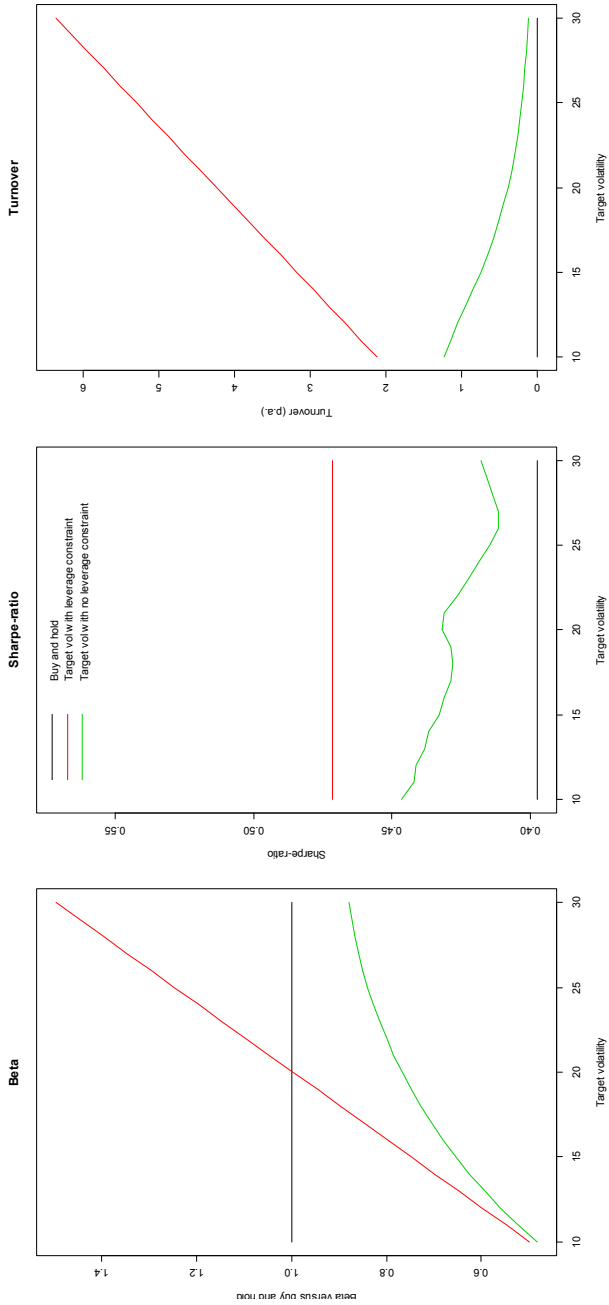


Figure 7.1. Target volatility and leverage

Let us illustrate this point using excess returns on the S&P 500. In contrast to the previous sections, we rebalance daily using a 20 day rolling realized volatility to construct the target volatility portfolio. Figure 7.1 displays the results where we compare symmetric and asymmetric target volatility (leverage constraints does not allow nominal weights to exceed 100%) to a buy and hold strategy. We can establish the following regularities:

1) Target volatility with leverage constraints runs at a lower beta than buy and hold. Only when target volatility becomes very large this beta gap narrows (as this forces target volatility to remain fully invested almost all the time).

2) For low target volatility, a leverage constraint has little impact on beta. This changes for high target volatilities. Investors need to ensure target volatility and leverage constraints are consistent.

3) Even target volatility without leverage constraint set at the long run S&P volatility of 21% will display a beta lower than 1 due to the imperfect correlation with the S&P.

4) Volatility scaling will lead to higher Sharpe ratios (unless low volatility is followed by low returns; empirical examples for this can be found but they are not the norm in equities)

5) Imposing leverage constraints on volatility targeting creates an asymmetry in using volatility information. This leads to a decrease in Sharpe ratio.

6) The decrease in Sharpe ratio is smaller, if volatility targets are low (i.e. the volatility target normal portfolio already contains cash, thereby reducing the above mentioned asymmetry)

As a consequence, investors should avoid imposing asymmetric leverage restrictions in order not to hurt long-run performance.

7.7. Target volatility across asset classes

So far, we have focused on (US) equity markets. How do our results carry over to global returns on equities, fixed income, commodities and currencies? For this, we use (excess) returns on 50 liquid futures contracts starting in January 1980 (where available) and ending in August 2013. First, we look at cross-sectional effect of volatility targeting in Figure 7.2. For assets with unconditional risk premium (equities and bonds), there is a significantly negative and tight relationship between volatility and Sharpe ratio. Low volatility assets offer high historical Sharpe-ratio assets. This is the empirical regularity risk parity investing leverages on (overweight high Sharpe ratio assets due to their low volatility). Its theoretical basis is the notion of leverage aversion by Frazzini and Pederson [FRA 10]. However, we also see that

this regularity does not carry over to assets with conditional risk premium (currencies and commodities)⁷. There the relation is positive and sketchy. Secondly, we investigate the impact of volatility scaling on the (time series) performance of target volatility strategies in Figure 7.3. The left hand panel displays the relation between buy and hold and target volatility performance (Sharpe-ratio) for realistic volatility forecasting (last month realized volatility as predictor of future volatility) while the right hand panel displays the less realistic case of perfect foresight volatility. We see that perfect volatility forecasts lead to substantial Sharpe-ratio increases while natural volatility persistence is not enough to engineer target volatility Sharpe-ratios larger than buy and hold strategies. Thirdly, we repeat predictive regressions where we use last month's realized volatility as a predictor of next month returns. Results from these regressions (t -values on the regression coefficient for realized volatility) are displayed in Figure 7.4. Only two contracts (natural gas and wheat) show significantly negative t -values (low volatility is followed by high returns) while six contracts display significantly positive t -values (low volatility is followed by low returns). In summary, we find little evidence that volatility targeting improves Sharpe-ratios. This is similar to our results on US equity markets.

In Figure 7.2, we display the relation between volatility and Sharpe-ratio for alternative asset classes. All calculations are based on monthly futures returns for the period January 1980 to August 2013 (subject to availability).

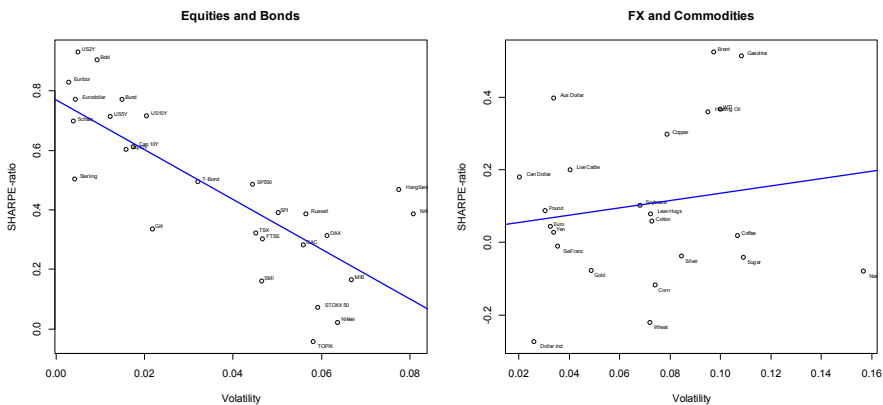


Figure 7.2. Cross-sectional relation between volatility and Sharpe-ratio

⁷ This should not surprise, as a conditional risk premium requires changing long short positions.

In Figure 7.3, we display the relation between Sharpe-ratios for target volatility versus buy and hold for alternative precision of volatility forecasts. The left hand panel displays the relation between buy and hold and target volatility performance (Sharpe-ratio) for realistic volatility forecasting (last month realized volatility as predictor of future volatility) while the right hand panel displays the less realistic case of perfect foresight volatility. All calculations are based on monthly futures returns for the period January 1980 to August 2013 (subject to availability).

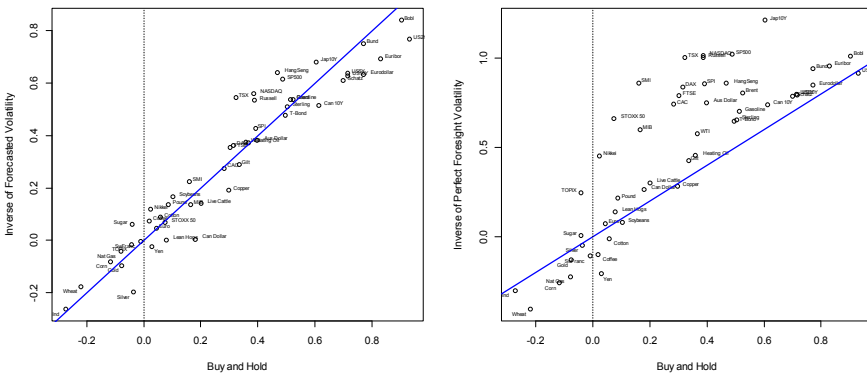


Figure 7.3. Volatility targeting versus buy and hold for forecasted versus perfect foresight volatility

In Figure 7.4, we display the t -values on the slope coefficient from a predictive regression relating month t volatility to month $t + 1$ returns. All calculations are based on monthly futures' returns for the period January 1980 to August 2013 (subject to availability).

7.8. Conclusions

At a minimum, volatility scaling to a target volatility offers – *cet.par.* – better diversification by avoiding random market timing and cutting down on tail risk. At the maximum, volatility scaling directly improves returns by creating informative leverage. So far the theoretical arguments. Empirically, we find little evidence for the assets under investigation that target volatility indeed improves performance. The notable exception is momentum investing for US Equities.

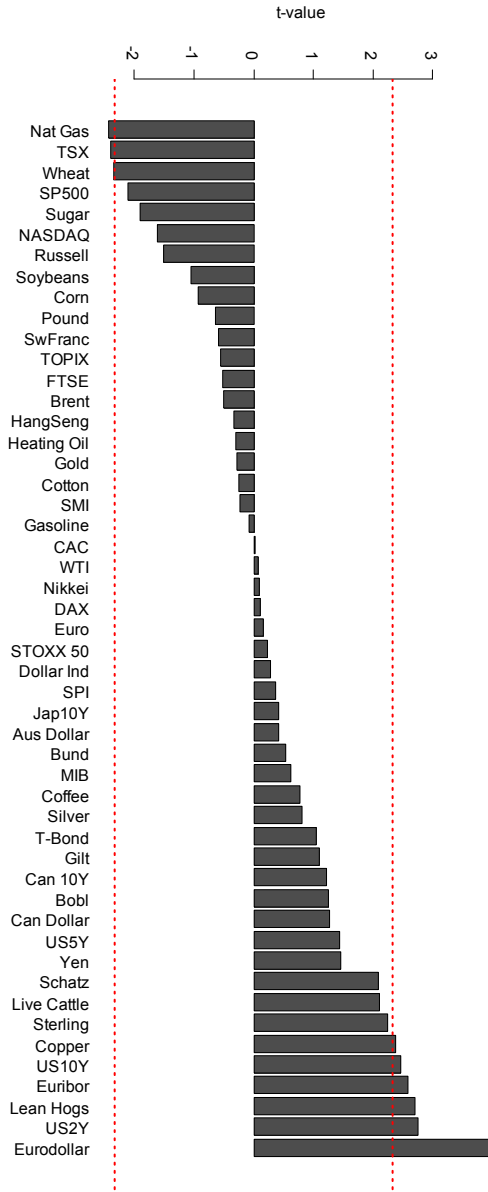


Figure 7.4. Predictive regressions

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Smart Beta Equity Investing Through Calm and Storm

Smart beta portfolios typically achieve a superior diversification than the benchmark market capitalization-weighted portfolio, but remain vulnerable to broad market downturns. We examine tactical investment decision rules to switch timely between equity and cash investments based on an underlying regime switching model with macroeconomic, macrofinancial and price momentum variables as drivers for the time-varying transition probabilities. A regression-based method is applied to select the relevant state variables. An extensive out-of-sample evaluation for the S&P 500 stocks over the period 1991–2014 shows the gains of smart beta portfolios, the usage of time-varying transition probabilities and the requirement that the expected return should exceed the time-varying threshold implied by a forward-looking extension of Faber's market timing strategy. The resulting investment decisions are more reactive to changes in the market conditions, tend to avoid equity investment in bearish market conditions and have a substantially better risk-adjusted performance and lower drawdowns.

8.1. Introduction

Tobin's separation theorem recommends investors to combine an investment in the maximum Sharpe ratio risky asset with an appropriate amount of cash. Based on the capital asset pricing model (CAPM) model, the investment community has traditionally put forward the market capitalization-weighted portfolio as the maximum Sharpe ratio portfolio. This traditional choice is increasingly criticized

because the underlying CAPM assumptions are often not met on real data. At the same time, a growing literature on smart beta portfolios shows that risk-based portfolios outperform the market capitalization-weighted portfolio in the long run in terms of risk-adjusted return [BAK 11].

Smart beta portfolios typically achieve a superior diversification than the benchmark market capitalization-weighted portfolio, but remain vulnerable to broad market downturns. This chapter proposes a pure quantitative tactical asset allocation framework that removes emotion and subjective decision-making by delegating the allocation decision to a simple rule: invest as long as the market is expected to be rising or flat and the estimated downside risk is below its maximum risk level. The predicted return and expected shortfall (ES) are computed under a regime switching model with an optimized set of macroeconomic, macrofinancial and price momentum variables driving the dynamic transition probabilities. The out-of-sample analysis examines the performance of the investment rules on the universe of S&P 500 constituents over the period 1991–2014 and compares the performance when the underlying equity strategy is the standard market capitalization-weighted portfolio with the alternative use of three smart beta equity investments, namely inverse volatility weighted, equally weighted and fundamental value-weighted portfolios invested in the S&P 500 constituents.

The proposed tactical allocation framework is closely linked to Faber’s [FAB 07] market timing model. It is founded on the time-tested intuition that market timing based on trend-following strategy is a risk-reduction technique that signals when an investor should exit a risky asset class in favor of risk-free investments. It exploits the information in the 10-month moving average price to stay invested in increasing markets and exit the risky investment in falling equity markets. It thus safeguards the investor against the well-known disposition effect of selling their winning investments too early and to hold on too long to their losing positions [SHE 85]. In addition, market timing can have a major impact on reducing volatility, avoiding large negative returns and (because of the asymmetric impact of large negative and positive returns on compound performance) it tends to lead to a higher long-term investment value. Similar results are obtained by Kritzman *et al.* [KRI 12] in an asset allocation framework, where they use a regime switching model for state variables and show that a tactical asset allocation strategy based on differentiating investments in high and low regimes of the state variables (e.g. financial stability vs. financial turbulence, high and low inflation, economic growth vs. recessions) leads out-of-sample to higher investment returns, lower volatility, lower value at risk (VaR) and a lower maximum drawdown, compared to constant mix strategies.

The proposed market timing model with downside risk control requires a forward-looking mean and risk estimate, which we obtain under parametric assumptions on the return generating process. An appropriate model for equity

returns needs to accommodate the stylized facts of time-varying volatility, skewness and kurtosis of stock returns. The workhorse method in applied financial time series is to use rolling estimation samples to accommodate the gradual changes in the return distribution. Such a method is, however, inherently slow in adapting to abrupt changes in the return distribution. These may happen when the economy's endowment switches between high and low economic growth [CEC 93], in case of asset pricing bubbles and collapses [BLA 82] or in times of transitions between exogenous and endogenous risk regimes [DAN 11]. In order to account for these sudden changes in the return distribution, we will use Hamilton's [HAM 89] regime switching model in which, conditionally on the regime, the stock return is normally distributed. As shown by Guidolin and Timmermann [GUI 08], such a regime switching model is able to accommodate fat tails and skewness in the return distribution, and since it is estimated on rolling estimation samples, it is also robust with respect to the more persistent volatility dynamics observed in return data.

In addition to the across-sample dynamics in the model parameters captured by the rolling sample estimation, we expect that the transition probabilities of the regime switching model change within the sample as a function of the changing market conditions. We follow the standard approach to generate dynamics in the transition probabilities by using lagged variables as the source of time variation [DIE 94, SCH 97]. These authors use one state variable. We will take a composite of state variables that we obtain based on an underlying linear regression model to forecast returns and where variable selection techniques are used to determine the relevant variables. The fitted return (a linear combination of the factors) is used as the state variable to drive the time-variation in the transition probabilities.

We then show how the predicted return and risk under the proposed regime switching mean-variance model for the smart beta equity returns can be used as input for tactical investment decision rules to switch timely between equity and cash investments based. The proposed market timing strategies can be interpreted as a forward-looking alternative for the backward-looking market timing model of Faber [FAB 07].

The empirical analysis on the S&P 500 stocks shows that, over the out-of-sample period 1991–2014, the inverse volatility-weighted portfolio offers the best risk-adjusted performance among all smart beta portfolios considered. This ranking is robust to the analysis that controls for the Fama–French–Carhart factor exposures, since we find that the alpha is the highest for the inverse volatility-weighted portfolio. Second and third best in terms of alpha is the equally weighted portfolio and fundamental value-weighted portfolio, while the market capitalization-weighted portfolio thus has the worst alpha of all portfolio weightings considered.

A common drawback of all pure equity portfolios is that they have large drawdowns that range between 36% for the inverse volatility-weighted portfolio and

58% for the fundamental value-weighted portfolio. We find that the active strategy of switching between the equity portfolio and the cash position (with switches driven by the regime switching model predicted return and risk of the equity investment and the use of forward-looking minimum required return target) reduces substantially the largest drawdown, which now ranges between 15 and 20%. It also improves the other performance measures in terms of a higher return, significantly lower volatility and lower downside risk than their equity-only counterparts. Finally, the analysis of the turnover shows that the gains from switching are high enough to compensate the transaction costs.

The rest of the chapter is organized as follows. Section 8.2 presents the regime switching model based-approach to market timing the smart beta equity investment. Section 8.3 describes the sample and variables used in the analysis. Our findings are presented in section 8.4. Finally, section 8.5 concludes. Section 8.7 contains technical details on the implementation of the regime switching investment model.

8.2. A regime switching approach to market timing

Faber [FAB 07] shows the good performance of a simple, but effective investment rule based on a trend signal extracted from rolling prices. We first review this approach and then present an alternative investment decision framework based on a regime switching model for the risky asset returns. We limit ourselves to the equity timing decision and do not consider approaches (such as volatility target or portfolio insurance investment strategies) that aim to construct an optimally weighted portfolio of cash and equity investment.

8.2.1. Faber's timing model based on rolling price averages

Faber [FAB 07] evaluates in detail the performance of a trend-following market timing model. He emphasizes that, in order to avoid behavioral bias in investment decisions, a quantitative investment model is needed. He recommends a trend-following model which invests in a risky asset over the period $[t-1, t]$ when the price of the risky asset at time $t-1$ is greater than the 10-month simple moving average. The condition that P_{t-1} exceeds the simple average of $P_{t-1}, P_{t-2}, \dots, P_{t-10}$ can be reverse engineered to find the lower bound constraint on the asset return in month $t-1$ needed for this condition to be satisfied. More precisely, it can be shown that the condition that P_{t-1} exceeds the simple average of $P_{t-1}, P_{t-2}, \dots, P_{t-10}$ is equivalent to requiring that the asset return in month $t-1$ exceed the following dynamic threshold:

$$\kappa_{t-1} = \frac{\frac{1}{9} \sum_{i=2}^{10} P_{t-i}}{P_{t-2}} - 1. \quad [8.1]$$

Otherwise, the investment is in cash. This dynamic threshold κ_{t-1} is shown in the short-dashed line in Figure 8.1. We see that in bullish markets, the threshold becomes negative, while in bearish markets the threshold becomes relatively high and the condition that r_{t-1} exceeds κ_{t-1} becomes thus very restrictive. By lowering the threshold in bullish equity markets, the probability of investing in equities increases, and *vice versa* in bearish markets. This explains why Faber's strategy is qualified as a trend-following investment strategy. Note in Figure 8.1 that the dynamic threshold κ_{t-1} of Faber [FAB 07] can become relatively extreme.

The advantage of Faber's momentum strategy is that it is also a risk management model. By avoiding the market downtrends, a significant reduction in volatility is achieved. Faber [FAB 07] shows that the performance gains in terms of lower volatility and higher return are not a result of data mining, since they are found for different markets including stocks, bonds, commodities and the real estate market, and over many time periods.

The disadvantage of Faber's momentum strategy is that it is too mechanical and does not exploit the information in state variables predicting the future return distribution. A solution for this is our timing model based on the predicted return distribution from a mean-variance regime switching model with state variables driving transition probabilities presented in the next section.

A further drawback is that the investment decision is backward-looking (invests in equity over the horizon $[t-1, t]$ if the past return r_{t-1} exceeds the market trend-following threshold κ_{t-1} defined in [8.1]). We will consider a forward-looking alternative, which invests in equity when the predicted return for the period $[t-1, t]$ ($\mu_{t|t-1}$) is such that the predicted stock price exceeds the (predicted) 10-month simple moving average, i.e. that $P_{t-1}(1+\mu_{t|t-1})$ exceeds the simple average of $P_{t-1}(1+\mu_{t|t-1}), P_{t-1}, \dots, P_{t-9}$. As such a forward-looking alternative for the threshold κ_{t-1} can be obtained by deriving the minimum expected return required for $P_{t-1}(1+\mu_{t|t-1})$ to exceed the simple average of $P_{t-1}(1+\mu_{t|t-1}), P_{t-1}, \dots, P_{t-9}$. In order to avoid extreme conditions, the obtained threshold is further truncated at -2 and $+2\%$. As such risky equity investments are avoided when the predicted return is less than -2% , while the equity investment is enforced when the predicted return is above 2% . More precisely, we invest in equities when the predicted return $\mu_{t|t-1}$ exceeds the forward-looking return target threshold $\tilde{\kappa}_{t-1}$ given by:

$$\tilde{\kappa}_{t-1} = \min \left\{ \max \left\{ \frac{\frac{1}{9} \sum_{i=1}^9 P_{t-i}}{P_{t-1}} - 1, -2\% \right\}, 2\% \right\}. \quad [8.2]$$

The resulting time series of forward-looking minimum return target is plotted in the long-dashed line in Figure 8.1. We see that during the bullish market, it usually stays at -2% and it is close to 2% in bearish markets. The corresponding

investment rule of switching from equities to the safe haven cash investment when $\mu_{r|t-1} < \tilde{k}_{t-1}$ protects the investor against the disposition effect, by stimulating the investment in bullish markets and enforcing the investor to cut losses in bearish markets.

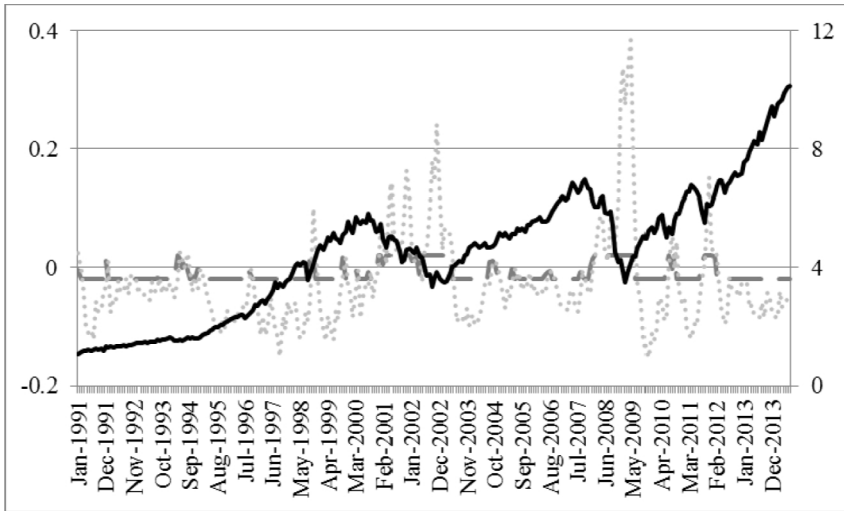


Figure 8.1. Time series plot of the monthly minimum return target of Faber [FAB 07] and the forward-looking return threshold in [8.2] when the corresponding equity investment is the market capitalization-weighted portfolio

COMMENT ON FIGURE 8.1.— The short-dashed line, with corresponding axis on the left-hand side, shows the time series of minimum required return rate under Faber’s strategy with the market capitalization-weighted portfolio as the risky asset, as defined in [8.1]. The long-dashed line presents the forward-looking return threshold for the expected return, as defined in [8.2]. The full line is the cumulative compound return of the market capitalization-weighted portfolio, with corresponding axis on the right-hand side.

8.2.2. Timing model based on the predicted return distribution from a mean-variance regime switching model with state variables driving the transition probabilities

We first introduce the model in sections 8.2.2.1–8.2.2.2 and then discuss the investment decision process in section 8.2.2.3.

8.2.2.1. Regime switching model for equity returns

Let y_t be the monthly stock return ($t = 1, 2, \dots, T$) whose distribution depends on a state variable S_t . Assume further that there are two states ($S_t=1$ for the first regime and $S_t=2$ for the second regime), and that, conditionally on the state S_t , the return distribution is normal with mean μ_{S_t} and variance $\sigma_{S_t}^2$:

$$y_t = \mu_{S_t} + \varepsilon_t; \varepsilon_t \sim iid N(0, \sigma_{S_t}^2). \quad [8.3]$$

The regimes can be interpreted as good and bad regimes. Note that, even though, conditionally on the regime, returns are normal, the unconditional distribution is often non-normal [ANG 12, MAR 92].

We follow Hamilton [HAM 08] and assume the states to be unobserved. Likelihood-based filters are used to infer the state S_t from the observed y_t 's under the additional assumption that the latent state variable S_t is a realization of a Markov chain with time-varying transition probabilities:

$$\Pr(S_t=j|S_{t-1}=i, S_{t-2}=k, \dots) = \Pr(S_t=j|S_{t-1}=i) = p_{ij,t}. \quad [8.4]$$

The specification in [8.4] assumes that the probability of a change in regime depends on the past only through the most recent regime. The properties of $p_{ij,t}$ are: $\sum_{j=1}^2 p_{ij,t} = 1$; and $p_{ij,t} \geq 0 \forall i, j \in \{1, 2\}$.

The diagonal elements of this matrix are parameterized using the logit transformation of an underlying time-varying process $c_1 + d_1 x_{t-1}$, $c_2 + d_2 x_{t-1}$ that depends on the state variables V_1, V_2, \dots, V_N through the following calibration: $x_{t-1} = \hat{h}_{t-1}(V_1, V_2, \dots, V_N)$ Diebold *et al.* [DIE 94]:

$$p_{11,t} = \exp(c_1 + d_1 \hat{h}_{t-1}(V_1, V_2, \dots, V_N)) / [1 + \exp(c_1 + d_1 \hat{h}_{t-1}(V_1, V_2, \dots, V_N))], \quad [8.5]$$

$$p_{22,t} = \exp(c_2 + d_2 \hat{h}_{t-1}(V_1, V_2, \dots, V_N)) / [1 + \exp(c_2 + d_2 \hat{h}_{t-1}(V_1, V_2, \dots, V_N))], \quad [8.6]$$

where the parameters c_j and d_j are the intercept and coefficient modeling the impact of the lagged value of the state variable on the time-varying transition probability for state j ($j= 1, 2$); the function \hat{h}_{t-1} corresponds to the estimated function of the future return y_t based on the linear regression model:

$$y_t = \beta_0 + \sum_{i=1}^N \beta_i V_{i,t-1} + \varepsilon_t, \quad [8.7]$$

where the variables in the prediction function are a subset of those in Table 8.1, selected as the subset having the lowest Bayesian information criterion (BIC). The predicted return under this selected model is used as the driver for the transition

probabilities, both in-sample and out-of-sample to forecast return and transition probabilities for the next period. As such, the time-variation in the transition probabilities is driven by the state of the financial and economic system. Of course, from [8.5] and [8.6], we have $p_{12,t} = 1 - p_{11,t}$, and $p_{21,t} = 1 - p_{22,t}$.

8.2.2.2. Estimation

The model parameters are estimated by maximum likelihood on rolling estimation samples of 6 years of monthly observations. For each estimation window, eight parameters are estimated by maximum likelihood techniques (the parameter vector $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, c_1, c_2, d_1, d_2)$ where μ_j and σ_j are the mean and volatility of the risky asset return in each regime, and c_j and d_j are the intercept and slope coefficient in time-varying transition probability matrix, respectively; 1, 2 stand for the good and bad regime)¹. The likelihood is calculated under the weighted likelihood approach in which the more recent observations receive a higher weight. The motivation is that relatively long estimation samples are needed to calibrate the regime switching model (6 years) and by using the exponentially decaying weights, the obtained predictions are more robust to the in-sample changes in the parameters. Following Meucci [MEU 13], the weights are defined as follows:

$$w_t = \frac{g_t}{\sum_{t=1}^{\bar{t}} g_t}, \quad [8.8]$$

$$g_t = e^{-\frac{\ln(2)}{\tau}|t-\bar{t}|}, \quad [8.9]$$

where \bar{t} denotes the most recent observation (72 in our research) and $\tau > 0$ is the half-life of the exponential decay, which we set to 36.

Based on the estimated parameters, the predicted probability that the smart beta return is in each regime in the next period can then readily be computed. Assume $\xi_{i,t-1|t-1}$ to be an inference of probability in regime i on date $t-1$ based on information up to date $t-1$ (obtained using the Hamilton's filter, as described in section 8.7), the predicted probability of regime j on date t is given by:

$$\xi_{j,t|t-1} = \sum_{i=1}^2 p_{ij,t} \xi_{i,t-1|t-1}, \quad [8.10]$$

where $p_{ij,t}$ denotes the transition probability from regime i to regime j on date t .

¹ The log-likelihood function of the regime-switching (RS) model can have local optima. In order to avoid the gradient-based optimization method to be stuck in a local optimum, we first use the stochastic global optimizer differential evolution [PRI 06] to find the best starting value. Based on these starting values, the log-likelihood function is then optimized using the quasi-Newton Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm, using transformations to ensure the parameters are well defined.

8.2.2.3. Investment decision

Our investment decision is based on two factors: predicted return under regime switching model and expected shortfall level. Under the two-regime specification, the predicted return is the weighted expected return of each regime, with weights equal to the predicted probability of each regime:

$$\mu_{t|t-1} = \sum_{j=1}^2 \xi_{j,t|t-1} \mu_j, \quad [8.11]$$

where $\xi_{j,t|t-1}$ is the predicted probability of regime j on date t based on the return information available at time $t-1$, and μ_j : mean of return under regime j .

The conditional density of the one-step ahead return is a mixture of two Gaussian densities with weights $\xi_{1,t|t-1}$ and $\xi_{2,t|t-1}$. We compute the expected shortfall under this mixture distribution, as detailed in section 8.7.

In our report, two types of market timing strategies are tested: Faber's market timing rule and the regime switching approach. Under Faber's market timing rule, the decision is to invest in the equity as long as its price level is greater than the 10-month simple moving average price. Otherwise, a bear market is detected and the portfolio is fully invested in London Interbank Offered Rate (LIBOR).

Under the regime switching approach, the objective is to invest in the risky equity asset as long as the market is expected to be rising or flat, i.e. when $\mu_{t|t-1}$ is above the minimum expected return target. Two types of return target will be tested: (1) the fixed return target at -2% and (2) the forward-looking return threshold $\tilde{\kappa}_{t-1}$ in [8.2], which is more restrictive in bearish markets ($\tilde{\kappa}_{t-1}$ is around 2%) than in bullish markets ($\tilde{\kappa}_{t-1}$ is around -2%). Otherwise, the portfolio is invested in 1-month LIBOR, in USD.

In addition, we consider the same market timing strategy, but with a downside risk management overlay. As in Yamai and Yoshida [YAM 05], we restrict ourselves to the equity market as long as the expected market return is above the minimum expected return and the expected shortfall at 95% is less than 10%. Otherwise, the portfolio is invested in the LIBOR. To summarize, we will consider three investment decisions based on the regime switching model (all of them have parameters estimated with the predictive log likelihood in which more recent observations have higher weights):

- 1) RS_static: market timing strategy based on the regime switching model with static transition probabilities;
- 2) RS_dynamic: market timing strategy based on the regime switching model with time-varying transition probabilities;

3) RS_EScontrol: market timing strategy based on the regime switching model with time-varying transition probabilities and with a control of downside risk (expected shortfall calculated at 95%).

8.2.3. Performance evaluation

For each time series of out-of-sample monthly return, we report five performance measures to describe the gross performance (before transaction costs): (1) annualized average return; (2) annualized volatility; (3) annualized Sharpe ratio; (4) maximum drawdown; and (5) historical expected shortfall at 95%. For the first three measures, we will test whether the observed differences are statistically significant².

In order to account for differences in transaction costs, we also report three descriptive measures on the frequency and cost of switching between the risky asset and cash, namely: (1) the average portfolio turnover; (2) the percentage of months for which the investment is in cash; and (3) the two-way break-even transaction cost. Similarly as in Chandrashekar [CHA 05] and Kritzman *et al.* [KRI 12], the break-even transaction cost is defined as the fee (expressed as a percentage of the amount traded) that makes the annualized Sharpe ratio of the out-performing strategy (in our case: the market timing portfolio) equal to the annualized Sharpe ratio of the benchmark portfolio (in our case: the buy-and-hold equity portfolio). Such a break-even transaction fee is of course only reported for the outperforming market timing strategies in terms of a higher Sharpe ratio with respect to the benchmark portfolio.

For the evaluation of the smart beta portfolios, we further control for style risk by computing the alpha of the portfolios using the Fama–French–Carhart four-factor model to decompose excess returns of the smart beta portfolios into its abnormal return component and the return explained by the exposure to the market, size, value and momentum factors. The estimated abnormal return α (alpha) is the least squares estimation of the intercept in the regression of the excess portfolio return (ER_t) on the four factors in Fama and French [FAM 92] and Carhart [CAR 97]:

$$ER_t = \alpha + \beta_1 MKT_t + \beta_2 SMB_t + \beta_3 HML_t + \beta_4 MOM_t + \varepsilon_t, \quad [8.12]$$

² For this, we follow Engle and Colacito [ENG 05] by testing significant differences between the monthly portfolio returns and squared returns using a Diebold and Mariano [DIE 95] type test. It regresses the monthly differences between the performance measures of two portfolio methods on a constant, and tests whether the estimated constant is significantly different from zero using a Newey–West standard error (that accounts for the serial correlation and heteroskedasticity in the return series). The significance of the difference in Sharpe ratios is evaluated using the test of Jobson and Korkie [JOB 81], Memmel [MEM 03] and Ledoit and Wolf [LED 08] in which Newey–West standard errors are also used.

where MKT_t denotes the market excess return on date t ; SMB_t is the size factor on date t (i.e. is the average return on the three small portfolios minus the average return on the three big portfolios); HML_t stands for the return on the book-to-market factor on date t (i.e. is the average return on the two value portfolios minus the average return on the two growth portfolios) and MOM_t is the return on the momentum factor on date t (i.e. is the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios) (see, for example, Bauer *et al.* [BAU 05] and Barber and Lyon [BAR 97]). Data of the four factors are for the U.S. stock market and obtained from the K. French data library. It is worth noting that alpha in [8.12] can typically be interpreted as a measure of out or underperformance relative to market proxy and the size, value and momentum risk factors. The statistical significance of the estimated alpha is evaluated using the t -test with Heteroskedasticity and Autocorrelation Consistent (HAC) standard errors.

8.3. Sample and variable description

8.3.1. Choice of risky asset

The investment universe consists of the S&P 500 stocks over the period January 1985–July 2014. Return data series of these stocks are computed using adjusted price data from COMPUSTAT. We will consider four types of equity weighting: the traditional market capitalization-weighted portfolio, the inverse volatility-weighted portfolio [LEO 12], the equally weighted portfolio [DEM 09] and the fundamental-weighted portfolio [ARN 05]. Below, we discuss each of them in detail:

– *Market capitalization-weighted portfolio.* The market capitalization represents a broadly invested portfolio, which has the advantage of a low turnover and can be interpreted as an equilibrium portfolio [PER 07]. The popularity of the market capitalization-weighted portfolio originates from the CAPM which states that the market capitalization-weighted portfolio is the maximum Sharpe ratio portfolio under very strict assumptions. In practice, the assumptions are often violated, leading to the observed mean-variance domination of the market capitalization-weighted portfolio over long evaluation windows [BAK 11].

– *Inverse volatility-weighted portfolio.* Among others, Baker and Haugen [BAK 12] and Dutt and Humphery-Jenner [DUT 13] show that low volatility stocks earn higher returns compared to high volatility stocks. They find that this phenomenon is consistent both over time and over different markets. In our application, we follow the S&P Dow Jones index [DOW 14] methodology and compute the low risk portfolio as the inverse volatility-weighted portfolio invested in the 100 least volatile stocks. The volatilities are estimated over a 252-day moving window.

– *Equally weighted portfolio*. The equally weighted portfolio represents a naively diversified portfolio, in which all assets (whatever their size, value or risk characteristics) receive the same weight. While the equally weighted portfolio is highly diversified in terms of weights, the diversification in terms of risk is often limited.

– *Fundamental value-weighted portfolio*. Arnott *et al.* [ARN 05] popularized the approach of fundamental value weighting (also called fundamental indexation (FI)). Under this approach, assets are selected and weighted based on fundamental metrics of company size. In our application, we consider similar fundamental characteristics as Arnott *et al.* [ARN 05], namely: book value of common equity, dividends, net operating cash flow³ and sales. Each of them is computed on the basis of 5-year rolling averages. The metrics are aggregated into a composite weight by: first, dividing each metric by the total value of the metrics over all firms in the universe (standardization) and then taking the average value of these weights. The fundamental data are retrieved from the COMPUSTAT database on an annual basis from 1984 to 2014. The metrics are lagged by one quarter to ensure data availability.

8.3.2. Variables in the multivariate regression model

Table 8.1 shows the different state variables that we consider as possible drivers for the time-varying transition probabilities. Consistent with previous research of Giot and Petitjean [GIO 11], De Boer and Norman [DEB 14] and many other researchers, we classify them into three groups: macroeconomic variables, macrofinancial variables and price momentum variables. The first two groups are exactly the same among smart beta portfolios, while the last group is the momentum specific to each of them. All variables considered are required to be prespecified and do not depend on the estimated regime switching model. This excludes the use of duration as a driver for time-varying transition probabilities [MAH 00]. All variables are considered at their end-of-month value. To avoid look-ahead bias, all variables series used in multivariate regression model are always lagged by a month compared to the time series of monthly returns⁴.

³ We follow Kothari *et al.* [KOT 05] in defining net operating cash flow as the difference between the operating income before depreciation and total accruals. The accrued liabilities at time t are the change in current assets minus the change in cash and short-term investments, minus the change in current liabilities excluding long-term debt minus the amount of depreciation and amortization scaled at the lagged value of total assets.

⁴ The macroeconomic and macrofinancial variables are retrieved from the Website of the Federal Reserve Bank of St. Louis. The CAY series is downloaded from http://faculty.haas.berkeley.edu/lettau/data_cay.html and the CAPE series is downloaded from Shiller's Website at <http://www.econ.yale.edu/~shiller/data.htm>.

The estimation window used for the multivariate regression is also 6 years (72 monthly observations) and matches thus with the in-sample window of the regime switching model. Based on real data availability of all series, some of them miss data before 1990. Any series that has a missing observation in the 72-month rolling sample will be omitted. To reduce the impact of outliers, the state variables are winsorized at a lower and upper bound corresponding to their in-sample median \pm two times the median absolute deviation. In each estimation window, the most predictive variables are selected according to the BIC [KON 08]. Then, the predicted return from the selected regression model is used as the state variable driving the time-varying transition probabilities of the regime switching model.

8.4. Results

In this section, first, we present the results of the impact of the choice of portfolio weighting method (market capitalization, inverse volatility, equal and fundamental value weighting schemes) on the out-of-sample performance on the universe of S&P 500 stocks. We will show that these portfolios still suffer from extreme downside risks. Then, we will analyze market timing strategies using cash (1-month LIBOR-based USD) as a safe haven investment in case the trend and/or risk signals indicate that the equity investment is estimated to be decreasing in value and/or is too risky.

Variable description	References
<i>Macroeconomic variables</i>	
Real GDP growth: y-o-y change of real GDP	[DEB 14]
Inflation: y-o-y change of consumer price index (CPI)	[DEB 14, EST 98, FAM 77, FAM 90]
PMI: purchasing manager index	[EST 98, WOL 04]
<i>Macrofinancial variables</i>	
TED spread: difference between 3-month USD Libor rate and 3-month T-Bill yield	[BOU 12, DEB 14]
Short-term yield: the difference between short-term interest rate (3-month T-bill yield) and a 12-month backward-looking moving average	[EST 98, FAM 90, GIO 11]
Long-term yield: the difference between long-term government bond yield (10-year bond) and a 12-month backward-looking moving average	[EST 98, FAM 90, GIO 11]

Term spread: logarithm of difference between long-term government bond yield and short-term interest rate	[EST 98, GIO 11, FAM 90]
Credit spread: difference between Moody's seasoned Baa corporate bond yield and Fed fund rate	[GIL 09, NOR 09]
VIX: CBOE's implied volatility index of S&P 500 index options	[BOU 12, GUO 06]
Monthly change of VIX index	[BOU 12]
Variance risk premium: the difference between implied variance and realized variance, where implied variance is the squared return of VIX and realized variance is the squared of last 3-month return of SP500	[BOL 09]
CAY: consumption wealth ratio is defined as the log of consumption in the US divided by aggregate wealth	[LET 01, BOL 09]
CAPE: cyclically adjusted price-earnings: price level of the S&P500 divided by the average of earnings adjusted for inflation	[TAB 11]
Fear index: the difference between VIX data and annualized standard deviation of 3-month daily returns	[SHA 14]
<i>Price momentum variables</i>	
1-month return	[VAN 99]
Rolling 3-month average returns	[VAN 99]
Price spread: difference between the current price and its simple moving average over 10 months	[FAB 07]

Table 8.1. State variables used in the predictive return model to construct the composite state variable driving the time-variation in the transition probabilities

Invest 100% in equity if	Benchmark (buy-and-hold)	Faber's strategy ($r_{t-1} > \kappa_{t-1}$)	RS_static		RS_dynamic		RS_EScontrol	
			$\mu_{t t-1} > -2\%$	$\mu_{t t-1} > \kappa_{t-1}$	$\mu_{t t-1} > -2\%$	$\mu_{t t-1} > \kappa_{t-1}$	$\mu_{t t-1} > -2\%$	$\mu_{t t-1} > \kappa_{t-1}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Market capitalization weighted portfolio								
Ann.ret	10.32%	11.13%	9.77%	11.40%	11.45%	11.67%	10.45%	10.71%
Ann.sd	14.53%	***10.41%	13.58%***	***10.65%	***12.11%**	***10.45%	***11.27%	***10.26%
Sharpe	0.71	*1.07	0.72*	*1.07	0.95	**1.12	0.93	1.04
Max.DD	50.05%	17.03%	42.68%	23.72%	32.21%	16.40%	32.21%	16.40%
His.ES	9.30%	5.95%	8.83%	6.16%	7.90%	6.37%	7.38%	6.34%
Turnover	NA	4.24%	1.41%	4.95%	8.83%	8.83%	7.42%	8.48%
Percentage_in_cash	NA	22.26%	3.89%	20.85%	22.26%	30.39%	31.8%	37.1%
Breakeven transaction cost (two-way)								
	NA	3.13%	0.30%	2.71%	1.17%	1.73%	1.23%	1.49%
Inverse volatility weighted portfolio								
Ann.ret	11.45%	11.39%	11.45%	10.81%	11.87%	11.72%	12.24%	12.08%
Ann.sd	10.92%	***8.53%	10.92%***	***8.86%**	**10.28%***	***8.42%	**10.12%**	***8.24%
Sharpe	1.05	1.34	1.05	1.22	1.15	*1.39	1.21	**1.47
Max.DD	36.26%	16.22%	36.26%	23.24%	36.26%	15.00%	36.26%	15.00%
His.ES	6.64%	4.46%	6.64%	4.88%	6.18%	4.39%	6.18%	4.38%
Turnover	NA	4.59%	0%	4.95%	3.89%	7.42%	5.30%	8.83%
Percentage_in_cash	NA	17.67%	0%	16.25%	6.01%	20.85%	9.19%	24.03%
Breakeven transaction cost (two-way)								
	NA	1.82%	NA	1.10%	1.03%	1.32%	1.15%	1.37%
Equally weighted portfolio								
Ann.ret	12.94%	11.19%	11.77%	10.71%	14.73%*	13.42%	11.09%	11.74%
Ann.sd	16.51%	***11.47%	15.18%***	***11.66%**	**14.12%**	***12.08%	***12.8%	***11.67%
Sharpe	0.78	0.98	0.78	0.92	1.04	1.11	0.87	1.01
Max.DD	55.60%	19.88%	41.47%	19.88%	38.35%	19.88%	29.79%	19.88%
His.ES	10.65%	6.36%	9.79%	6.74%	7.75%	6.10%	7.22%	6.05%
Turnover	NA	6.71%	2.12%	7.24%	7.77%	10.78%	9.36%	11.48%
Percentage_in_cash	NA	19.43%	3.89%	18.73%	16.25%	25.8%	29.68%	34.98%
Breakeven transaction cost (two-way)								
	NA	1.21%	NA	0.82%	1.72%	1.39%	0.44%	0.88%
Fundamental value weighted portfolio								
Ann.ret	11.73%	11.72%	10.82%	12.02%	12.30%	13.23%	11.29%	12.77%
Ann.sd	15.42%	***10.43%	13.83%***	***10.4%	**12.09%*	***10.54%	***11.42%	***10.28%
Sharpe	0.76	*1.12	0.78	*1.16	1.02	**1.26	0.99	**1.24
Max.DD	58.72%	21.18%	40.86%	15.13%	29.04%	15.13%	32.39%	15.13%
His.ES	10.30%	5.82%	9.07%	5.73%	7.03%	5.64%	6.87%	5.51%
Turnover	NA	5.30%	1.41%	5.12%	5.65%	6.36%	7.07%	7.07%
Percentage_in_cash	NA	19.43%	3.89%	19.08%	18.37%	25.09%	28.27%	32.51%
Breakeven transaction cost (two-way)								
	NA	2.46%	0.77%	2.82%	1.95%	2.91%	1.35%	2.52%

Table 8.2. Out-of-sample performance of different smart beta portfolios and portfolios applying market timing strategies

COMMENT ON TABLE 8.2.— Out-of-sample period from January 1991 to July 2014. ***, ** and * on the left-hand side: comparison between market timing portfolios and the benchmark (column 2) at 1, 5 and 10%, significance level, respectively; ***, ** and * on the right-hand side: comparison between the market timing portfolios using regime switching model versus the market timing portfolios using Faber’s strategy (column 3) at 1, 5 and 10% significance level, respectively; expected shortfall (ES) is the historical ES estimate calculated at 95%; portfolio turnover of switching between risky asset and cash (turnover) in the out-of-sample period (283 months); `percentage_in_cash`: number of months in cash in the out-of-sample period; break-even transaction cost is calculated on the basis of two-way rule and defined as the rate that makes annualized Sharpe ratio of the market timing portfolio and the benchmark indifferent; NA: not applicable.

8.4.1. Impact of choice of smart beta equity strategies on portfolio performance

How does the equity weighting affect the portfolio performance? This is the question we investigate in the second column of Table 8.2. We find a clear confirmation of the low risk anomaly: the inverse volatility-weighted portfolio has a higher annualized return and lower annualized volatility than the market capitalization-weighted portfolio (annualized return: 11.45 vs. 10.32% and annualized volatility: 10.92 vs. 14.53%). Its downside risk (maximum drawdown and historical expected shortfall) is also lower. This can also be seen in Figure 8.2 where we report the histogram of the negative portfolio returns. The black bars correspond to the buy-and-hold portfolios. Note that the inverse volatility-weighted portfolio has more small monthly losses (e.g. [−2%, 0%), [−4%, −2%]) and less extreme monthly losses (e.g. a monthly return below −10%) than the market capitalization-weighted portfolio. These results are similar to those reported by Baker *et al.* [BAK 11], Baker and Haugen [BAK 12] and Leote De Carvalho *et al.* [LEO 12].

The equally weighted portfolio has the highest annualized return of all four weighting strategies (12.94%). It turns the initial investment of \$1 in January 1991 into \$17.65 at the end of July 2014, compared to \$10.14, \$12.88 and \$13.68 for the market capitalization weighted, inverse volatility weighted and fundamental value-weighted portfolio. In terms of risk, the equally weighted and fundamental value-weighted portfolios have a higher annualized volatility than the market capitalization-weighted portfolio (16.51 and 15.42% vs. 14.53%). Also, their maximum drawdown levels are higher than the one of the market capitalization-weighted portfolio (55–58% vs. 50%). The higher risks of extreme negative returns for these portfolios are illustrated in Figure 8.2 where we show the histogram of negative returns for the equity-only investment in the first column of each block. We see that extreme losses of equal

weight and fundamental value-weighted portfolio are more frequent than that of the market capitalization and inverse volatility-weighted portfolio.

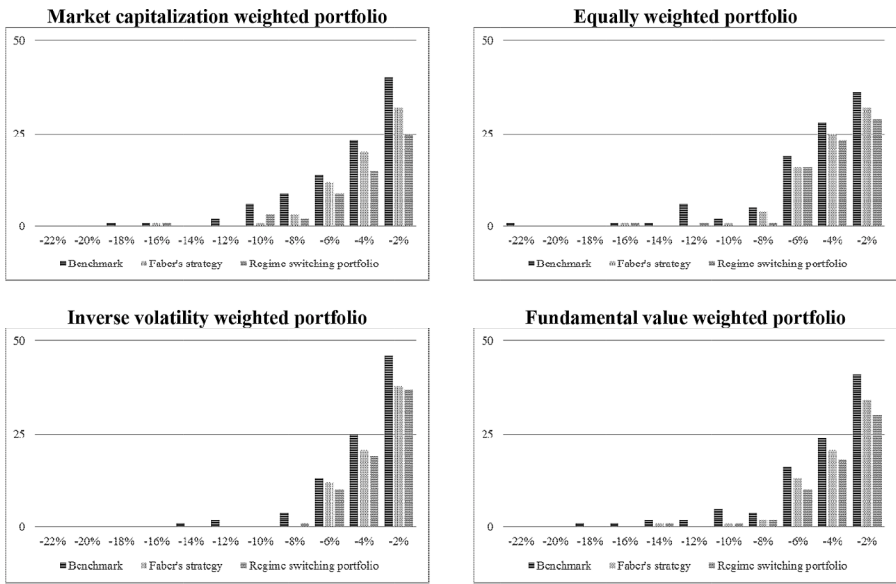


Figure 8.2. Histogram of monthly negative returns for the equity-only benchmark portfolios, as well as the switching portfolios using rolling price averages (Faber’s strategy) and the market timing portfolio using the regime switching model with dynamic transition probabilities and the forward-looking return target

COMMENT ON FIGURE 8.2.– In the charts, the frequency of the left tail of the return distribution is shown, on the grid $[-22\%, -20\%)$, $[-20\%, -18\%)$, ..., $[-2\%, 0)$, respectively. The regime switching portfolios shown in the four charts are those corresponding to market timing based on the RS model with dynamic transition probabilities and the forward-looking return threshold (RS_{dynamic}).

Portfolio	Market cap. Weighted	Inverse volatility weighted	Equally weighted	Fundamental value weighted
(1)	(2)	(3)	(4)	(5)
Alpha	0.0432	0.2226**	0.1715**	0.1029**
(Se)	(0.0323)	(0.1129)	(0.0682)	(0.0454)

Table 8.3. Alpha of smart beta portfolios obtained under the Fama–French–Carhart four-factor model

COMMENT ON TABLE 8.3.— Alpha and HAC standard error (denoted as Se and put in parentheses) are multiplied by 100. ***, **and * indicate significance at the 1, 5 and 10% level, respectively.

The return outperformances of smart beta portfolios are confirmed by the alpha-performance (which corrects for the exposure of the investments with respect to the Fama–French–Carhart four-factor model) presented in Table 8.3. They show that both three smart beta portfolios have significantly positive alpha after adjusting for both four risk factors. Their alphas range from 0.1–0.2% and are significant at the 5% level. The alpha of inverse volatility-weighted portfolio is the highest among the three smart beta portfolios considered.

Overall, we see that smart beta portfolios achieve higher returns than the market capitalization-weighted portfolio. But, they remain vulnerable to broad market downturns, as they still have large drawdowns (36–58%). Let us now investigate whether market timing reduces these drawdowns and whether there is a cost in terms of expected returns to be paid.

8.4.2. Impact of market timing strategies on portfolio performance

Is it possible to increase the performance both in terms of returns, risk and drawdown by switching the portfolio allocation between equities and a cash investment? This is the next question that we study. To answer this question, first we will present performance of portfolios using Faber’s method and then consider six market timing investment strategies based on the expected return (and risk) of three estimated RS models with two types of return target. The first RS model-based investment assumes static transition probabilities and is denoted by RS_static. The second one (RS_dynamic) innovates by considering time-varying transition probabilities based on a composite index of state variables. RS_static and RS_dynamic invest in equities when the market is flat or rising. The third model is based on the same model as RS_dynamic but adds an additional risk control layer in the investment and invests in equities only if the market is flat or rising and the expected shortfall calculated at 95% is below 10% (RS_EScontrol). For each of these three regime switching strategies, two types of minimum return target are studied: a fixed return target of –2% and a forward-looking (time varying) return target $\tilde{\kappa}_{t-1}$ in [8.2]. The main results are reported in Table 8.2, where we report the different return and risk performance measures on a gross performance basis. In order to describe the corresponding transaction costs, we also calculate the statistics on the frequency and persistence of the switches between equities and cash. More precisely, the last three rows in Table 8.2 report: (1) the portfolio turnover, (2) the percentage of months the strategy invests in cash (1-month LIBOR-based USD) and (3) break-even transaction cost (two-way) that equalizes annualized Sharpe ratio of

the market timing strategy to the benchmark (buy-and-hold strategy) during the out-of-sample period (1991–2014).

8.4.2.1. Market timing based on Faber's method using rolling price averages

The performance of the portfolios using Faber's method is presented in column three of Table 8.2. Regarding the market capitalization-weighted portfolio, the portfolio using Faber's method has a higher annualized return (11.13 vs. 10.32%), significantly lower annualized volatility (10.41 vs. 14.53%), significantly higher Sharpe ratio (1.07 vs. 0.71) and lower drawdown (maximum drawdown is 17 vs. 50% of the benchmark). This result is in line with the research of Faber [FAB 07]. When Faber's method is applied to the three smart beta portfolios (inverse volatility, equal and fundamental value-weighted portfolio), the resulting market timing portfolios have slightly lower annualized returns (11.19–11.72% vs. 11.45–12.94%), but also significantly lower annualized volatilities (8.53–11.47% vs. 10.92–16.51%), substantially higher Sharpe ratios (0.98–1.34 vs. 0.76–1.05) and much lower drawdowns (maximum drawdowns between 16 and 21% vs. 36 and 55%) than the buy-and-hold benchmarks.

In terms of portfolio stability, we find that the portfolios applying Faber's strategy have a turnover between 4.2% (the market capitalization-weighted portfolio) and 6.7% (the equally weighted portfolio). Overall, the portfolio is invested in cash for 20% of all months. The better risk-adjusted performance in terms of Sharpe ratio is robust to transaction costs, since the break-even transaction cost that makes their Sharpe ratio equal to the Sharpe ratio of the corresponding buy-and-hold benchmark is as high as 1.21–3.13% per transaction.

Overall, using Faber's method, portfolios have similar returns and much lower volatilities and drawdowns. By consequence, the risk-adjusted returns are all higher with respect to the benchmarks. In the next section, we investigate whether the market timing strategies using the regime switching models improve the portfolio performance.

8.4.2.2. Market timing based on three different regime switching models

In this section, we study the impact of the design of the regime switching-based investment on the portfolio performance.

8.4.2.2.1. Model 1: the constant (static) transition probabilities regime switching model

The results for the regime switching model with static transition probabilities and a static -2% target in terms of minimum required return are reported in column four in Table 8.2. Note that the RS portfolios using static transition probabilities have

similar gross performance to the benchmarks (the equity-only investment in column two in Table 8.2) in terms of a similar annualized return, volatility and Sharpe ratios. The similar performance is due to the low turnover value (0–2.12%) and low percentage of time investing in cash (less than 4%). For the inverse volatility-weighted portfolio, there is even no switch at all using the static transition probabilities regime switching model. The net performance after transaction costs measured by break-even transaction cost is only applicable for the market capitalization-weighted portfolio (0.3%) and the fundamental value-weighted portfolio (0.77%). In comparison with Faber’s method, portfolios using this market timing strategy underperform (annualized returns are lower, volatilities are significantly higher and Sharpe ratios are also lower). Overall, the market timing strategy using static transition probabilities and a fixed -2% return threshold only slowly reacts to the market changes.

A solution is to apply the forward-looking return target $\tilde{\kappa}_{t-1}$ as can be seen in column five in Table 8.2. Portfolios using static transition probabilities regime switching model then have better performance than using the fixed return target in terms of risk-adjusted return and drawdowns. The impact on annualized return is mixed (higher for the market capitalization portfolio and fundamental value-weighted portfolio but lower for others) but the annualized volatilities are significantly lower and the Sharpe ratios are substantially higher. Also, the drawdowns are improved compared to the buy-and-hold in the equity benchmarks. The higher Sharpe ratios are calculated on the gross return basis. From the large values for the break-even transaction costs for all four equity indices (0.82–2.82%), it can be expected that the out-performance in terms of Sharpe ratio also exists when computed on the portfolios’ net returns (after transaction costs). The risk-adjusted returns (Sharpe ratios) are nevertheless lower than those using Faber’s method.

The results of regime switching model with static transition probabilities are, therefore, not satisfactory yet. It motivates us to study the usage of dynamic transition probabilities in regime switching model to do market timing.

8.4.2.2.2. Model 2: time-varying (dynamic) transition probabilities regime switching model

As described in section 8.3.2, we consider 17 candidate state variables as drivers for the transition probabilities. Before analyzing the performance of the market timing strategy with dynamic transition probabilities regime switching model, we investigate which variables are selected as drivers for time-varying transition probabilities and when.

Variables selected as drivers for the transition probabilities

It is natural to expect that the variable selection will depend on the market regime. In Table 8.4, we show, for the market capitalization-weighted portfolio, the obtained variable selection in terms of number of months for the bullish and bearish periods in our sample, January 1991–July 2014⁵.

Period	January 1991- June 1998	July 1998- August 1998	September 1998- August 2000	September 2000- September 2002	October 2002- October 2007	November 2007- February 2009	March 2009- July 2014	Full sample
Duration (in months)	90	2	24	25	61	16	65	283
Macroeconomic variables								
Real GDP growth	3	0	1	19	13	12	16	64
Inflation	6	0	9	3	1	9	24	52
PMI	3	0	0	0	15	2	4	24
Macrofinancial variables								
TED spread	18	0	0	9	6	0	1	34
Short-term yield	0	0	4	3	15	0	23	45
Long-term yield	13	0	1	0	6	0	5	25
Term spread	9	1	1	7	10	4	11	43
Credit spread	13	1	18	0	2	5	37	76
VIX	0	1	20	8	35	0	0	64
Change of VIX	0	0	0	0	1	0	56	57
Variance risk premium	0	0	0	0	0	0	49	49
CAY	6	1	21	23	58	12	34	155
CAPE	33	0	21	23	57	16	52	202
Fear index	6	0	3	0	26	9	63	107
Price momentum variables								
1-month return	16	0	1	9	1	0	1	28
Average of 3-month return	0	0	1	0	0	1	0	2
Price spread	44	2	8	16	3	2	45	120

Table 8.4. Variable selection in the multivariate regression model that predicts the market capitalization-weighted portfolio returns based on lagged state variables. For each variable, the number of months the variable is selected is reported

⁵ We only present the table of variable selection for the market capitalization-weighted portfolio. Those of other equity portfolios are similar and available upon request.

COMMENT ON TABLE 8.4.— The order of variables matches with the order in Table 8.1; the second, fourth and sixth periods are bearish and others are bullish; variables are selected in terms of BIC.

We follow Ellis [ELL 05] in defining a bearish period for the US equities market as the period over which the S&P 500 index declines by more than 12%. This leads to three bearish equity market regimes: July 1998–August 1998; September 2000–September 2002 and November 2007–February 2009 with compound losses of 15.32%, 42.68% and 50.05%, respectively. Table 8.4 shows that during the bullish periods, macrofinancial variables and price momentum variables are mostly selected, while the group of macroeconomic variables appears in the selection during the bearish periods (September 2000–September 2002 and November 2007–February 2009), particularly the real GDP growth variable in this group is usually selected. Two variables (CAPE and CAY) in the group of macrofinancial variables are actively selected in all periods. Overall, macrofinancial variables (consumption wealth ratio (CAY), cyclically adjusted price-earnings (CAPE) and fear index) and price momentum variables (price spread) play the most important roles in predicting stock return.

Performance gains of market timing strategy using time-varying (dynamic) transition probabilities

Let us now return to the key question of this part: is the strategy of dynamic transition probabilities regime switching better than the buy-and-hold strategy and Faber's strategy in terms of improving the portfolio performance? We find that, compared to the benchmark (column two in Table 8.2), the portfolios that invest based on the expected return under the regime switching model with dynamic transition probabilities regime switching model and the fixed return target (column six in Table 8.2) perform better than the benchmarks with insignificantly higher annualized returns (0.5–3%) and significant lower annualized volatility (0.6–3% at 1 and 5% significant level), their Sharpe ratios are insignificantly higher than the benchmarks. Their downside risks are also lower than the benchmarks (maximum drawdowns are in the range of 29–38% vs. 36–58%) (except for the inverse volatility-weighted portfolio, their maximum drawdown is 36%, the same as the benchmark). The outperformance in terms of Sharpe ratios on gross returns is expected to be robust to transaction costs, since the corresponding break-even transaction costs are relatively high (1.03–1.95%). Compared to the market timing strategy of Faber, portfolios using this strategy have higher annualized return as well as significantly higher annualized volatilities, so their Sharpe ratios are indeed lower than those using Faber's strategy (except for the equally weighted portfolio). In addition to the higher volatility, they also have higher average portfolio turnovers. Therefore, their break-even transaction cost values are usually lower than those using Faber's strategies (1.03–1.95% vs. 1.21–3.13%).

Column seven in Table 8.2 shows the performance of the portfolios invested in equities based on the forward-looking return target in [8.2]. These portfolios consistently yield higher annualized returns, significantly lower annualized volatilities, significantly higher Sharpe ratios and lower drawdowns than the corresponding buy-and-hold benchmarks. In comparison with portfolios using Faber's strategy, these portfolios always have higher annualized returns (11.67–13.42% vs. 11.13–11.72%), higher Sharpe ratios (1.11–1.39 vs. 0.98–1.34), comparable volatility (8.42–12.08% vs. 8.53–11.47%) and slightly lower drawdowns (maximum drawdown of 15–19.88% vs. 16.22–21.18%). The relative outperformance in terms of gross returns comes at the price of higher turnover than in Faber's strategy (6.36–10.78% vs. 4.24–6.71%). The relative performance as measured by the break-even transaction costs depends on the equity index. For the market capitalization and inverse volatility-weighted portfolio, the break-even transaction costs are lower for the regime switching model than for Faber's timing model (1.73 vs. 3.13% and 1.32 vs. 1.82%). For the equally weighted and fundamental value-weighted portfolio, the reverse is observed.

The bottom line of this analysis is to confirm that the usage of dynamic transition probabilities regime switching and the forward-looking return target improves the portfolio performance compared to the buy-and-hold strategy (both gross and net performance) and Faber's strategy (especially the gross performance). Next, we will investigate the benefits of risk control in the market timing strategy using dynamic transition probabilities regime switching model.

8.4.2.2.3 Model 3: time-varying (dynamic) transition probabilities regime switching with a risk control model

We have seen the benefit of using dynamic transition probabilities in market timing strategy in the previous sections. In this part, we will investigate the benefits of including also a limit on downside risk in the investment by answering the question: does the market timing strategy using the dynamic transition probabilities perform better than the buy-and-hold strategy and Faber's strategy?

First, we will check this strategy with the usage of fixed return target. Compared to the benchmarks, the portfolios using dynamic transition probabilities regime switching with the control of expected shortfall (column eight in Table 8.2) have mixed results of annualized returns and significant lower annualized volatilities than the benchmarks. Their downside risks are also lower than the benchmarks. Since the break-even transaction costs are relatively high (0.44–1.35%), it can be expected that the outperformance with respect to the buy-and hold benchmarks subsists after transaction costs. Compared to the competing market timing portfolio based on Faber's rule, the strategy based on the regime switching model underperforms in

terms of lower returns (except for the inverse volatility-weighted portfolios), lower Sharpe ratios, higher volatilities and drawdowns and a higher turnover.

Let us now study the improvements in performance obtained when, instead of investing based on comparing the predicted mean $\mu_{t|t-1}$ with the fixed level of -2% , we compare the conditional mean $\mu_{t|t-1}$ with the forward-looking return threshold ($\tilde{\kappa}_{t-1}$). The resulting performance is shown in column nine of Table 8.2. Using this strategy, portfolios usually have higher annualized returns (except for the equally weighted portfolio), higher Sharpe ratios and lower annualized volatility and drawdowns than the benchmarks. Their performance is slightly better than those using Faber's strategy in terms of higher annualized returns (except for the market capitalization-weighted portfolio), higher Sharpe ratio (1.01–1.47 vs. 0.98–1.34) and lower annualized volatility (8.24–11.67% vs. 8.53–11.47%), drawdowns (15–19.88% vs. 16.22–21.18%). In terms of portfolio turnover, this strategy has much higher turnovers (7.07–11.48% vs. 4.24–6.71%) and slightly higher Sharpe ratios. As a result, the break-even transaction costs (which make the Sharpe ratios on net returns equal to those of the buy-and-hold benchmarks) are lower than Faber's strategy (except for the fundamental value-weighted portfolio: 2.52 vs. 2.46%).

Compared to portfolios using strategy without risk control (column seven in Table 8.2), portfolios with risk control have lower volatility. These results are expected as this market timing strategy focuses on controlling potential risk. The lower risk comes at the expense of lower average portfolio returns (except for the inverse volatility-weighted portfolio where the return is higher: 11.72 vs. 12.08%). Note also that the risk-controlled strategy tends to be more frequently invested in cash.

In summary, we find first that the usage of the forward-looking return target is better than the fixed one. Second, among the three regime switching-based investment models considered (static transition probabilities, dynamic transition probabilities and dynamic transition probabilities with a risk control), the market timing strategy using dynamic transition probabilities performs best. It yields higher annualized returns, Sharpe ratios and lower drawdowns with respect to the simple trend-following rules based on moving average price of Faber's strategy. We thus find that, for our sample of S&P 500 stocks over the period 1991–2014, the complexity of the regime switching model-based tactical allocations seems to pay off compared to the computationally more simple strategy of Faber [FAB 07].

To get an insight into the superior performance of the market timing strategy using dynamic transition probabilities regime switching model (with forward-looking return threshold) over Faber's strategy, Figure 8.3 compares the buy-and-hold cumulative returns for the four equity indices (market capitalization, inverse volatility, equally and fundamental value-weighted portfolio) with the timing

decisions by showing their relative performance versus the benchmarks of these two strategies. It shows that these two strategies make different timing decisions. The strategy using dynamic transition probabilities regime switching model (with forward-looking return threshold) usually has more timely reactions to the market change during down-trend markets, leading to their superior return performance compared to Faber’s strategy over the whole period.

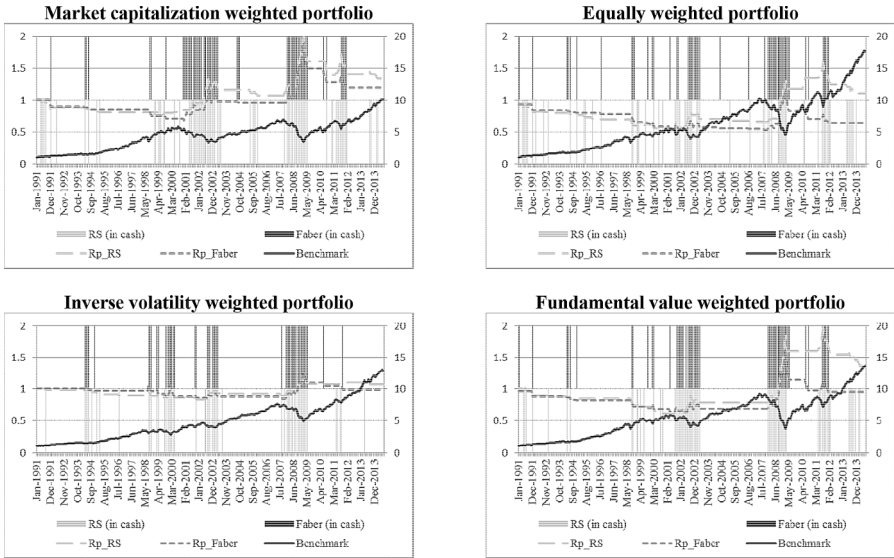


Figure 8.3. The cumulative return of the benchmarks and relative performance of the switching portfolios based on rolling prices (Faber) and the regime switching model with dynamic transition probabilities and the forward-looking return target

COMMENT ON FIGURE 8.3.– Each figure shows, on the right-hand side axis, the full line is the cumulative return of the equity-only-weighted portfolio (the benchmark). On the left-hand side axis, the long-dashed line and short-dashed line are the relative performance of the dynamic regime switching portfolio with the forward-looking return target (Rp_RS) and Faber’s strategy (Rp_Faber) versus the benchmark, respectively. The cash investments of portfolios using dynamic transition probabilities regime switching model and forward-looking return target are indicated with the lower vertical bars (RS (in cash)), while the upper vertical bars indicate the cash investments of portfolios using Faber’s strategy (Faber (in cash)).

8.5. Conclusion

Smart beta portfolios are increasingly popular. By an alternative weighting and selection with respect to the market capitalization-weighted portfolio, they typically achieve a superior diversification, but remain vulnerable to broad market downturns. We examine tactical rules to switch timely between equity and cash investments based on an underlying regime switching model with macroeconomic, macrofinancial and momentum variables as drivers for time-varying transition probabilities. The investment universe analyzed consists of the S&P 500 stocks over the period 1991–2014. We answer two questions. First, how does an alternative smart beta weighting scheme affect the portfolio performance compared with the market capitalization-weighted portfolio. Second, we investigate whether the smart beta portfolio performance can be improved by switching the portfolio allocation between equities and a cash investment.

We find that, compared to the traditional market capitalization-weighted portfolio, the equally weighted, inverse volatility-weighted and fundamental value-weighted portfolios yield a higher compound return over the period. Except for the inverse volatility-weighted portfolio, this comes at higher investment risks. The higher return and lower risk of the inverse volatility-weighted portfolio are consistent with the ample literature documenting the low risk anomaly for the universe we analyze. We further show that the tactical investment decisions based on the regime switching model with time-varying transition probabilities and the forward-looking return target improve the portfolio performance. It leads to investment decisions that are more reactive to changes in the market conditions and a substantially better risk-adjusted performance and lower drawdowns.

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8.7. Appendix

In this section, we provide more details on the inference regarding the probabilities to be in each regime using Hamilton's filter and the calculation of the expected shortfall.

8.7.1. Hamilton's filter

Hamilton's filter aims at estimating from the observed returns y_t the probability to be in each regime, given all information available in the return data up to time t :

$$\xi_{j,t|t} = \Pr(S_t = j | \Omega_t; \theta), \quad [8.13]$$

where $\Omega_t = \{y_t, y_{t-1}, \dots, y_1, y_0\}$ denotes the set of observations obtained as of date t ; and θ is a vector of population parameters. The key magnitudes to perform this iteration are the densities under the regimes:

$$\eta_{j,t} = f(y_t | S_t = j, \Omega_{t-1}; \theta) = \frac{1}{\sqrt{2\pi}\sigma_{S_t}} \exp\left(-\frac{(y_t - \mu_{S_t})^2}{\sigma_{S_t}^2}\right). \quad [8.14]$$

Given the input from [8.13], we can iteratively compute:

$$\xi_{j,t|t} = \frac{\sum_{i=1}^2 p_{ij,t} \xi_{i,t-1|t-1} \eta_{j,t}}{f(y_t | \Omega_{t-1}; \theta)}, \quad [8.15]$$

where $p_{ij,t}$ is the transition probability from regime i to regime j on date t and $f(y_t | \Omega_{t-1}; \theta)$ is the conditional density of the t -th observation:

$$f(y_t | \Omega_{t-1}; \theta) = \sum_{i=1}^2 \sum_{j=1}^2 p_{ij,t} \xi_{i,t-1|t-1} \eta_{j,t}. \quad [8.16]$$

8.7.2. Calculation of expected shortfall of stock returns under the RS model

As we assume the returns follow a normal distribution in the two regimes, the conditional return distribution is a mixture of normals, and, as explained in [BRO 11], expected shortfall can be computed in two steps. First, it requires us to compute the corresponding quantile based on numeric techniques. Then, an explicit expression for the expected shortfall is calculated. More precisely, let μ_j, σ_j be the mean and sigma of the return series y_t under regime j ; $y_t \sim N(\mu_j, \sigma_j^2)$; ($j = 1, 2$). The cdf corresponding to [8.16] is:

$$F(y_t | \Omega_{t-1}; \theta) = \sum_{j=1}^2 \xi_{j,t|t-1} \Phi(y_t; \mu_j, \sigma_j^2), \quad [8.17]$$

where $\xi_{j,t|t-1}$ stands for the predicted probability in regime j on date t (as defined in [8.10]). It follows that the γ -quantile of y_t , $q_{y_t, \gamma}$, can be determined by solving the equation $\gamma - F(q_{y_t, \gamma}; \theta) = 0$ (throughout the chapter γ is set to 0.05, corresponding to a 95% ES calculation). Letting $c_{j,t} = (q_{y_t, \gamma} - \mu_j) / \sigma_j$, $\Phi(c_{j,t})$ and $\phi(c_{j,t})$ are the

standard normal distribution and density function evaluated at $c_{j,t}$, respectively. Then, the expected shortfall of the return is given by:

$$ES_{\gamma}(y_t; \mu_j, \sigma_j, \xi_{j,t|t-1}) = \sum_{j=1}^2 \frac{\xi_{j,t|t-1} \Phi(c_{j,t})}{\gamma} \left\{ \mu_j - \sigma_j \frac{\varphi(c_{j,t})}{\Phi(c_{j,t})} \right\}. \quad [8.18]$$

8.8. Bibliography

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Solving the Rebalancing Premium Puzzle¹

Volatility is usually considered as a synonym for risk. However, recently developed investment strategies based on the concept of volatility harvesting claim to use volatility as a source of additional return. Proponents of these strategies refer to the additional return they offer over buy-and-hold investments as rebalancing premium or rebalancing bonus. In this chapter our analysis of performance of rebalanced portfolios is consistent with multi-period capital growth theory. We identify that over short horizons there is a risk premium associated with rebalancing and zero expected additional growth over buy-and-hold portfolios. We point out that at longer time horizons a bonus from rebalancing does appear under some conditions. We clearly identify these conditions when comparing the expected growth rate of optimally rebalanced portfolio with the expected growth rate of the best asset in this portfolio. Finally, we provide insights on how and when it is possible to add value from rebalancing in active portfolio management. As fire can be either dangerous, if uncontrolled, or useful to run a mechanical engine if controlled, in the same way it should be possible to put volatility to work in a controlled manner in order to produce growth.

9.1. Introduction

Over the last decade, convincing evidence has led to a growing consensus that part of the alpha of active managers could be explained by the existence of systematic

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factor risk premia [ANG 15]. A good review on factor investing is given in [CAZ 14, HAR 15, ANG 15].

In a quest to apply alternative risk premia in asset allocation and portfolio construction framework, an observation that rebalancing volatile assets in a portfolio may produce an extra return with respect to a passive buy-and-hold portfolio awakes the interest of a number of authors [BOU 12, HAL 14, PAL 13]. Indeed, the idea that rebalancing a portfolio to some predefined fixed weight may increase the long-term growth of capital is not new. Historically, it can be traced back to optimal-growth portfolio strategies in the 1960s by Claude Shannon, the father of information theory. Although Shannon never published on the subject, he gave a historic talk at the MIT in the mid-1960s on the topic of maximizing the growth rate of wealth. By using a simple Wiener example, he detailed a method on how to grow portfolio wealth by rebalancing weights to some predefined allocation. This followed the line of thought of the works by Kelly [KEL 56] and Breiman [BRE 61] relating the information asymmetry to optimal bet sizing in order to minimize the time necessary for the wealth to achieve a specific goal.

In this quickly developing area of quantitative asset management, there is a significant confusion in terminology and often a lack of agreement about when rebalancing premium exists and when it does not. No wonder that the general investment community is left perplexed and is somewhat distrustful of the innovations in this area. In our views, the confusion arises principally from the fact that the rebalancing premium is by construct a multi-period effect, while traditional tools and metrics to tackle premia (e.g. beta and alpha) are based on a single-period approach.

Indeed, it is clear that properties related to volatility harvesting and rebalancing premium are intimately related to the concept of optimal growth portfolios [MAC 10]. One of the first formalisms used to address volatility harvesting is Stochastic portfolio theory [FER 02, ODE 13]. In this context, it can be shown that if the market remains diverse (i.e. none of the stocks dominates the market in terms of capitalization), then eventually a rebalancing effect can be observed.

In this chapter, we aim to debunk some common misconceptions and long-standing issues concerning rebalancing. In doing this, to foster intuition, we focus initially on the two-asset case and drop Fernholz diversity condition, i.e. in our context, a cap-weighted market could be dominated by the best performing asset.

We focus on capital growth dynamics of two strategies in stationary markets: a contrarian strategy based on a constantly rebalanced portfolio, and a trend following strategy corresponding to a buy-and-hold portfolio. In this, we will abstract from more challenging issues of real non-stationary markets and the effects of transaction costs. We believe that additional layers of complexity should be added only once the

right level of intuition is established. Our objective is to extract some simple rules on regimes allowing the relative dominance of one strategy with respect to the other.

We approach the problem from two horizon limits. In section 9.2, we show that on short-time horizon, there is no expected rebalancing bonus but a risk premium related to rebalancing emerges. In this case, frequent local gains are compensated by a negative skew of the resulting distribution of relative in-sample growth rates. In section 9.3, we probe the applicability of the short-time limit in realistic scenarios. When we extend the time horizon and the cumulative growth becomes large, in some cases the rebalancing bonus emerges. In section 9.4, we identify regimes when a rebalanced portfolio is expected to grow faster than the best asset in this portfolio, in doing this we lay down the path to explaining the low-risk anomaly. Armed with our results, we arrive at a number of interesting theoretical and practical conclusions in terms of optimal allocation in section 9.5, where we summarize when rebalancing makes sense and provide a diagram of the best investment choices for the case of two risky and one risk-free asset and perfect information. In order to not dilute concepts with technicalities, we omit obvious algebra and relegate non-trivial derivations until the appendix in section 9.7.

9.2. Rebalancing as a risk premium

Authors analyzing rebalancing return and its components usually consider the limit of small variations in asset prices over the considered time horizon. This approximation is implicit in the derivation of the so-called dispersion discount in [HAL 14]. Yet, the approximation is not often explicitly pointed out while it bears some remarkable consequences as we will outline in the sequel. In this chapter, we refer to this regime as the short term or short sample limit.

Another common misunderstanding in the literature regarding rebalancing return arises from the confusion between model parameters and in-sample realization of these parameters. This distinction becomes especially important in the short sample limit. We will show here that in the short-term approximation, rebalancing premium does not come as a free lunch but genuinely behaves as a risk premium, i.e. frequent-limited gains are counterbalanced by rare but potentially severe losses drawn from a left-skewed distribution of returns.

To gain an easy insight into our thesis, we start from a very simple two asset set-up. We then build two reference portfolios: a buy-and-hold portfolio where weights fluctuate according to asset returns; a rebalanced portfolio, the weights of which are reset to the initial ones at the end of each rebalancing period. We start with the buy-and-hold portfolio initially identical to the rebalanced portfolio with weights ω_{1o} and ω_{2o} . Both portfolios have the same return over the first period $t = 1$.

Instead of starting with a predefined breakdown of volatility premium and dispersion discount, we simply find a second-order expansion of realized growth rates of rebalanced and buy-and-hold portfolios. In our simple two assets case considering T rebalancing periods (derivation for N assets is given in section 9.7), we analyze the realized growth rate g_p per rebalancing period². The realized growth rate for the rebalanced portfolio g_p^{rb} is given by:

$$g_p^{rb} = \frac{1}{T} \sum_{t=1}^T g_{p_t}^{rb} = \frac{1}{T} \sum_{t=1}^T \log(1 + \omega_{1o}r_{1t} + \omega_{2o}r_{2t}) \quad [9.1]$$

$$\stackrel{\text{2nd Order}}{\approx} \bar{r}_p - \frac{1}{2} \left(\frac{T-1}{T} \widehat{\sigma}_p^2 + \bar{r}_p^2 \right)$$

where the sample mean return of the rebalanced portfolio \bar{r}_p is given by:

$$\bar{r}_p = \frac{1}{T} \sum_{t=1}^T (\omega_{1o}r_{1t} + \omega_{2o}r_{2t})$$

and sample rebalanced portfolio variance $\widehat{\sigma}_p^2$ is defined as:

$$\widehat{\sigma}_p^2 = \frac{1}{T-1} \sum_{t=1}^T (\omega_{1o}r_{1t} + \omega_{2o}r_{2t} - \bar{r}_p)^2$$

where r_{1t} and r_{2t} are the simple returns of the two assets over period t .

Similarly for the realized growth rate g_p^{bh} of the buy-and-hold portfolio, we get:

$$g_p^{bh} = \frac{1}{T} \log \left(1 + \omega_{1o} \left(\prod_{t=1}^T (1 + r_{1t}) - 1 \right) + \omega_{2o} \left(\prod_{t=1}^T (1 + r_{2t}) - 1 \right) \right)$$

$$= \frac{1}{T} \log \left(\omega_{1o} \prod_{t=1}^T (1 + r_{1t}) + \omega_{2o} \prod_{t=1}^T (1 + r_{2t}) \right) \quad [9.2]$$

$$\stackrel{\text{2nd Order}}{\approx} \bar{r}_p - \frac{T}{2} \bar{r}_p^2 - \frac{T-1}{2T} \left(\omega_{1o} \widehat{\sigma}_1^2 + \omega_{2o} \widehat{\sigma}_2^2 \right) + \frac{T-1}{2} (\omega_{1o} \bar{r}_1^2 + \omega_{2o} \bar{r}_2^2)$$

where \bar{r}_i and $\widehat{\sigma}_i$ are the realized mean return and sample standard deviation of asset i .

While to get a usable expression for the rebalanced portfolio, we only need $\sigma_i \ll 1$ for the second-order expansion in equation [9.1], we must assume $T\bar{r}_i \ll 1$ in order to drop cross terms of the higher order in the first step of equation [9.2]. In the appendix, section 9.7, we show in detail why this approximation is

² Note that the growth rate per period is not the same as geometric average return $r_p^{geom} = \exp(g_p) - 1$, although these two quantities are often confused.

needed and point out that it is identical to $\sqrt{T}\sigma_i \ll 1$. For the difference of the realized growth rates, we get:

$$g_p^{rb} - g_p^{bh} = \frac{1}{2} \frac{T-1}{T} \left[\omega_{1o} \left(\widehat{\sigma}_1^2 - T\bar{r}_1^2 \right) + \omega_{2o} \left(\widehat{\sigma}_2^2 - T\bar{r}_2^2 \right) - \left(\widehat{\sigma}_p^2 - T\bar{r}_p^2 \right) \right] \quad [9.3]$$

This result is quite similar to the one obtained by Hallerbach in [HAL 14]. Importantly, we identify the limits of applicability $\sqrt{T}\sigma_i \ll 1$ of both equation [9.3] and the result in [HAL 14]. The expression in equation [9.3] can be grouped into terms identified as volatility premium and dispersion discount as in [HAL 14]. However, we find that such a grouping makes it difficult (if not impossible) to compare premium and discount terms³, we will thus proceed in an alternate way.

While the difference of the realized growth rates in equation [9.3] can be either positive or negative, it is important to make assessment of this value on average over multiple observations. Therefore, we define rebalancing bonus as the expected value of the difference $g_p^{rb} - g_p^{bh}$.

Up to this point, we made no assumption about the distributions of asset returns r_i in this section. If we assume geometric Brownian motion (normally distributed returns with zero mean and no serial dependence $N(\sigma_i, \mu_i = 0)$), we gather that there is no rebalancing bonus in the short term. By taking the expected value of equation [9.3] and noting that the expected value of the square of the sample realized mean \bar{r}_i over T observations is:

$$\mathbb{E}(\bar{r}_i^2) = \frac{\mathbb{E}(\widehat{\sigma}_i^2)}{T} = \frac{\sigma_i^2}{T} > 0$$

we get the result that on average in the limit $\sqrt{T}\sigma_i \ll 1$ the rebalancing bonus is zero⁴:

$$\mathbb{E}(g_p^{rb} - g_p^{bh}) = 0 \quad [9.4]$$

Once again, we emphasize the importance of distinguishing between the parameters of return distributions σ_i, μ_i and the sample observations $\widehat{\sigma}_i, \bar{r}_i$. Over multiple observation periods, $\widehat{\sigma}_i$ will have an expected value σ_i and independent observations will be distributed according to χ^2 distribution with $(T-1)$ degrees of freedom. At the same time, \bar{r}_i^2 will have an expected value of σ_i^2/T and will be

³ Note that our expression contains a correction for volatility premium, consistent with the fact that for $T=1$ the difference in the growth rates should be zero by definition.

⁴ It is obvious from above that if returns have a serial dependence characteristic to a mean-reverting process, a positive rebalancing bonus will be observed.

distributed as χ^2 with one degree of freedom. The resulting combined distribution describing realized observations of growth rate differences will then be characterized by a strong negative skew. If we attempt to profit from rebalancing in the short term, we will observe frequent small positive returns over buy-and-hold strategy, which will eventually be offset by rare but large negative returns. This is well evidenced by a numerical experiment for two assets and $T=10$ repeated 100,000 times. The histogram of the realized difference of growth rates $g_{rb} - g_{bh}$ and approximation of this difference by equation [9.3] is shown in Figure 9.1.

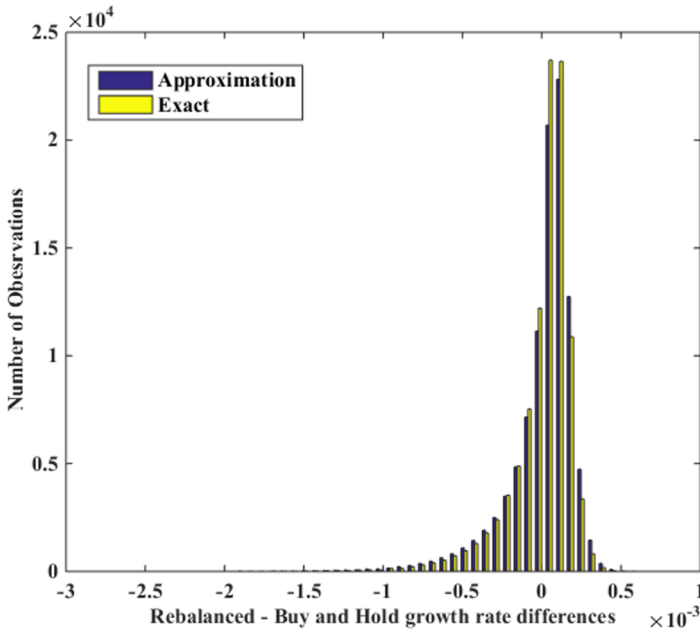


Figure 9.1. Distribution of the realized growth rate difference $g_p^{rb} - g_p^{bh}$ for a portfolio of two uncorrelated assets with normally distributed returns with zero means and 2% per period volatilities. 100,000 simulations over $T = 10$ rebalancing periods are considered

What if we consider assets with returns following geometric Brownian motion but with non-zero expected returns? It appears that for proponents of rebalancing, things only get worse. In this case:

$$\mathbb{E}(\bar{r}_i^2) = \mu_i^2 + \frac{\sigma_i^2}{T}$$

Therefore, on average we have:

$$\mathbb{E}(g_p^{rb} - g_p^{bh}) = -T\omega_1\omega_2(\mu_1 - \mu_2)^2 \quad [9.5]$$

When expected returns are different, there is a small (second order in difference of μ) rebalancing discount even when we do not take into account transaction costs. From this result, it would appear that, if we intend to keep our portfolio allocations for a short time, it is better not to rebalance the portfolio.

This result seemingly contradicts conclusions from a number of previous studies claiming that regular rebalancing leads to a better performance. In the next section, we will evaluate what happens when the approximation $\sqrt{T}\sigma_i \ll 1$ breaks down.

9.3. Probing the limits: when simulations provide more insight

In order to check the case $\sqrt{T}\sigma_i \simeq 1$, we can proceed with the expansion of buy-and-hold growth rate to the fourth order. After some tedious algebra, we indeed obtain a positive expected value for the rebalancing premium but choose not to show it here to save space. Instead, we rely on a simple Monte Carlo simulation to probe when the effective rebalancing premium comes into force and the approximation breaks. We analyze this by plotting the expected difference between the growth rates of rebalanced and buy-and-hold portfolios in the numerical experiment described below when the number of rebalancing periods increases. The parameters used for this exercise are estimated from the total return series of Starbucks and Apple stocks over a horizon from January 1993 to May 2015. Obviously, the following example is not general but it provides some useful insights.

We select i.i.d. normally distributed asset returns with the daily expected return of 11.4 and 12.1 basis points for assets 1 and 2, while daily volatility is 2.6 and 3%, respectively. The correlation between asset returns is 0.25. Initially, the portfolio holds 50% of each asset. We compare the difference between daily expected growth rates of a portfolio rebalanced daily and buy-and-hold portfolio for horizons between 2 weeks and 1,000 years. The results of simulations are summarized in Figure 9.2. Each point is an average over 200,000 independent simulations. Observed rebalancing bonus per day for Apple and Starbucks over this horizon is also shown in the figure.

Despite the simplicity of the numerical experiment described above, we can make some important conclusions. First, the short horizon approximation used in section 9.2 has very limited applicability. In our example, the result in equation [9.3] works for horizons up to 5 months and then it breaks down as the difference between approximated and exact calculations becomes statistically significant. Many authors consider implicitly the second-order expansion of buy-and-hold portfolio's multi-period growth rate and apply the results of this expansion to multi-year periods.

In the light of our example, conclusions of these authors should be re-examined. As we show, second-order results apply only for very short horizons when realistic asset parameters are considered. Second, we demonstrate that, while it is futile to look for advantages of rebalanced over buy-and-hold portfolios over a short horizon in case of i.i.d normally distributed returns, the rebalancing bonus may indeed appear when longer horizons $\sqrt{T}\sigma_i \simeq 1$ are considered.

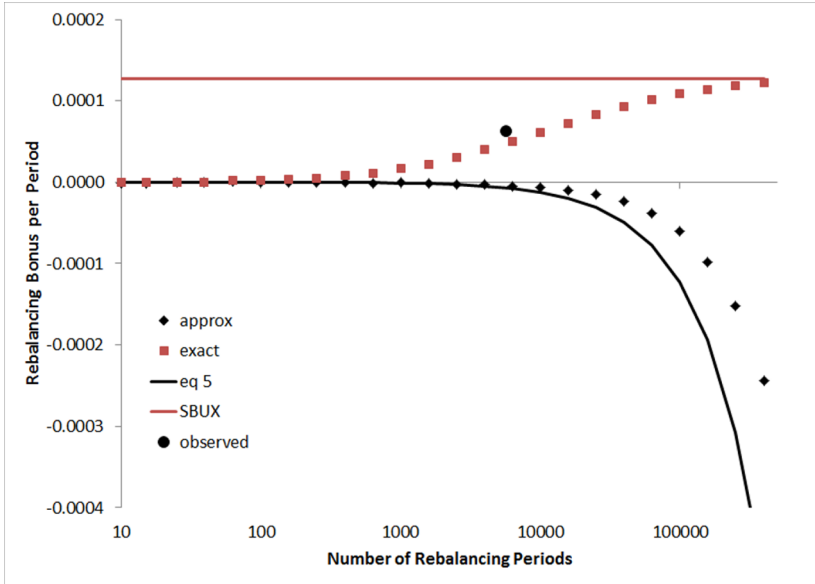


Figure 9.2. *Rebalancing bonus per day versus horizon for two assets with normally distributed daily returns $N(0.00114, 0.026)$ and $N(0.00121, 0.03)$ and correlation 0.25. Each point represents an average over 200,000 independent simulations. Black diamonds represent approximate results using equation [9.3], while grey squares show results of the exact calculation. Black line is the expectation according to equation [9.5] and grey line shows expected difference between the growth rate of rebalanced portfolio and that of the best asset (SBUX). Black dot shows the actual rebalancing bonus observed for AAPL and SBUX over a horizon from January 1993 to May 2015*

Another important observation concerns the limiting value of the rebalancing bonus at long horizons. The expected growth rate of a buy-and-hold portfolio approaches the expected growth rate of the best asset in the portfolio in the long term. The asset with the higher expected growth rate in a portfolio thus plays an important role in defining the maximum possible value of rebalancing premium.

Before moving on to consider when the expected growth of the rebalanced portfolio is higher than the expected growth of the best asset, we show a couple of

easy extensions of the numerical experiment defined above. Can we compare rebalancing bonus at different rebalancing frequencies? We already noticed that rebalancing bonus appears when $\sqrt{T}\sigma \simeq 1$. If instead of rebalancing daily, we do this once a month (roughly once every 20 days), the number of rebalancing periods over T days will decrease to $T/20$, while the volatility of monthly returns will increase to $\sigma\sqrt{20}$. Overall, we see that if $\sqrt{T}\sigma = 1$, then $\sqrt{\frac{T}{20}}\sigma\sqrt{20} = 1$. It appears that in a frictionless market, the rebalancing bonus should not depend on rebalancing frequency. We verify this observation in numerical simulations summarized in Figure 9.3.

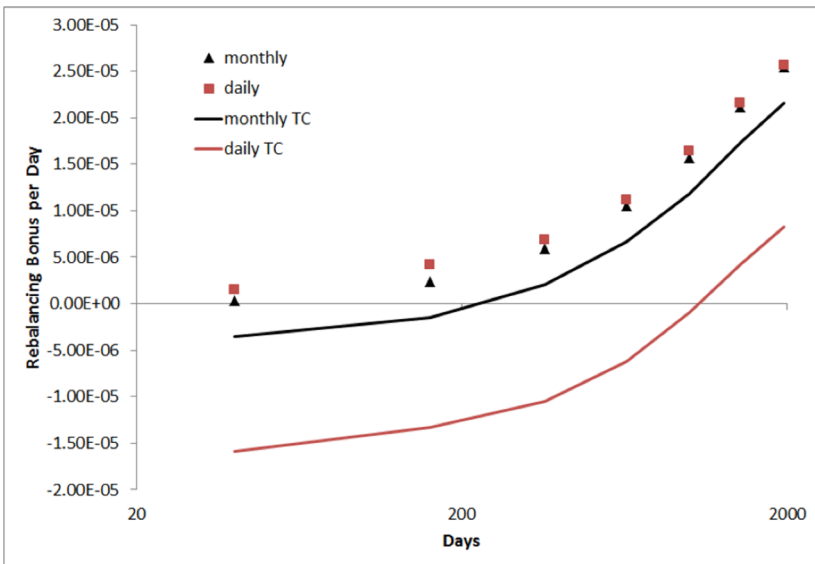


Figure 9.3. *Rebalancing bonus for two assets with normally distributed returns $N(0.00114, 0.026)$ and $N(0.00121, 0.03)$ and correlation 0.25. Daily (grey squares) and monthly (black triangles) rebalancing frequencies are compared in a frictionless market. Lines of corresponding color show impact of 30 bps two-way transaction costs on rebalancing bonus*

When we include realistic two-way transaction costs of 30 bps, the situation obviously changes. Monthly rebalancing becomes profitable for horizons longer than 1 year, while daily rebalancing bonus is only sufficient to offset transaction costs after 4 years or more. Before embracing rebalancing as a source of excess returns, we need to be aware that even if it is positive, it may take a while before we observe the benefit in practice.

9.4. Beating the best asset: a path to the low-risk anomaly explanation

As explained in the previous sections, analytical Taylor series expansion of the growth rate of buy-and-hold portfolios is inadequate when horizon T becomes large (i.e. approaches the limit $\sqrt{T}\sigma \simeq 1$). Can we obtain additional insight into rebalancing problem without resorting to numerics? It turns out that we can say a lot more when comparing rebalanced portfolio growth rates with the growth rates of individual assets. In fact, when we move into the long horizon limit, this becomes the most relevant question to ask. In the long term, the growth of the buy-and-hold portfolio will be determined by its fastest growing asset and the dependence on the initial allocation of weights in the buy-and-hold portfolio will eventually disappear.

As before, the growth rate of the rebalanced portfolio is given by equation [9.1]. We assume that asset 1 has a higher expected growth rate and that one-period returns of both assets are small $|r_{it}| \ll 1$

$$g_1 = \frac{1}{T} \sum_{t=1}^T \log(1 + r_{1t}) \stackrel{\text{2nd Order}}{\approx} \frac{1}{T} \sum_{t=1}^T \left(r_{1t} - \frac{1}{2} r_{1t}^2 \right) \quad [9.6]$$

Before proceeding further to the expected value of the difference between growth rates of the rebalanced portfolio and the fastest growing asset, we make an assumption of stationary i.i.d. normally distributed asset i returns with mean μ_i and standard deviation σ_i over one period:

$$r_{it} = \mu_i + \sigma_i \epsilon_{it} \quad [9.7]$$

where ϵ_{it} is a standard normal i.i.d. variable $N(0, 1)$ with zero mean and unit variance. Assuming correlation between the two assets to be ρ , we have the following expressions for the expected values:

$$\begin{aligned} \mathbb{E}\left(\sum_{t=1}^T \epsilon_{it}\right) &= 0 \\ \mathbb{E}\left(\sum_{t=1}^T \epsilon_{it}^2\right) &= 1 \\ \mathbb{E}\left(\sum_{t=1}^T \epsilon_{it}\epsilon_{jt}\right) &= \rho \\ \mathbb{E}\left(\sum_{t=1}^T \sum_{t \neq k} \epsilon_{it}\epsilon_{jk}\right) &= 0 \end{aligned} \quad [9.8]$$

After some simple algebra, we obtain expected growth rates:

$$\begin{aligned}\mathbb{E}(g_1) &= \mu_1 - \frac{1}{2}(\mu_1^2 + \sigma_1^2) \\ \mathbb{E}(g_{rb}) &= \omega_{1o}\mu_1 + \omega_{2o}\mu_2 - \frac{1}{2}\omega_{1o}^2(\mu_1^2 + \sigma_1^2) - \frac{1}{2}\omega_{2o}^2(\mu_2^2 + \sigma_2^2) \\ &\quad - \omega_{1o}\omega_{2o}(\mu_1\mu_2 + \sigma_1\sigma_2\rho)\end{aligned}\tag{9.9}$$

Assuming full investment $\omega_{1o} + \omega_{2o} = 1$ and comparing expected value of the rebalanced portfolio growth rate with that of asset 1, we determine the weight limits when the rebalanced portfolio is expected to outperform the best asset in the long run:

$$0 < \omega_{2o} < 2 \frac{\mu_2 - \mu_1 + \mu_1^2 + \sigma_1^2 - \mu_1\mu_2 - \sigma_1\sigma_2\rho}{\mu_1^2 + \sigma_1^2 + \mu_2^2 + \sigma_2^2 - 2\mu_1\mu_2 - 2\sigma_1\sigma_2\rho}\tag{9.10}$$

The optimal weights, for which the expected growth rate of the rebalanced portfolio is maximized, are given by:

$$\begin{aligned}\omega_{1o} &= \frac{\mu_1 - \mu_2 + \mu_2^2 + \sigma_2^2 - \mu_1\mu_2 - \sigma_1\sigma_2\rho}{\mu_1^2 + \sigma_1^2 + \mu_2^2 + \sigma_2^2 - 2\mu_1\mu_2 - 2\sigma_1\sigma_2\rho} \\ \omega_{2o} &= \frac{\mu_2 - \mu_1 + \mu_1^2 + \sigma_1^2 - \mu_1\mu_2 - \sigma_1\sigma_2\rho}{\mu_1^2 + \sigma_1^2 + \mu_2^2 + \sigma_2^2 - 2\mu_1\mu_2 - 2\sigma_1\sigma_2\rho}\end{aligned}\tag{9.11}$$

Before moving further, let us consider two special cases for the parameters in equations [9.11].

In the case where $\mu_1 = \mu_2$, the optimal growth portfolio turns out to be the rebalanced MV portfolio between two assets, shading some light on the so-called low-risk anomaly in [HAU 91] (see also [HAU 12]). In this respect, the anomaly disappears as the dominance of a MV allocation framework emerges as a natural phenomenon implicit in the compounding process. In the absence of any valuable information on future expected returns, the rebalanced MV portfolio turns out to be the most sensible choice if we target the long-term growth of capital. In a separate paper, we go a step further and obtain full efficient frontier of Markovitz portfolio theory based on information theoretic considerations and the result in equation [9.11] [DUB 15].

Another special case, which is even more relevant for the discussion about rebalancing, concerns a combination of one risky and one risk-free asset. If we set $\sigma_2 = 0$ and denote $\mu_2 = r_f$ the risk-free return, we obtain the optimal weight for the risky asset to be held with the risk-free asset in an optimal rebalanced portfolio. To

simplify notation, we drop the subscript for the risky asset in what follows: $\mu_1 = \mu$, $\sigma_1 = \sigma$, and $\omega_{1o} = \omega_o$

$$\omega_o = \frac{(\mu - r_f)(1 - r_f)}{\sigma^2 + (\mu - r_f)^2} \quad [9.12]$$

The optimal expected growth of this rebalanced portfolio is then given by:

$$\begin{aligned} \mathbb{E}(g_{rb}) &= r_f + \omega_o(\mu - r_f) - \frac{1}{2}\omega_o^2\sigma^2 - \frac{1}{2}(r_f + \omega_o(\mu - r_f))^2 \\ &= r_f - \frac{1}{2}r_f^2 + \frac{1}{2}\frac{(\mu - r_f)^2(1 - r_f)^2}{\sigma^2 + (\mu - r_f)^2} = g_f + \frac{1}{2}\frac{SR^2(1 - r_f)^2}{\tau + SR^2} \end{aligned} \quad [9.13]$$

where growth rate of the risk-free asset is $g_f = r_f - \frac{1}{2}r_f^2$, Sharpe ratio of the risky asset is $SR = \sqrt{\tau}\frac{\mu - r_f}{\sigma}$ and τ is the number of rebalancing periods per year⁵. An important property of the rebalanced portfolio is immediately obvious from equation [9.13]. If we have to choose one risky asset from many, we should select the asset with the highest Sharpe ratio as it yields the highest growth rate in the optimally rebalanced portfolio, which includes a risk-free asset. Incredibly, from a solution to a simple two-asset rebalancing problem, we directly obtain one of the most important results in modern portfolio theory. Indeed, this is a multi-period case of the Tobin's mutual fund theorem [TOB 58]. Note that this result does not require an *ad hoc* assumption that investors are averse to asset price volatility. In our problem setting, investors are only concerned with maximizing the expected growth rate of their wealth. For such an investor, risk is no longer simply synonymous with volatility. More generally and more intuitively, the risk can be associated with the probability of negative returns or negative growth rates and, thus, it is related to the accuracy of forecasting return expectations. For a more detailed discussion on this, see [DUB 15].

Another important characteristic of equation [9.13] is that optimal expected growth is positive even for assets with negative Sharpe ratio (when expected return of the risky asset is smaller than the risk-free return). In this case, according to equation [9.12], it is optimal to short the risky asset and invest the proceeds at the risk-free rate. In addition, if optimal weight in equation [9.12] is greater than 1, the optimal strategy is to borrow money at the risk-free rate and to leverage investment into the risky asset. It is easy to prove that if short-selling and leverage are allowed and if borrowing at the risk-free rate and trading for free are possible, it is more profitable to invest in a constant-weight portfolio of a risk-free and risky asset than in the risky asset alone. In this framework, knowledge of future expected returns and variance fully determines

⁵ Note that $\sqrt{\tau}$ scaling factor is needed to annualize return and volatility as Sharpe ratio is usually defined for annualized quantities.

the choice of leverage. There is no need to assume a certain arbitrary trade-off between risk and variance to find an optimal leverage.

If neither leverage nor shorting is allowed, we can determine the limits when rebalancing is more profitable than holding the risky asset. From equation [9.12], in a long-only non-leveraged case, when $\omega_o \leq 0$ we hold risk-free asset instead of rebalancing and when $\omega_o \geq 1$ it is better to fully invest in the risky asset⁶.

$$\begin{cases} r_f < \mu < \sigma^2/2 & \iff \mathbb{E}(g_1) < g_f < \mathbb{E}(g_{rb}) \\ \sigma^2/2 < \mu - r_f < \sigma^2 & \iff g_f < \mathbb{E}(g_1) < \mathbb{E}(g_{rb}) \\ \sigma^2 < \mu - r_f & \iff g_f < \mathbb{E}(g_{rb}) < \mathbb{E}(g_1) \end{cases} \quad [9.14]$$

Equation [9.14] provides an important insight into a problem of allocating capital between a risky and a risk-free asset. Let us take $r_f = 0$ as is proper in the current investment environment.

If the expected return of the risky asset is less than zero, it does not make sense to invest. This should not be a surprise to anyone. However, positive arithmetic mean μ does not guarantee growth of the investment in the risky asset. In fact, if $0 < \mu < \sigma^2/2$, the buy-and-hold strategy will eventually lose all initial investments. While this result is familiar to many investors, very few actually know that it is possible to obtain positive growth for a regularly rebalanced portfolio of cash and a risky asset even if by itself the risky asset has a negative expected growth rate. It is even less obvious that an optimal rebalanced portfolio will in a long run outperform any buy and hold combination of a risky asset and cash if $\mu < \sigma^2$. Finally, if the expected return is large enough $\mu > \sigma^2$, the investor should fully invest in the risky asset to maximize the long-term wealth.

9.5. When rebalancing pays off

We should point out that the methodology adopted in section 9.4 is readily extendable to any number of assets. Here, we consider the case of two risky assets and cash given perfect knowledge about future expected returns, volatilities and correlations. Short-selling and leverage are not allowed. This case is practically useful and is well suited for an intuitive graphical representation [LAU 09]. We clearly separate the parameter space into regions where rebalancing adds value from the regions where it is best to hold one asset with the highest growth rate. Schematically, the regions are shown in the diagram in Figure 9.4. A point on this diagram, showing excess expected returns for the two assets, will determine what investment choice will produce the highest growth in the long term. As is well known

⁶ Here, we dropped usually much smaller $(\mu - r_f)^2$ terms.

in option pricing, asset variance turns out to be a useful measuring stick in the return space.

Below, we focus on explaining optimal investment choices corresponding to each region. Light gray areas in the parameter space is where rebalancing adds value. In white areas, it is better to buy-and-hold a single asset.

Region A covers the most obvious case – when expected excess returns of both risky assets are negative, we should just hold the risk-free asset. Regions B, C and D have a common characteristic in that they require rebalancing between the risk-free asset and risky assets. As shown in equation [9.13], the risky asset with maximum Sharpe ratio maximizes expected growth when combined with the risk-free asset. Region B corresponds to the set of parameters, for which asset 2 has the highest Sharpe ratio, in region C – it is asset 1 that dominates. In region D, the risky holding with the highest Sharpe ratio is a rebalanced portfolio of two risky assets. Clearly, the risky assets have to be held in proportions that produce the maximum Sharpe ratio in order to maximize the expected growth rate in equation [9.13]. Region D is the only region, where all three assets are held and rebalanced. In region G, we should hold an optimal combination of risky assets according to equation [9.11]. In regions E and F, the best expected growth is produced by fully investing into the corresponding risky asset. The lines separating region B from E and region C from F are defined by 100% optimal weight of the risky asset in equation [9.12]. Similarly, the lines separating region G from E and F correspond to 100% weight of one of the assets in equation [9.11]. Within region G, we identify the lines reflecting some well-known portfolio construction choices. The dashed line corresponds to MV portfolio of two risky assets, while the dotted line corresponds to their equal weight combination.

Having identified regions for optimal investment in the parameter space, we turn to a simple example to illustrate how much difference the correct selection can make. We consider total returns of two stocks: Apple Inc. as asset 1 (AAPL), and Starbucks Corporation as asset 2 (SBUX) over the period from January 1993 to May 2015. Both companies were big success stories over the period considered. If you were to invest \$100 in AAPL in January 1993, your investment would grow to \$6800, while initial \$100 investment in SBUX would be worth \$9800 today. Holding a monthly rebalanced equal weight combination of the two assets demonstrates the power of rebalancing in the long term. \$100 invested into this rebalanced portfolio would be worth \$17000 today. The difference appears to be impressive. However, let us consider another case, where we make investment decisions roughly every 3 years. The points identified in Figure 9.4 by two numbers show where the realized returns of AAPL and SBUX fall over the period specified. What if every 3 years we were able to correctly predict the region in the parameter space where the expected returns of the pair of stocks would fall? Now, we are not claiming the perfect knowledge but just an ability to correctly identify the region for each 3-year period. Also, we do not expect to hold the optimal weight but choose equal weight combinations in relevant

regions. If we were to invest according to such region selection, we would do so as follows: from 1993 to the end of 1996 and from 2000 to 2002 we would hold SBUX, from 1997 to 1999 and from 2003 to 2005 we would hold AAPL, from 2006 to 2008 we would rebalance between 50% in cash and 50% in AAPL and only starting from 2009 we would have a monthly rebalanced equal weight combination of the two assets. Incredibly, correct identification of the regions in the parameter space would help us turn our initial investment of \$100 in 1993 into \$370000 today or about 54 times more than the investment into AAPL would bring. The cumulative wealth curves are shown for each case considered above in Figure 9.5. 30 bps two-way transaction costs were included in calculation.

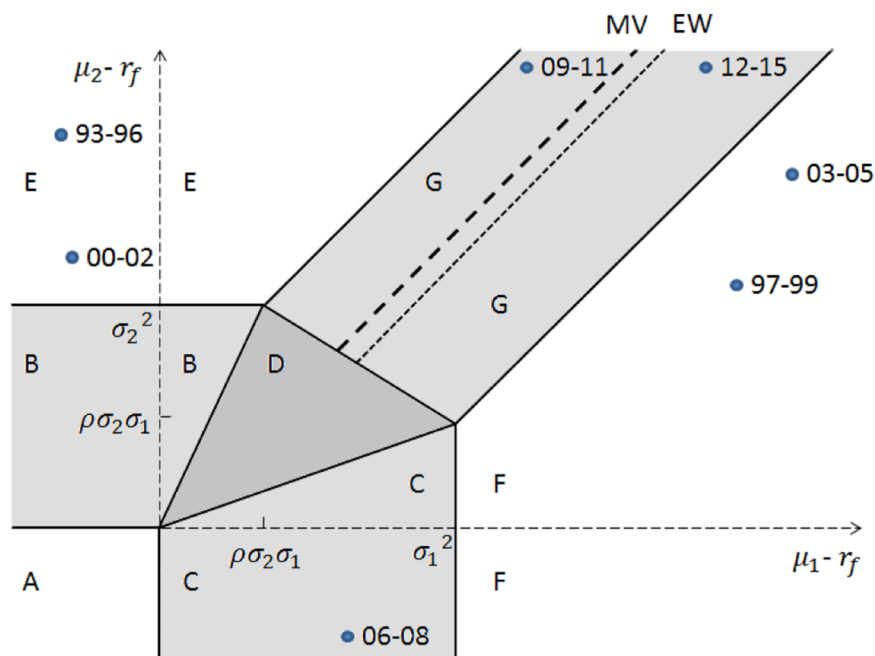


Figure 9.4. Optimal investment choices given perfect information for two risky and one risk-free asset. Gray regions – rebalancing yields higher expected growth (region B – between risk-free and asset 2, region C – between risk-free and asset 1, region G – between risky assets, region D – risk-free and the maximum Sharpe ratio combination of risky assets). White regions – single asset holding is optimal (region A – risk-free asset, region E – asset 2, region F – asset 1). Dots with numbers show realized returns of a pair (AAPL and SBUX) over corresponding periods

Realistically, even the correct prediction of a region for a set of assets may not be possible. However, schematic representation in Figure 9.4 may also guide us in

portfolio construction if we take into account uncertainty in expected return forecasts. If instead of given μ_1 and μ_2 we have a distribution of likely outcomes, we can construct a portfolio that maximizes growth taking an integral over such a distribution. This can be easily accomplished even in a multiple stock setting by a Monte Carlo integration technique. An application of this approach to portfolio construction is a topic of a separate study.

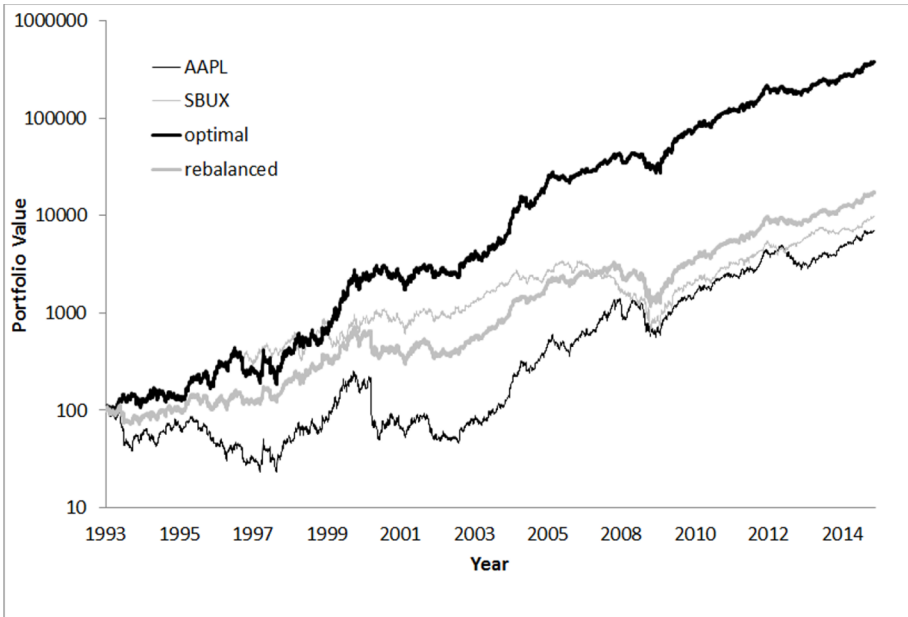


Figure 9.5. Cumulative wealth growth curves for investments into cash, AAPL and SBUX from January 1993 to May 2105. Thin black (AAPL) and grey (SBUX) lines show 100% buy-and-hold investments into corresponding stocks. Thick grey line shows growth of capital of an equal-weighted monthly rebalanced portfolio of AAPL and SBUX. Thick black line shows potential performance if regions in Figure 9.4 were to be correctly identified every 3 years

It is well known that expected returns are difficult to forecast accurately. However, we can better forecast variances and to a certain degree correlations. One relevant question to ask then is which assets we should select from the point of view of their volatilities and correlations in order to ensure that rebalancing adds value. What we strive to achieve is to make our regions where rebalancing is optimal (gray areas) bigger in the parameter space. Selecting very volatile assets is one way to do this. Hence, the name volatility harvesting. If you have volatile assets you are more likely to end up in a region where periodic rebalancing is beneficial. One way to expand regions

D and G is to choose assets with low correlations. Thus, portfolios well diversified into assets with low pairwise correlations are more likely to benefit from rebalancing.

Our simple graphical representation of optimal investment choices has already yielded a number of insights. Another observation concerns certain risk-based portfolio construction techniques. It is often claimed that risk-based investing takes no views on expected returns. However, in order to be optimal from the point of view of expected growth rate maximization, each risk-based methodology implies a certain view on expected returns [QIA 05, MAI 10, JUR 15]. As pointed out above, rebalanced minimum variance portfolio (MV) construction implies that the expected returns are assumed to be equal. If they are, then in region G MV portfolio is the optimal growth portfolio. Therefore, in Figure 9.4, MV line corresponds to $\mu_1 = \mu_2$.

Similarly, equally weighted portfolio (EW) fully invested in risky assets implies that the expected growth rates of the two assets are the same:

$$\mu_1 - \frac{\sigma_1^2}{2} = \mu_2 - \frac{\sigma_2^2}{2} \quad [9.15]$$

EW portfolio is in the middle of the region G in Figure 9.4. This means that EW portfolio is ideal if we expect assets to have equal growth rates. Such a view expressed by equation [9.15] is also least sensitive to misspecification. Among all portfolios where rebalancing is expected to dominate, departures from equal growth rate expectations are least likely to bring us into a regime where buy-and-hold is the best strategy.

Equal risk contribution (ERC) portfolio is identical to maximum diversification portfolio in the two-asset case. This portfolio lies between MV and EW lines in Figure 9.4 and it implies the following expectation on future mean returns:

$$\mu_1 - \mu_2 = \frac{(1 + \rho)\sigma_1\sigma_2(\sigma_1 - \sigma_2)}{\sigma_1 + \sigma_2} \quad [9.16]$$

It is clear that in the two-asset case for the ERC portfolio to be expected growth optimal, it is necessary to assume that expected returns of assets are proportional to their volatilities⁷.

9.6. Conclusions

This work started as an attempt to clarify a widespread confusion about rebalancing premium. In order to do so, we looked at a simplified case of only two

⁷ Note that this assumption is necessary but not sufficient for optimality.

assets. In the process, we discovered that the insights we obtain in this simple setting allow us to tackle some of the long-standing controversies over return from rebalancing. We showed how the confusion arises principally from the fact that the rebalancing premium is by construct a multi-period effect, while conventional tools and metrics to describe rebalancing premia are single-period averages. By using an adapted formalism and by carefully considering the difference between the one-period model set-up and the statistics of in sample realizations, the premium appears as a genuine risk premium with frequent small gains counterbalanced by rare but large losses. Our results allowed us to describe market regimes for which rebalancing works and regimes for which it does not and enabled us to clarify the implicit return assumptions of fashionable allocation schemes such as the ERC and MV portfolios.

It is worthwhile to sketch out future research work within the framework we propose. The concept should be extended to the multi-asset case, take into account transaction costs and realistic asset properties such as mean reversion. In particular, it will be of great interest to analyze optimal rebalanced portfolios with the use of results from information theory by changing the focus from risk aversion to more general expected return uncertainty.

9.7. Appendix

9.7.1. Rebalancing premium: the multi-asset case

Average growth rate g_p^{rb} over T periods, we will have:

$$\begin{aligned}
 g_p^{rb} &= \frac{1}{T} \sum_{t=1}^T g_{p_t}^{rb} = \frac{1}{T} \sum_{t=1}^T \log \left(1 + \sum_{i=1}^N \omega_{io} r_{it} \right) \\
 &\stackrel{\text{2nd Order}}{\approx} \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \omega_{io} r_{it} - \frac{1}{2T} \sum_{t=1}^T \left(\sum_{i=1}^N \omega_{io} r_{it} \right)^2 \\
 &= \bar{r}_p - \frac{1}{2} \left(\widehat{\sigma}_p^2 + \bar{r}_p^2 \right)
 \end{aligned} \tag{9.17}$$

where the sample mean return of the rebalanced portfolio \bar{r}_p is given by:

$$\bar{r}_p = \frac{1}{T} \sum_{t=1}^T \sum_{i=1}^N \omega_{io} r_{it}$$

and the following relation is used for the sample rebalanced portfolio variance $\widehat{\sigma}_p^2$:

$$\widehat{\sigma}_p^2 = \frac{1}{T-1} \sum_{t=1}^T \left(\sum_{i=1}^N \omega_{io} r_{it} - \bar{r}_p \right)^2 = \frac{T}{T-1} \left(\frac{1}{T} \sum_{t=1}^T r_{pt}^2 - \bar{r}_p^2 \right)$$

Similarly for the growth rate g_p^{bh} of the buy-and-hold portfolio, we get:

$$\begin{aligned}
 g_p^{bh} &= \frac{1}{T} \log \left(1 + \sum_{i=1}^N \omega_{io} \left(\prod_{t=1}^T (1 + r_{it}) - 1 \right) \right) \\
 &\stackrel{\text{2nd Order}}{\approx} \frac{1}{T} \log \left(1 + \sum_{i=1}^N \omega_{io} \left(\sum_{t=1}^T r_{it} + \sum_{t>k} r_{it} r_{ik} \right) \right) \\
 &\stackrel{\text{2nd Order}}{\approx} \frac{1}{T} \sum_{i=1}^N \sum_{t=1}^T \omega_{io} r_{it} - \frac{T}{2} \left(\sum_{i=1}^N \omega_{io} \bar{r}_i \right)^2 + \frac{1}{2T} \sum_{i=1}^N \omega_{io} \sum_{t \neq k} r_{it} r_{ik} \quad [9.18] \\
 &= \bar{r}_p - \frac{T}{2} \bar{r}_p^2 + \frac{T}{2} \sum_{i=1}^N \omega_{io} \bar{r}_i^2 - \frac{1}{2T} \sum_{i=1}^N \omega_{io} \sum_{t=1}^T r_{it}^2 \\
 &= \bar{r}_p - \frac{T}{2} \bar{r}_p^2 - \frac{T-1}{2T} \sum_{i=1}^N \omega_{io} \hat{\sigma}_i^2 + \frac{T-1}{2} \sum_{i=1}^N \omega_{io} \bar{r}_i^2
 \end{aligned}$$

In the second line of the expression [9.18], we make a strong assumption $T\bar{r}_i \ll 1$ to take only the terms of the first and second order in r_{it} . The need for this assumption arises due to larger number of terms associated with higher orders in r_{it} . Thus, we have T terms of the first order, $T(T-1)/2$ terms of the second order and $T(T-1)(T-2)/6$ terms of the third order. In order to drop third-order terms, we need to assume $r_{it}(T-2)/3 \ll 1$ or $\bar{r}_i(T-2)/3 \ll 1$. For $T \gg 1$ and a stationary process with the assumption of small mean compared to volatility, we have a simpler requirement $\sqrt{T}\sigma_i \ll 1$ for all assets. The necessity of this approximation may have been overlooked by other authors studying the expansion of rebalancing premium to the second order.

Note that in order to arrive to the final expression in [9.18], we used the following steps to simplify the double sum over cross-products of returns:

$$\begin{aligned}
 \sum_{t>k} r_{it} r_{ik} &= \frac{1}{2} \sum_{t \neq k} r_{it} r_{ik} + \frac{1}{2} \sum_{t=1}^T r_{it}^2 - \frac{1}{2} \sum_{t=1}^T r_{it}^2 \\
 &= \frac{1}{2} \left(\sum_{t=1}^T r_{it} \right)^2 - \frac{1}{2} \sum_{t=1}^T r_{it}^2 = \frac{T^2}{2} \bar{r}_i^2 - \frac{T-1}{2} \hat{\sigma}_i^2 - \frac{T}{2} \bar{r}_i^2 \quad [9.19]
 \end{aligned}$$

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Smart Betas: Theory and Construction¹

10.1. Introduction

Smart beta, also known as *rule-based strategies*, or *systematic strategies*, or *alternative beta*, or *exotic beta*, or *quantitative alpha*, or “*factors*” or “*risk premia*”, are portfolio construction techniques inspired by the key theorems of asset pricing, following the work of [ROS 85, FAM 93] and [GRI 99]. More recent reviews include [CAZ 14, AME 14] and [HOM 15].

The rationale behind smart beta is the belief in a persistent relationship similar to the Arbitrage Pricing Theory (APT) across each time period:

$$\text{performance} \quad \text{plain/"smart" beta} \quad \text{noise} \\ \Pi_{t \rightarrow t+1} - r_{t \rightarrow t+1}^{rf} \mathbf{v}_t = \beta_{1,t} Z_{1,t+1} + \dots + \beta_{\bar{k},t} Z_{\bar{k},t+1} + \mathbf{U}_{t+1}, \quad [10.1]$$

where the factors have positive premia:

$$\begin{pmatrix} \mathbb{E}\{Z_{1,t+1}\} \\ \mathbb{E}\{Z_{2,t+1}\} \\ \vdots \\ \mathbb{E}\{Z_{\bar{k},t+1}\} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{\bar{k}} \end{pmatrix} > 0. \quad [10.2]$$

In the above, the factors $Z_{k,t}$ range from the plain betas, namely well-known factors such as the Capital Asset Pricing Model (CAPM)-like excess market return to alternative/exotic/smart betas, which represent the value added by the portfolio manager.

Chapter written by Attilio MEUCCI*.

**KKR & Symmys*

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If the multi-period model [10.1] is identified, we can proceed to replicate the factors $Z_{k,t}$ with suitable investments, and cash the premia [10.2] λ_k period after period. Unearthing and investing in the betas is the ultimate goal of systematic investment.

The standard approach to systematic investment proceeds as follows: first, we guess potential predictive signals for the risk drivers; second, we transform the signals into factors, which we replicate *via* investable portfolios.

In section 10.2, we explain the notion of signal and we describe different classes of signals, such as fundamental, pricing and statistical. We refer to the longer version at symmys.com/node/3300 for the important, yet tedious step of signal filtering.

In section 10.3, we introduce the fundamental law of asset management, which inspires the construction of the factors.

In section 10.4, we describe how to build factors from signals via (flexible) characteristic portfolios, and how to backtest the factors.

In the longer version of this chapter at symmys.com/node/3300, we discuss in more depth the connections between smart beta and APT, and we point the readers toward more recent developments in building quantitative strategies, such as *machine learning* [WIK 15a] techniques. Furthermore, at symmys.com/node/3300, we provide all the codes.

10.2. Signals

Signals are the “Holy Grail” of quantitative portfolio management. Constructing signals is the main skill of quantitative portfolio managers.

In full generality, a *signal* s_t is the (possibly vector-valued) output of a function that summarizes relevant features from the information set \mathbf{i}_t available in the market at the current time t :

$$s_t \equiv \text{signal}_\phi(\mathbf{i}_t), \quad [10.3]$$

where ϕ is a set of parameters for the function *signal* that is used to build the signal.

A *signal cluster* $\mathbf{s}_t^{\text{cluster}}$ is a set of signals generated by common themes/characteristics, which are used in conjunction:

$$\mathbf{s}_t^{\text{cluster}} \equiv \begin{pmatrix} s_{1,t}^{\text{cluster}} \\ \vdots \\ s_{k,t}^{\text{cluster}} \end{pmatrix} \equiv \text{signal}_\phi(\mathbf{i}_t), \quad [10.4]$$

where $signal_\phi$ is now a vector-valued function parametrized by ϕ . To simplify the notation, we drop the apex “cluster” whenever we focus on a generic signal cluster.

EXAMPLE 10.1.— For a publicly traded stock, the value signal is the value-to-book ratio of the stock, i.e. the ratio of the traded value of the firm v_t (from the stock market), to the book value [WIK 15b] of the firm $book_val_t$ (from accounting data).

$$s_t^{value} \equiv \frac{v_t}{book_val_t}, \quad [10.5]$$

see [FAM 93] and later [ASN 13]. Value stocks are those for which the value signal is large. Growth stocks are those for which the value signal is small. The value signal cluster $s_t^{value} \equiv (s_{1,t}^{value}, \dots, s_{k,t}^{value})$ is the set of value signals from all the k stocks in a given market.

We now proceed to discuss these categories.

Fundamental signals rely on observable characteristics of the risk drivers in a given market. Fundamental signals are the archetypal signal, widely discussed in academia and applied in the industry.

Accounting signals are a class of fundamental signals that are available for each risk driver and allow for relative comparison across the risk drivers. Works in this direction include [ZHA 05, CHA 01] and [LEW 04].

EXAMPLE 10.2.— The value signal [10.5] is fundamental. Another fundamental signal, also considered in [FAM 93], is the size signal, defined as:

$$s_t^{size} = \ln\left(\frac{1}{h_t^{mkt} v_t}\right), \quad [10.6]$$

where h_t^{mkt} is the number of outstanding shares of the stock. Small capitalization stocks (“small caps”) are those for which the size signal is large. Large capitalization stocks (“large caps”) are those for which the size signal is small. The size signal cluster $s_t^{size} \equiv (s_{1,t}^{size}, \dots, s_{k,t}^{size})$ is the set of size signals from all the k stocks in a given market. The value signal [10.5] and the size signal [10.6] have become standard in the equity industry [INV 15].

Fundamental signals need not be defined in terms of accounting data. Among such fundamental signals, particularly important are carry, curve and macrosignals.

Carry signals are triggered by the carry of an instrument.

In the fixed-income carry trade, the manager purchases high-yielding long-maturity bonds by selling low-yielding short-maturity bonds, rolling down the curve and cashing in the profits from the positive carry.

In the foreign exchange carry trade, the manager benefits from the interest rate differential in two currencies.

Using the general definition of carry, we can implement carry signals across all asset classes, including volatility and commodities trading. Refer to the longer version at symmys.com/node/3300.

Curve/surface signals are similar in nature to carry signals, though they are pure heuristics driven by the shape (typically slope or curvature) of the term/smile structure in a given market.

In fixed-income trading, the curve signal is a linear combination of points on the interest rate curve. In commodities trading, simple curve signals include the *backwardation* [WIK 15c] and *contango* [WIK 15d] signals (spread of two prices for different times to maturity).

In volatility trading, the curve has two dimensions: the *calendar signal* is the difference of the implied volatility between two reference points with the same moneyness; the *smile signal* is the difference of the implied volatility with the ATM implied volatility with the same tenor; the *skew signal* is the difference of the in-the-money implied volatility with the out-of-the-money implied volatility with the same tenor.

Macrosignals are triggered by macroeconomic variables, such as gross domestic product, inflation, etc.

Carry, curve and macrosignals are better used after filtering the signal.

Pricing signals follow from measuring the dislocation between a driver and the value of the driver implied by a pricing model. The intuition behind pricing signals is that the real driver should converge eventually toward its fair value, as provided by the pricing model.

EXAMPLE 10.3.— *In fixed-income trading, a signal can be the difference between the realized yield with time to maturity τ and the same yield, as implied by the Vasicek pricing model:*

$$s_{\tau,t}^{Vas} \equiv y_t(\tau) - y^{Vas}(\tau; \theta_t). \quad [10.7]$$

EXAMPLE 10.4.— *In volatility space, a pricing signal can be constructed from the discrepancy between the actual implied volatility $\sigma_t(m, \tau)$ and the implied volatility $\sigma^{Hes}(m, \tau; \theta_t)$ consistent with the Heston pricing model, similarly to the fixed-income case in Example 10.3.*

Unlike pricing and fundamental signals, statistical signals are created by analyzing the time series of financial data by means of statistical techniques, such as *cointegration* and *spectral theory* [WIK 15e].

A *cointegration signal* is the z-score of a cointegrated, or mean-reverting, time series. When the z-score is positive, the mean-reverting cointegrated process is expected to decrease, when the z-score is negative, the mean-reverting cointegrated process is expected to increase. Cointegration signals generalize *pairs-trading* [WIK 15f], see [AVE 10].

EXAMPLE 10.5.— *Equity pairs trading is a cointegration strategy between two stocks, see [ALE 02].*

A *spectral signal* is a signal built on the analysis of time series in the frequency domain. Key tools for the construction of such signals are *Fourier theory* [WIK 15g] and *wavelet theory* [WIK 15h].

Similarly to cointegration and spectral signals, *technical signals* process one or more time series of financial data. Unlike statistical signals, technical signals are rules, not necessarily rooted in econometrics, spectral analysis or more in general statistical theory.

The simplest example of technical signal is *momentum*, studied in [CAR 97] and later, among others, in [JEG 01, HON 00, BAR 12] and [DAN 13]. A stock displays *momentum* if recent positive returns have positive impact on the return over the next period. A stock displays *reversal* if recent positive returns have negative impact on the return over the next period.

To build the momentum signal, let us consider a generic univariate series $\mathbf{x} \equiv (x_0, x_1, \dots)$. The *exponentially weighted moving average* (EWMA) is defined as:

$$ewma_t^\nu\{\mathbf{x}\} \equiv \gamma_t \sum_{s \leq t} e^{-\nu(t-s)} x_s, \quad [10.8]$$

where γ_t is a normalizing factor $\gamma_t \equiv 1/\sum_{s \leq t} e^{-\nu(t-s)} \approx 1 - e^{-\nu}$ and $\nu > 0$ determines the half-life of the moving average $\tau = (\ln 2)/\nu$.

Let $\Delta \mathbf{x}_n \equiv (\Delta x_{n,0}, \Delta x_{n,1}, \dots)$ be the time series of (the increments of) the generic n -th risk driver. The *momentum signal* is the EWMA of the time series:

$$s_{n,t}^{mom} \equiv ewma_t^\nu \{\Delta \mathbf{x}_n\}. \tag{10.9}$$

EXAMPLE 10.6.— *In equities trading, the risk drivers are the log-prices, or log-values, of each stock $X_{n,t} = \ln V_{n,t}$. Thus, the momentum signal [10.9] reads:*

$$s_{n,t}^{mom} = ewma_t^\nu \{\ln(\frac{v_{n,1}}{v_{n,0}}), \ln(\frac{v_{n,2}}{v_{n,1}}), \dots\}. \tag{10.10}$$

Much more complex signals can be created, using combinations of moving averages or more general filters. We refer the readers to the specialized literature, see, for example, [LO 00] or [WIK 15i].

10.3. Fundamental law of active management

Here, we present the fundamental law of active management in its rawest format, at the level of the risk drivers, rather than at the level of returns. We recall that the risk drivers for a given market are the variables $\mathbf{X}_t \equiv (X_{1,t}, \dots, X_{\bar{n},t})$ which drive the P&L or the return of the various instruments. For instance, risk drivers are log-prices for stocks, the term structure of interest rates for bonds, implied volatility surfaces for options, etc.

In order to be useful, a signal or, more in general, a signal cluster must be *predictive*, in that the signal cluster $\mathbf{S}_t \equiv (S_{1,t}, \dots, S_{\bar{k},t})'$ and the next-period risk-drivers $\mathbf{X}_{t+1} \equiv (X_{1,t+1}, \dots, X_{\bar{n},t+1})'$ are not independent:

$$\left(\begin{matrix} \mathbf{X}_{t+1} \\ \mathbf{S}_t \end{matrix} \right) \sim f(\mathbf{X}_{t+1}, \mathbf{S}_t) \neq f_{\mathbf{X}_{t+1}} f_{\mathbf{S}_t}. \tag{10.11}$$

EXAMPLE 10.7.— *In equities trading, the risk drivers are the log-prices, or log-values, of the stock $\mathbf{X}_t \equiv \ln \mathbf{V}_t$. The value signal cluster \mathbf{S}_t^{value} from Example 10.1 is predictive if the next-step risk-drivers \mathbf{X}_{t+1} , or equivalently the log-returns $\Delta \mathbf{X}_{t+1} \equiv \ln \mathbf{V}_{t+1} - \ln \mathbf{V}_t$, depend on some extent of \mathbf{S}_t^{value} .*

The fundamental law of asset management quantifies the aggregate predictiveness of a signal cluster on a market.

More precisely, let us assume that the joint distribution of risk drivers and signals [10.11], conditioned on all the information available at time t , *except* for the signal(s), is normal:

$$\left(\begin{matrix} \mathbf{X}_{t+1} \\ \mathbf{S}_t \end{matrix} \right) | \mathbf{i}_t \sim N\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{X};t} \\ \boldsymbol{\mu}_{\mathbf{S};t} \end{pmatrix}, \begin{pmatrix} \sigma_{\mathbf{X};t}^2 & \sigma_{\mathbf{X};t} \boldsymbol{p}_{\mathbf{X},\mathbf{S};t} \boldsymbol{\sigma}_{\mathbf{S};t} \\ \sigma_{\mathbf{S};t} \boldsymbol{p}'_{\mathbf{X},\mathbf{S};t} \boldsymbol{\sigma}_{\mathbf{X};t} & \sigma_{\mathbf{S};t}^2 \end{pmatrix} \right), \tag{10.12}$$

where $\sigma_{\mathbf{X};t} = \sigma'_{\mathbf{X};t}$ is the Riccati root of the positive definite covariance matrix of the risk drivers $\sigma^2_{\mathbf{X};t} \equiv \mathbb{C}v_t\{\mathbf{X}_{t+1}\}$; and similar for the signal cluster $\sigma^2_{\mathbf{S};t} \equiv \mathbb{C}v_t\{\mathbf{S}_t\}$. The linkage matrix $\mathbf{p}_{\mathbf{X},\mathbf{S};t} \equiv \mathbb{C}r_t\{\tilde{\mathbf{X}}_{t+1}, \tilde{\mathbf{S}}_t\}$ is the correlation between the standardized risk drivers $\tilde{\mathbf{X}}_{t+1} \equiv \sigma^{-1}_{\mathbf{X};t}(\mathbf{X}_{t+1} - \boldsymbol{\mu}_{\mathbf{X};t})$ and the standardized signals $\tilde{\mathbf{S}}_t \equiv \sigma^{-1}_{\mathbf{S};t}(\mathbf{S}_t - \boldsymbol{\mu}_{\mathbf{S};t})$, where $\boldsymbol{\mu}_{\mathbf{X};t} \equiv \mathbb{E}_t\{\mathbf{X}_{t+1}\}$ and $\boldsymbol{\mu}_{\mathbf{S};t} \equiv \mathbb{E}_t\{\mathbf{S}_t\}$.

EXAMPLE 10.8.– Consider the case of \bar{k} jointly standard normal driver-signal pairs $(X_{k,t+1}, S_{k,t})$ with positive correlation $\lambda > 0$, where all the pairs $(X_{k,t+1}, X_{k',t+1})$ and $(S_{k,t}, S_{k',t})$ are (conditionally) independent across time. Thus, the joint distribution [10.12] reads

$$\begin{pmatrix} \mathbf{X}_{t+1} \\ \mathbf{S}_t \end{pmatrix} | \mathbf{i}_t \sim N\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbb{I}_{\bar{k}} & \lambda \mathbb{I}_{\bar{k}} \\ \lambda \mathbb{I}_{\bar{k}} & \mathbb{I}_{\bar{k}} \end{pmatrix}\right), \tag{10.13}$$

and the linkage matrix is $\mathbf{p}_{\mathbf{X},\mathbf{S};t} = \lambda \mathbb{I}_{\bar{k}}$.

The linkage matrix $\mathbf{p}_{\mathbf{X},\mathbf{S};t}$ is the key to the predictiveness of the signal cluster.

Indeed, let us define the *information coefficient* $ic_{k,t}$ of the k -th signal in the cluster as the total (square) correlation accumulated by the signal over all the risk drivers:

$$ic_{k,t} \equiv \sqrt{\sum_{n=1}^{\bar{n}} [\mathbf{p}_{\mathbf{X},\mathbf{S};t}]^2_{n,k}}. \tag{10.14}$$

EXAMPLE 10.9.– Continuing from Example 10.8, the information coefficient [10.14] is the same across all signals and time:

$$ic_{k,t} \equiv ic = \lambda. \tag{10.15}$$

Given a signal occurrence \mathbf{s}_t , let us define the conditional excess signal-to-noise ratio achieved by a linear combination (identified by a vector \mathbf{v}) of the excess-risk drivers:

$$sn_{\mathbf{v}}(\mathbf{s}_t) \equiv \frac{\mathbb{E}_t\{X_{\mathbf{v}} | \mathbf{s}_t\}}{\text{Sd}_t\{X_{\mathbf{v}} | \mathbf{s}_t\}}, \tag{10.16}$$

where the excess variable is $X_{\mathbf{v}} \equiv \mathbf{v}'(\mathbf{X}_{t+1} - \boldsymbol{\mu}_{\mathbf{X};t})$.

EXAMPLE 10.10.– We continue from Example 10.9. The conditional distribution of the risk drivers given the signals reads:

$$\mathbf{X}_{t+1} | \mathbf{s}_t \sim N(\lambda \mathbf{s}_t, (1 - \lambda^2) \mathbb{I}_{\bar{k}}). \tag{10.17}$$

Hence, since normal distributions are elliptical [WIK 15j], we have $\mathbf{v}'(\mathbf{X}_{t+1}|\mathbf{s}_t - \boldsymbol{\mu}_{\mathbf{X}}) \sim N(\lambda(\mathbf{v}'\mathbf{s}_t), (1 - \lambda^2) \|\mathbf{v}\|^2)$. Therefore, the conditional excess signal-to-noise ratio [10.16] reads:

$$sn_{\mathbf{v}}(\mathbf{s}_t) = \frac{\mathbb{E}_t\{X_{\mathbf{v}}|\mathbf{s}_t\}}{\mathbb{S}d_t\{X_{\mathbf{v}}|\mathbf{s}_t\}} = \frac{\lambda}{\sqrt{1-\lambda^2}\|\mathbf{v}\|}(\mathbf{v}'\mathbf{s}_t). \tag{10.18}$$

Thus, we define the *maximal conditional signal-to-noise ratio*:

$$max_sn(\mathbf{s}_t) \equiv \max_{\mathbf{v}} sn_{\mathbf{v}}(\mathbf{s}_t), \tag{10.19}$$

which depends on the realization of the signal \mathbf{s}_t ; and the (*L2 expected*) *maximal conditional signal-to-noise*:

$$\|max_sn\|_2 \equiv \sqrt{\mathbb{E}\{[max_sn(\mathbf{S}_t)]^2\}}. \tag{10.20}$$

EXAMPLE 10.11.– We continue from Example 10.10. The maximal conditional signal-to-noise [10.19] reads:

$$max_sn(\mathbf{s}_t) = \frac{\lambda}{\sqrt{1-\lambda^2}} \|\mathbf{s}_t\|. \tag{10.21}$$

Hence, the (*L2 expected*) *maximal conditional signal-to-noise* [10.20] reads:

$$\|max_sn\|_2 = \sqrt{\mathbb{E}\{\frac{\lambda^2}{1-\lambda^2} \mathbf{S}'_t \mathbf{S}_t\}} = \frac{\lambda}{\sqrt{1-\lambda^2}} \sqrt{\bar{k}}, \tag{10.22}$$

where we used the fact that $\mathbb{E}\{\mathbf{S}'_t \mathbf{S}_t\} = \text{tr}(\mathbb{E}\{\mathbf{S}_t \mathbf{S}'_t\}) = \text{tr}(\mathbb{I}_{\bar{k}}) = \bar{k}$.

The *fundamental law of active management* states that, when the signal is weak, i.e. $ic_{k,t} \ll 1$, the maximal predictability provided by a signal cluster, as represented by the maximal signal-to-noise ratio achievable, reads:

$$\|max_sn\|_2 \approx \sqrt{ic_{1,t}^2 + \dots + ic_{k,t}^2} \tag{10.23}$$

In particular, if all the signals in the cluster have the same information coefficient [10.14], or $ic_{k,t} \equiv ic_t$, then the maximum predictability, and thus ultimately profitability, is the product of (1) the information coefficient of the signal cluster [10.14], chosen by the portfolio managers using their skills and (2) the *breadth* of the cluster, as represented by the square root of the number of signals in the cluster:

$$\|max_sn\|_2 \approx \underset{\substack{\uparrow \\ \text{max-predictability}}}{ic_t} \times \underset{\substack{\uparrow \\ \text{skill}}}{ic_t} \times \underset{\substack{\uparrow \\ \text{breadth}}}{\sqrt{\bar{k}}}. \tag{10.24}$$

EXAMPLE 10.12.— We continue from Example 10.11. Suppose $\lambda \ll 1$ so that $1 - \lambda^2 \approx 1$. If we consider one driver (think of a stock) with one related signal ($\bar{k} = 1$), we can achieve a maximum signal-to-noise ratio of the order $\|max_sn\|_2 \approx \lambda \ll 1$, as follows from the fundamental law [10.23]. Instead, if we consider a large portfolio of independent drivers, each with an independent signal, we can achieve a maximum signal-to-noise ratio $\|max_sn\|_2 \approx \lambda\sqrt{\bar{k}} > 1$, up to consider a large enough market ($\bar{k} \gg 1$).

The fundamental law of asset management [10.53] connects the maximum signal-to-noise (Sharpe) ratio of a portfolio with the strength of a signal. A *different, much simpler* result states that a portfolio of uncorrelated bets has larger Sharpe ratio than any of the individual bets, without any mention of signals or predictiveness. It is surprising how often the simpler result is mistakenly advertised as the fundamental law of asset management.

EXAMPLE 10.13.— Consider the case of \bar{k} independent normal risk drivers with mean λ and unit variance:

$$\mathbf{X}_{t+1} \sim N(\lambda \mathbf{1}_{\bar{k}}, \mathbb{I}_{\bar{k}}). \tag{10.25}$$

Thus, the signal-to-noise ratio of each driver $X_{t,k}$ is λ , whereas the signal-to-noise ratio of an equally weighted portfolio $\frac{1}{\bar{k}} \sum_{k=1}^{\bar{k}} X_{t,k}$ is $\lambda\sqrt{\bar{k}}$.

10.4. Factors construction

In Chapter 9, we have obtained signals such as, say, momentum s_t^{mom} , defined in [10.9]. In this chapter, we use signals for portfolio construction, i.e. to build factors as in [10.1]. Intuitively, a factor is a strategy constructed from one signal, which gives rise to stream of P&L's with positive mean [10.2].

More precisely, a *factor* is a self-financing strategy, namely a feasible sequence of portfolio allocations $\{\mathbf{h}_t^{signal}\}_{t \geq 0}$, where each allocation $\mathbf{h}_t^{signal} \equiv (h_{1,t}^{signal}, \dots, h_{\bar{n},t}^{signal})'$ is held over the generic period $(t, t + 1]$ and is a function of information available at time t , as summarized by one cluster of signals \mathbf{s}_t , or $\mathbf{h}_t^{signal} \equiv h(\mathbf{s}_t)$.

The strategy is rebalanced *without* transaction costs. The strategy generates the stream of P&L's $\Pi_{t \rightarrow t+1}^{signal} = \sum_{n=1}^{\bar{n}} h_{n,t}^{signal} \Pi_{n,t \rightarrow t+1}$, where $\Pi_{n,t \rightarrow t+1}$ is the P&L generated by one unit of the n -th instrument over the interval $(t, t + 1]$:

$$\underbrace{\mathbf{s}_t}_{\text{signal}} \quad \rightarrow \quad \underbrace{\mathbf{h}_t^{signal}}_{\text{factor portfolio}} \quad \rightarrow \quad \underbrace{\Pi_{t \rightarrow t+1}^{signal} = \mathbf{h}_t^{signal} \Pi_{t \rightarrow t+1}}_{\text{P\&L}}. \tag{10.26}$$

The nomenclature “factor” is justified because the portfolios $\{\mathbf{h}_t^{signal}\}_{t \geq 0}$ are used to extract the factors of an APT-like linear factor model.

The factor stream of P&L's $\{\Pi_{t \rightarrow t+1}^{signal}\}_t$ in [10.26] is an ergodic process, i.e. a process whose future realized mean value is the same as the past realized (positive) mean value.

We emphasize that a factor is a strategy that cannot be implemented. What can be implemented is a strategy that pays transaction costs and is adversely affected by *market impact* [WIK 15k].

In section 10.4.1, we postulate the allocation \mathbf{h}_t directly from the signal cluster \mathbf{s}_t .

In section 10.4.2, we first associate with the signal cluster \mathbf{s}_t a set of expected returns for the instruments in the market, and then we compute the factor-replicating characteristic portfolio.

To overlay constraints to the approach in the previous section, and build more flexible characteristic portfolios we refer to the longer version of this chapter at symmys.com/node/3300.

10.4.1. Direct construction

In standard applications, the dimension of the signal cluster is the same as the dimension of the risk drivers, or $\bar{k} = \bar{n}$. In such situations, the signals are called *relative value* or *cross-sectional*.

There exists a simple, effective way to build cross-sectional portfolios from signals, first pioneered by Rosen [ROS 85] and Fama [FAM 93]. From the definition of signal \mathbf{s}_t in [10.11], we *postulate* that a positive signal implies a bullish view on the respective risk driver, and a negative signal implies a bearish view on the respective risk driver. Accordingly, let us denote by $\mathbf{h}_t \equiv (h_{1,t}, \dots, h_{\bar{n},t})'$ the holdings (number of shares, notional of bonds, etc.) of a portfolio. The direct construction proceeds as follows:

$$\mathbf{h}_t^{signal} : \begin{cases} \text{long equal-weight portfolio with large signal entry} \\ \text{short equal-weight portfolio with small signal entry} \end{cases}, \quad [10.27]$$

where long and short legs have the same market value with opposite sign in order to build a *dollar neutral* portfolio:

$$\mathbf{h}_t^{signal} \mathbf{v}_t \equiv 0. \quad [10.28]$$

EXAMPLE 10.14.— A typical dollar neutral portfolio which is directly built in terms of signals as in [10.27] is the High Minus Low portfolio \mathbf{h}_t^{HML} , where the long leg is obtained going long on stocks whose value signal $s_{n,t}^{value}$ [10.5] is larger than the signal of the first 70% of the ranked stocks, and the short leg going short on stocks whose value signal $s_{n,t}^{value}$ [10.5] is smaller than the first 30%. Refer to [FAM 93] for details.

10.4.2. Characteristic portfolios

Here, we discuss the factor construction based on characteristic portfolios. We generalize the standard construction discussed, for example, in [GRI 98] to the case of general instruments, with general risk drivers.

Before proceeding, we introduce some nomenclature.

A *characteristic* is a vector of features $\beta_t^{char} \equiv (\beta_{1,t}^{char}, \dots, \beta_{\bar{n},t}^{char})'$, one for one unit of each instrument in a given \bar{n} dimensional market.

The *exposure* to the characteristics β_t^{char} of a portfolio defined by the holdings $\mathbf{h}_t \equiv (h_{1,t}, \dots, h_{\bar{n},t})'$ is the sum:

$$\beta_{\mathbf{h},t}^{char} \equiv \mathbf{h}_t' \beta_t^{char} = \sum_{n=1}^{\bar{n}} h_{n,t} \beta_{n,t}^{char}, \quad [10.29]$$

where the holdings can be long ($h_{n,t} > 0$) or short ($h_{n,t} < 0$).

The *characteristic portfolio* \mathbf{h}_t^{char} for arbitrary characteristics β_t^{char} is the minimum-variance portfolio with unit exposure to the characteristics $\mathbf{h}_t^{char'} \beta_t^{char} = 1$, which reads:

$$\mathbf{h}_t^{char} \equiv \frac{(\boldsymbol{\sigma}_{\Pi;t}^2)^{-1} \beta_t^{char}}{\beta_t^{char'} (\boldsymbol{\sigma}_{\Pi;t}^2)^{-1} \beta_t^{char}}, \quad [10.30]$$

where $\boldsymbol{\sigma}_{\Pi;t}^2 \equiv \mathcal{C}v_t\{\Pi_{t \rightarrow t+1}\}$ is the ex-ante covariance matrix of the P&L's $\Pi_{t \rightarrow t+1} \equiv (\Pi_{1,t \rightarrow t+1}, \dots, \Pi_{\bar{n},t \rightarrow t+1})'$ of the instruments in the market, conditioned on all the information available at time t , *except* for the signal(s).

The characteristic portfolio has unit exposure to the characteristic:

$$\mathbf{h}_t^{char'} \beta_t^{char} = 1, \quad [10.31]$$

and its ex-ante conditional variance reads:

$$\mathbb{V}_t\{\mathbf{h}_t^{char'} \boldsymbol{\Pi}_{t \rightarrow t+1}\} = (\boldsymbol{\beta}_t^{char'} (\boldsymbol{\sigma}_{\boldsymbol{\Pi};t}^2)^{-1} \boldsymbol{\beta}_t^{char'})^{-1}. \quad [10.32]$$

Now, we are ready to outline the steps of factor construction based on characteristic portfolios.

Step 1. We estimate the distribution of the risk drivers, and we extract the first and second moments conditioned on all the information available at time t , *except* for the signal(s):

$$\boldsymbol{\mu}_{\mathbf{X};t} \equiv \mathbb{E}_t\{\mathbf{X}_{t+1}\}, \quad \boldsymbol{\sigma}_{\mathbf{X};t}^2 \equiv \mathbb{C}v_t\{\mathbf{X}_{t+1}\}. \quad [10.33]$$

Often, but not always, we can assume that the risk drivers are approximately a martingale, i.e. $\boldsymbol{\mu}_{\mathbf{X};t} \equiv \mathbf{x}_t$.

EXAMPLE 10.15.– *Continuing from Example 10.14, in the case of equities, $\mathbf{X}_t \equiv \ln \mathbf{V}_t$, the conditional expectation in [10.33] can be proxied as follows $\boldsymbol{\mu}_{\mathbf{X};t} \approx \ln \mathbf{v}_t + r_{t \rightarrow t+1}^{rf} \mathbf{1}_{\bar{n} \times 1}$.*

Step 2. We model the joint distribution of the risk drivers and the predictive signal [10.11] as a normal, potentially time-dependent distribution with constant correlation structure:

$$\begin{pmatrix} \mathbf{X}_{t+1} \\ \mathbf{S}_t \end{pmatrix} | \mathbf{i}_t \sim N\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{X};t} \\ \boldsymbol{\mu}_{\mathbf{S};t} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\sigma}_{\mathbf{X};t}^2 & \boldsymbol{\sigma}_{\mathbf{X},\mathbf{S};t} \\ \boldsymbol{\sigma}'_{\mathbf{X},\mathbf{S};t} & \boldsymbol{\sigma}_{\mathbf{S};t}^2 \end{pmatrix}\right), \quad [10.34]$$

where the conditioning information \mathbf{i}_t is all the information available at time t , *except* for the signal(s); and:

$$\begin{aligned} \begin{pmatrix} \boldsymbol{\sigma}_{\mathbf{X};t}^2 & \boldsymbol{\sigma}_{\mathbf{X},\mathbf{S};t} \\ \boldsymbol{\sigma}'_{\mathbf{X},\mathbf{S};t} & \boldsymbol{\sigma}_{\mathbf{S};t}^2 \end{pmatrix} &\equiv \text{Diag} \begin{pmatrix} \boldsymbol{\sigma}_{\mathbf{X};t}^{vol} \\ \boldsymbol{\sigma}_{\mathbf{S};t}^{vol} \end{pmatrix} \begin{pmatrix} \mathbf{c}_{\mathbf{X};t}^2 & \mathbf{c}_{\mathbf{X};t} \mathbf{p}'_{\mathbf{X},\mathbf{S};t} \mathbf{c}_{\mathbf{S};t} \\ \mathbf{c}_{\mathbf{S};t} \mathbf{p}'_{\mathbf{X},\mathbf{S};t} \mathbf{c}_{\mathbf{X};t} & \mathbf{c}_{\mathbf{S};t}^2 \end{pmatrix} \\ &\times \text{Diag} \begin{pmatrix} \boldsymbol{\sigma}_{\mathbf{X};t}^{vol} \\ \boldsymbol{\sigma}_{\mathbf{S};t}^{vol} \end{pmatrix}; \end{aligned} \quad [10.35]$$

and $\mathbf{c}_{\mathbf{X};t} = \mathbf{c}'_{\mathbf{X};t}$ is the Riccati root of the correlation matrix $\mathbf{c}_{\mathbf{X};t}^2 \equiv \mathbb{C}r_t\{\mathbf{X}_{t+1}\}$, and similar for $\mathbf{c}_{\mathbf{S};t}^2$; and the matrix $\mathbf{p}_{\mathbf{X},\mathbf{S};t}$ is the correlation of the standardized drivers $\tilde{\mathbf{X}}_{t+1} \equiv \mathbf{c}_{\mathbf{X};t}^{-1} \text{Diag}(\boldsymbol{\sigma}_{\mathbf{X};t}^{vol})^{-1} (\mathbf{X}_{t+1} - \boldsymbol{\mu}_{\mathbf{X};t}) \sim N(\mathbf{0}, \mathbb{I}_{\bar{n}})$ with the standardized signals $\tilde{\mathbf{S}}_t \equiv \mathbf{c}_{\mathbf{S};t}^{-1} \text{Diag}(\boldsymbol{\sigma}_{\mathbf{S};t}^{vol})^{-1} (\mathbf{S}_t - \boldsymbol{\mu}_{\mathbf{S};t}) \sim N(\mathbf{0}, \mathbb{I}_{\bar{n}})$.

We *postulate* that each normalized driver is equally affected by one and only one normalized signal, and thus their cross-correlation matrix is a multiple λ of the identity, as in Example 10.8:

$$\mathbf{p}_{\mathbf{X},\mathbf{S};t} \equiv \mathbb{C}r_t\{\tilde{\mathbf{X}}_{t+1}, \tilde{\mathbf{S}}_t\} = \lambda_t \times \mathbb{I}_{\bar{n}}, \quad [10.36]$$

Note that, by flipping the sign in the signal cluster, we can always assume that the cross-correlation is positive $\lambda_t > 0$. Furthermore, the information cross-correlation is small $0 < \lambda_t \ll 1$ because the signal is typically weak, see Figure 10.1. The cross-correlation λ_t is also the information coefficient [10.14], which is the same for each signal in the cluster, $ic_t \equiv ic_{n,t} = \lambda_t$.

Also, we *postulate* that the correlation matrices of the risk drivers and the signals are the same, i.e. $\mathbf{c}_{\mathbf{X};t} = \mathbf{c}_{\mathbf{S};t}$. Finally, we assume that the signals \mathbf{S}_t are filtered as discussed at symmys.com/node/3300, so that $\boldsymbol{\mu}_{\mathbf{S};t} = \mathbf{0}$ and $\boldsymbol{\sigma}_{\mathbf{S};t}^{vol} = \mathbf{1}_{\bar{n}}$.

The conditional expectation of the risk drivers given the signal is a function of the signal:

$$\mathbb{E}_t\{\mathbf{X}_{t+1}|\mathbf{s}_t\} = \boldsymbol{\mu}_{\mathbf{X};t} + \boldsymbol{\sigma}_{\mathbf{X};t}^{vol} \circ \mathbf{s}_t \times ic_t. \quad [10.37]$$

However, the conditional covariance is almost unaffected by the signal:

$$\mathbb{C}v_t\{\mathbf{X}_{t+1}|\mathbf{s}_t\} \approx \mathbb{C}v_t\{\mathbf{X}_{t+1}\} = \boldsymbol{\sigma}_{\mathbf{X};t}^2. \quad [10.38]$$

EXAMPLE 10.16.— *Continuing from Example 10.15, for equities the expectation conditioned on the signals \mathbf{s}_t reads:*

$$\mathbb{E}_t\{\ln \mathbf{V}_{t+1}|\mathbf{s}_t\} = \ln \mathbf{v}_t + r_{t \rightarrow t+1}^{rf} + (\boldsymbol{\sigma}_{\ln \mathbf{V};t}^{vol} \circ \mathbf{s}_t) \times ic_t. \quad [10.39]$$

Step 3. From the Taylor approximation of pricing, over the short run the P&L is approximately a linear function of the next-period risk-drivers:

$$\boldsymbol{\Pi}_{t \rightarrow t+1}|\mathbf{i}_t \approx \boldsymbol{\theta}_t + \text{Diag}(\boldsymbol{\delta}_t)(\mathbf{X}_{t+1}|\mathbf{i}_t - \mathbf{x}_t), \quad [10.40]$$

where $\boldsymbol{\theta}_t$ and $\boldsymbol{\delta}_t$ are the suitable \bar{n} -dimensional vectors.

Thus, the conditional expectation and covariance of the P&L of the market instruments follow from their counterparts for the risk drivers [10.37]–[10.38] affine equivariance of the expectation and covariance operators:

$$\boldsymbol{\mu}_{\boldsymbol{\Pi};t} \equiv \mathbb{E}_t\{\boldsymbol{\Pi}_{t \rightarrow t+1}|\mathbf{s}_t\} = \boldsymbol{\theta}_t + \text{Diag}(\boldsymbol{\delta}_t)(\boldsymbol{\mu}_{\mathbf{X};t} + \boldsymbol{\sigma}_{\mathbf{X};t}^{vol} \circ \mathbf{s}_t \times ic_t - \mathbf{x}_t) \quad [10.41]$$

$$\boldsymbol{\sigma}_{\boldsymbol{\Pi};t}^2 \equiv \mathbb{C}v_t\{\boldsymbol{\Pi}_{t \rightarrow t+1}|\mathbf{s}_t\} \approx \mathbb{C}v_t\{\boldsymbol{\Pi}_{t \rightarrow t+1}\} = \text{Diag}(\boldsymbol{\delta}_t) \boldsymbol{\sigma}_{\mathbf{X};t}^2 \text{Diag}(\boldsymbol{\delta}_t). \quad [10.42]$$

We can write the P&L [10.40] as an APT-like linear factor model:

$$\mathbf{\Pi}_{t \rightarrow t+1} \mathbf{s}_t - r_{t \rightarrow t+1}^{rf} \mathbf{v}_t = \boldsymbol{\alpha}_t + \boldsymbol{\beta}_t^{signal} Z_{t+1}^{signal} + \mathbf{U}_{t+1} \mathbf{s}_t. \quad [10.43]$$

In the above model, the excess drift without the signal is defined as $\boldsymbol{\alpha}_t \equiv \boldsymbol{\theta}_t + \text{Diag}(\boldsymbol{\delta}_t)(\boldsymbol{\mu}_{\mathbf{X};t} - \mathbf{x}_t) - r_{t \rightarrow t+1}^{rf} \mathbf{v}_t$ and is assumed null $\boldsymbol{\alpha}_t \approx \mathbf{0}$, as in the APT (see symmys.com/node/3300); the loadings are defined as:

$$\boldsymbol{\beta}_t^{signal} \equiv \text{Diag}(\boldsymbol{\delta}_t)(\boldsymbol{\sigma}_{\mathbf{X};t}^{vol} \circ \mathbf{s}_t). \quad [10.44]$$

The factor's premium is the information coefficient:

$$\mathbb{E}_t\{Z_{t+1}^{signal}\} = ic_t, \quad [10.45]$$

which is positive in accordance to [10.2]; and the residuals \mathbf{U}_{t+1} are assumed to have zero expectations and satisfy the systematic-idiosyncratic properties, as in the APT.

Furthermore, the information coefficient ic_t can be computed equivalently as follows:

$$ic_t = \frac{1}{\gamma_t} \text{tr}(\text{Cov}_t\{\mathbf{\Pi}_{t \rightarrow t+1}, \mathbf{B}_t^{signal}\}), \quad [10.46]$$

where $\mathbf{B}_t^{signal} \equiv \text{Diag}(\boldsymbol{\delta}_t)(\boldsymbol{\sigma}_{\mathbf{X};t}^{vol} \circ \mathbf{S}_t)$ and $\gamma_t \equiv \text{tr}(\text{Cov}_t\{\mathbf{B}_t^{signal}\})$.

EXAMPLE 10.17.— *Continuing from Example 10.16, from the P&L pricing function for equities we have $\boldsymbol{\theta}_t = \mathbf{0}$ and $\boldsymbol{\delta}_t = \mathbf{v}_t$, and thus we can write $\boldsymbol{\alpha}_t = \mathbf{0}$ and $\boldsymbol{\beta}_t^{signal} = \text{Diag}(\mathbf{v}_t)(\boldsymbol{\sigma}_{\ln \mathbf{V};t}^{vol} \circ \mathbf{s}_t)$.*

Step 4. We compute the characteristic portfolio [10.30] arising from the signal-induced characteristics $\boldsymbol{\beta}_t^{signal}$ [10.44]:

$$\mathbf{h}_t^{signal} \equiv \frac{(\boldsymbol{\sigma}_{\mathbf{\Pi};t}^2)^{-1} \boldsymbol{\beta}_t^{signal}}{\boldsymbol{\beta}_t^{signal'} (\boldsymbol{\sigma}_{\mathbf{\Pi};t}^2)^{-1} \boldsymbol{\beta}_t^{signal}}. \quad [10.47]$$

Note that the characteristic portfolio has in general non-zero value:

$$\mathbf{v}_t^{signal} \equiv \mathbf{h}_t^{signal'} \mathbf{v}_t. \quad [10.48]$$

To overlay constraints to the characteristic portfolio [10.47], and build more flexible factors (constant volatility, zero-investment, uncorrelated with stock market, etc.) we refer to the longer version of this chapter at symmys.com/node/3300.

EXAMPLE 10.18.— Continuing from Example 10.17 and considering the expression of the P&L covariance matrix $\sigma_{\Pi;t}^2$ [10.42], the characteristic portfolio [10.47] reads:

$$h_t^{signal} = \frac{(Diag(1./v_t)Diag(1./\sigma_{\ln V;t}^{vol})(c_{\ln V;t}^2)^{-1})s_t}{s_t'(c_{\ln V;t}^2)^{-1}s_t}, \tag{10.49}$$

where $c_{\ln V;t}^2 \equiv \mathbb{C}r_t\{\ln V_{t+1}\}$.

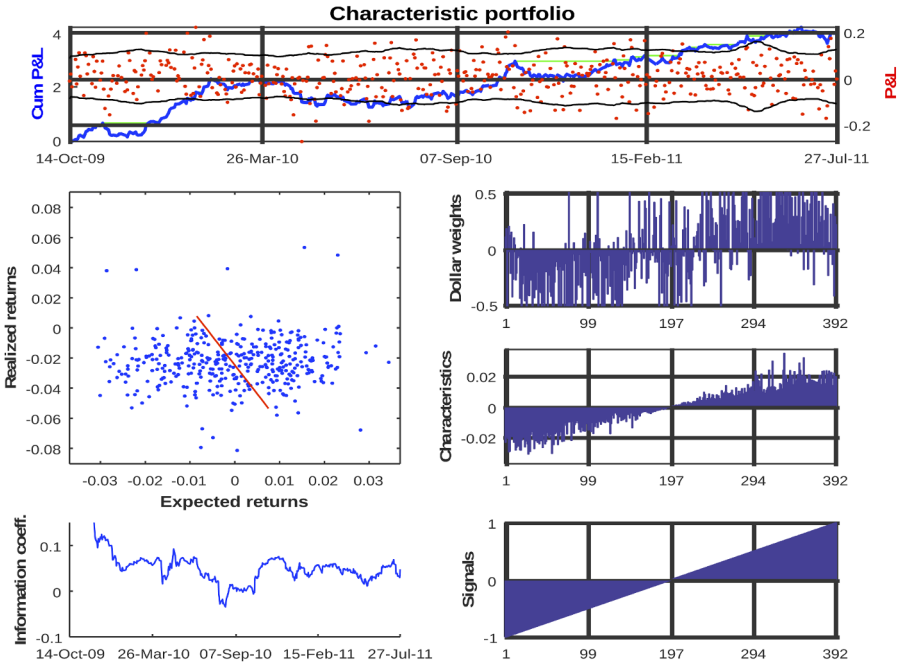


Figure 10.1. Play clip: characteristic portfolio strategy for the stocks of the S&P500 index based on reversal signals

Given the factor construction rule [10.26], we can go back in time and *backtest* the rule:

$$\left\{ \begin{array}{l} s_t \\ \text{signal} \end{array} \mapsto \begin{array}{l} h_t^{signal} \\ \text{factor portfolio} \end{array} \mapsto \begin{array}{l} \pi_{t \rightarrow t+1}^{signal} \equiv h_t^{signal} \pi_{t \rightarrow t+1} \\ \text{realized P\&L} \end{array} \right\}_{t \in \mathcal{T}}, \tag{10.50}$$

where \mathcal{T} is the whole available history of observations, or a subset thereof. Thus, we can study the time series properties of the backtest P&L $\{\pi_{t \rightarrow t+1}^{signal}\}_{t \in \mathcal{T}}$.

Arguably, the most relevant feature is that signal and factor P&L be *stationary and ergodic*, in such a way that we can rely on the backtest to infer the future behavior of the factor.

We define the *realized Sharpe ratio* of the realized excess performance over the risk-free investment:

$$\widehat{sr}_t \equiv \frac{ewma_t^\nu \{ \pi_{\cdot \rightarrow \cdot + 1}^{signal} - r_{\cdot + 1}^{rf} v_{\cdot}^{signal} \}}{ewm_sd_t^\nu \{ \pi_{\cdot \rightarrow \cdot + 1}^{signal} \}}, \quad [10.51]$$

where we used the moving average mean $ewma_t^\nu$ and the moving average standard deviation $ewm_sd_t^\nu$.

Another relevant feature is the *realized information coefficient*, which is defined as the moving average “correlation” between the realized P&L in the market and the ex-ante signal characteristics β_t^{signal} in [10.44]:

$$\widehat{ic}_t \equiv \frac{\text{tr}(ewm_cv_t^\nu \{ \pi_{\cdot + 1}, \beta_{\cdot}^{signal} \})}{\text{tr}(ewm_cv_t^\nu \{ \beta_{\cdot}^{signal} \})}, \quad [10.52]$$

where we used the moving average covariance $ewm_cv_t^\nu$.

Thus, it becomes possible to test the fundamental law of asset management:

$$\widehat{sr}_t \approx \widehat{ic}_t \times \sqrt{\bar{n}}. \quad [10.53]$$

EXAMPLE 10.19.— Consider the characteristic portfolio h_t^{signal} [10.47] invested in $\bar{n} = 392$ stocks of the S&P500 index arising from the signal-induced characteristics β_t^{signal} [10.44] based on $\bar{n} = 392$ momentum signals [10.10], filtered as described at symmys.com/node/3300. In the top plot of Figure 10.1, we show the realized daily portfolio P&L $\pi_{t \rightarrow t+1}^{signal}$ and the ex-ante estimated uncertainty band, along with realized the cumulate P&L $\pi_{0 \rightarrow t}^{signal}$ from 14th October 2009 to 27th July 2011. In the left middle plot, we display the scatter plot of the realized returns $\pi_{t \rightarrow t+1} / v_t$ of each stock at time $t + 1$ against the “expected returns” β_t^{signal} / v_t at time t , along with the best fit regression line. In the three plots on the right, we show the ranked reversal signals \tilde{s}_t in increasing order, the dollar weights $h_t^{signal} \circ v_t$ and the value of the rescaled signals $\sigma_{X;t}^{vol} \circ \tilde{s}_t$, i.e. the normalized characteristics β_t^{signal} [10.44], both ordered with respect to the signals \tilde{s}_t . Finally, we show the plot (left bottom) of the realized information coefficient \widehat{ic}_t [10.52] of the strategy. Refer to symmys.com/node/3300 for the code.

10.5. Conclusions

We presented a practical multi-step recipe to build smart beta, or more in general systematic strategies, across all types of signals and asset classes.

We provided the theoretical support for such recipe and we showed the connection with the fundamental law of active management.

We applied the multi-step recipe to a real-life case study. Further refinements and the code can be found at symmys.com/node/3300.

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Low-Risk Anomaly Everywhere: Evidence from Equity Sectors

We give strong empirical evidence of a risk anomaly in equity sectors in a number of regions and countries of developed and emerging markets, with the lowest risk stocks in each activity sector generating higher returns than would be expected given their levels of risk, and the converse outcome for the riskier stocks. We believe this evidence is a likely consequence of the fact that equity analysts and active fund managers tend to specialize in particular sectors and mainly select stocks from these sectors. Additionally, constraints restricting the deviation of sector weights in active portfolios against their market capitalization benchmarks are often used by active fund managers, in particular by quantitative managers who tend to go as far as being sector neutral. As a result, we find that sector neutral, low-risk approaches appear more efficient at generating alpha than non-sector neutral approaches, with the latter showing strong sector allocation toward financials, utilities and consumer staples than sector neutral, at least when applied to developed countries in a global universe. We also discuss some properties of low-risk investing, such as tail risk, turnover and liquidity.

11.1. Introduction

Low-risk investing in equities has been in the spotlight in recent years probably due, in particular, to the disappointing performance of equity markets since the start of the new millennium and up until the 2008 crisis. The main focus of low-risk investing is to reduce portfolio risk, defending the portfolio in equity market

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downturns, while capturing the positive alpha from low-risk stocks to improve risk-adjusted returns. Indeed, the positive alpha found in low-risk stocks explains why the Sharpe ratio of strategies invested in these stocks has been larger than that for the market capitalization index. Low-risk investing also naturally excludes the riskier stocks which have been delivering the poorest risk-adjusted returns and have had significant negative alpha.

Low-risk investing dates back to the seminal paper of Haugen and Heins [HAU 72] with empirical evidence that, between 1926 and 1969, portfolios systematically investing in U.S. low-volatility stocks would have delivered much larger returns than expected from their low level of beta, while portfolios invested in high-volatility stocks would have delivered returns much below what should have been expected from their high level of beta. Brennan [BRE 71] and Black [BLA 72] showed that the violation of one of the assumptions behind the capital asset pricing model (CAPM) – that investors have no constraints, e.g. on leverage or borrowing – is sufficient to reduce the slope of the relationship between returns and beta. Blitz [BLI 14] has recently reviewed the academic literature and summarized the different effects that have been proposed by academics to explain the low-risk anomaly.

The low-risk anomaly appears almost universally. Haugen and Baker [HAU 12] demonstrated empirically that it can be found in the cross-section of stock returns of almost all developed and emerging market countries in the world. The comprehensive empirical analysis of De Carvalho *et al.* [DEC 14] strongly suggests that the low-risk anomaly goes beyond equity markets and can also be found in the cross-section of bond returns of all major segments of fixed-income markets and regions. Their results show that portfolios invested in low-risk bonds with the lowest beta generated the largest positive alpha, while portfolios invested in the riskier bonds with the highest beta generated the most negative alpha. This result was found for government bonds, quasi- and foreign government bonds, securitized and collateralized bonds, corporate investment-grade bonds, corporate high-yield bonds, emerging market bonds and aggregations of some of these universes, and for bonds in USD, EUR, GBP and JPY. Frazzini and Pedersen [FAR 14] suggest that the low-risk anomaly is also observed in commodities, currencies and at top-down level in fixed income and equities, i.e. in the cross-section of the returns of currency forwards, index futures, equity and treasury country indices, portfolios aggregated by ratings and in the cross-section of all these put together. Baker *et al.* [BAK 14] have recently looked at the decomposition of the low-risk anomaly into top-down country and industry contributions and bottom-up contributions. They found a risk anomaly in the cross-section of country returns and, to a lesser extent, in the cross-sectional of industry returns. Asness *et al.* [ASN 14] gave stronger evidence of a low-risk anomaly in the cross-section of industry returns by using more granular industry definitions.

The low-risk anomaly is not only found in the cross-section of asset classes but also in the time series of asset class premiums and in the time series of factor premiums. Perchet *et al.* [PER 14a] showed that the time series of asset class returns shows volatility clustering, i.e. the volatility forms two distinct volatility regimes, one with low volatility and high average returns and one with high volatility and low average returns, or even negative, for most asset classes. In turn, Perchet *et al.* [PER 14b] showed that the time series of value and momentum factor returns in equity, government bonds and currency markets also shows volatility clustering, with two distinct volatility regimes: higher returns for the low-volatility regime and lower returns for the high-volatility regime.

In this chapter, we aim (1) to investigate the universality of the risk volatility anomaly by focusing on the cross-section of stock returns in equity sectors in developed countries and emerging market countries, in aggregate and at individual country level, and (2) to compare sector neutral low-risk investing with the traditional sector-biased low-risk approaches that are typically overexposed to defensive sectors.

We also aim to shed additional light on the results of Baker *et al.* [BAK 14], who found that the risk anomaly is stronger at stock level by neutralizing industry exposure than in the cross-section of industry returns, contrary to what should have been expected from the suggestion by Samuelson [SAM 98] that stocks are priced more efficiently than industries because industries have fewer substitutes than stocks, an argument they used to motivate their research. The results of Asness *et al.* [ASN 14] also point in the same direction, i.e. that the risk anomaly can be more efficiently captured by neutralizing industry exposures than by investing at top-down level in low-risk industries and avoiding the riskier industries. Moreover, we did not find any explicit effect that could explain these results in the available literature.

In fact, we will argue that one possible explanation comes from the active management industry and the way active managers tend to pick stocks for their active portfolios. This explanation is thus closely related to what Blitz [BLI 14] calls “relative utility” and “agents maximize option value”, but is likely to be a result of the practicalities of how fund managers tend to operate and manage portfolios with the objective of outperforming a benchmark index.

11.2. Low volatility or low beta?

Neither the stock volatility nor the stock beta is constant over time. Hence, low-risk investing requires periodic rebalancing to take into account that some stocks

which have been low risk in the past may no longer be low risk in the future. A strategy periodically rebalancing the stock allocation toward the minimum variance portfolio is an example of a low-risk strategy that can be shown to have delivered higher risk-adjusted returns than expected from its low level of beta. However, as shown by De Carvalho *et al.* [DEC 12], the minimum variance portfolio can be replicated by simple portfolio strategies based on equally overweighting low-beta stocks and underweighting high-beta stocks. Thus, we prefer to use simpler strategies that involve selecting stocks from risk rankings to build low-risk portfolios, rather than using minimum variance strategies.

Research on the low-risk anomaly often relies on building portfolios invested in a selection of stocks with the lowest *ex ante* beta, e.g. [BAK 14] and [ASN 14], and often in a selection of stocks with the lowest *ex ante* volatility, e.g. [BAK 12] and [LI 14]. We chose to use *ex ante* volatility instead of *ex ante* beta for the reasons listed below.

We built two strategies and applied them to the MSCI World Index¹ stock universe. In the first strategy, stocks are first ranked every month by their level of *ex ante* beta² calculated at that point in time from a 2-year rolling regression of the stock total returns in excess of cash against the total returns of MSCI World Index in excess of cash, with returns in USD. Every month, we built an equally weighted portfolio invested in the stocks with the lowest *ex ante* beta at the start of the month holding this portfolio until the next monthly rebalancing. We kept only 10% of the stocks in the universe. The historical simulation of this strategy runs from January 1995 to August 2013 and its results are compared with a similar strategy, which differs only in the fact that instead of *ex ante* beta we used a 2-year rolling standard deviation of returns².

1 Due to licensing constraints, for data prior to August 2006, we use the global universe of stocks of developed countries in the Exshare database for which the market-cap allocation minimizes the tracking risk against the total returns of the MSCI World Index in U.S. dollars. Therefore, the universe for the period prior to August 2006 may not be exactly the same universe that underlies the MSCI World Index. We believe that our universe is likely to contain more stocks than those in the MSCI Index in the period January 1995–August 2006. In our view, however, the impact of not using exactly the MSCI World index universe on the results of this chapter should be minor.

2 Only stocks with at least 450 days of pricing data in the 2 years used in the estimation of *ex ante* volatility and beta are retained. Otherwise, they are excluded from the selection process. The results are not very sensitive to the length of the window used in the estimation of the *ex ante* volatility and beta. But for shorter windows, the error estimation increases which generates more turnover in the strategy, while for longer windows more stocks will be excluded for not having sufficient pricing data. A 2-year rolling window offers a good compromise between these two effects.

Low-volatility stocks have low beta because beta is simply the product of the stock volatility by the correlation of returns with the market returns divided by the market volatility. However, not all low-beta stocks have low volatility. Some higher volatility stocks can be low beta due to the low correlation with the market. If we look at the average overlap between the portfolios behind the two strategies, we find that it is high at 55%. This is in fact high knowing that there are about 1,700 stocks on average in the MSCI World index and that we retain only 10% of these stocks in each case. But despite being high, the universe of low-volatility stocks is not exactly the same as the universe of low-beta stocks. We also observe that the strategy based on low beta has a higher turnover at 19% (two-way) per month than the strategy based on low volatility at only 13%. This is a significant difference and shows that the persistence of beta is less strong than the persistence of volatility, which should have been expected since the beta will change in time not only because of changes in volatility but also because of changes in correlation with the index. Thus, we have included a third strategy whereby the selection is based on a Bayesian estimation of the beta, thus following the procedure proposed by Vasicek [VAS 73], which aims at improving the estimation of beta.

The results of the simulations can be found in Table 11.1. We use US T-bill 3-month rates obtained via FactSet as the proxy for the risk-free rate and no transaction cost or market impact was considered. As we can see, the differences among the strategies are not large, in particular if we take into account the length of the backtest. Nevertheless, we find that when selecting the lowest beta stocks, the strategy delivers a slightly lower beta and alpha than when selecting the lowest volatility stocks. In turn, the volatility is slightly lower when selecting the lowest volatility stocks than when selecting the lowest beta stocks. Not surprisingly, we also find that the results based on a Bayesian estimation of the beta are closer to those based on volatility than those based on the standard beta estimation.

	Low Beta		Low Volatility
	CAPM	Bayesian	
Annualized Excess return over Cash	7.6%	7.9%	8.1%
Volatility	11.4%	11.1%	10.9%
Sharpe Ratio	0.67	0.71	0.74
Annualized alpha	5.7%	6.0%	6.0%
Beta	0.52	0.51	0.55

Table 11.1 Annualized returns, volatility, Sharpe ratio, alpha and beta for monthly rebalanced low-risk strategies based on ranking approaches using beta and volatility estimators. Selected low-risk stocks are equally weighted. World universe. January 1995–August 2013

Selecting low-volatility stocks generates much lower turnover, creates marginally more alpha and results in a beta that is almost as low as when selecting by low beta. For these reasons, we will use volatility instead of beta for the selection of stocks in the remainder of this chapter.

An additional reason for using volatility instead of beta is the non-universality of beta. From a CAPM point of view, the beta should be based on the market portfolio. But for a portfolio manager benchmarked against a segment of the market portfolio, what really matters is the beta measured against the market capitalization-weighted portfolio for the stocks in that market segment. Thus, the relevant measure of beta is not the same for all market participants if we take into account their different objectives.

11.3. Sector-neutral low-risk investing

11.3.1. Motivation

The CAPM assumes that investors are risk-averse and maximize the expected utility of absolute wealth, caring only about the mean and variance of returns. This is a large assumption which does not actually apply to all investors. Professional active portfolio managers are appraised on their performance relative to a benchmark index, typically a market capitalization portfolio of a given segment of the equity market, usually a country or region. Consequently, these professional investors do not care about absolute wealth or risk, but only about the relative performance in excess of the benchmark and the tracking-error risk. They often have targets and constraints on the tracking-error risk they can take.

As argued by Falkenstein [FAL 09], if CAPM was observed, active portfolio managers would then maximize their utility by investing in high-beta stocks instead of low-beta stocks. Under CAPM, given two stocks with the same level of tracking-error risk, one with high beta and one with low beta, the portfolio manager preference would necessarily be for the high-beta stock which, with a beta higher than one, would be expected to outperform the market capitalization index in the medium-to-long term due to its higher exposure to the market risk premium. In turn, the low-beta stock, with beta below one, would be expected to underperform the market capitalization index due to its low market exposure.

The higher demand for high-beta stocks created by these investors should push up the prices of such stocks and make the low-beta stocks that are less in demand cheaper. As shown by Falkenstein [FAL 09], the expected return for each stock is then the same in equilibrium. Even if these investors represent just part of the universal investor population and other investors maximize the expected utility of

absolute wealth, a risk anomaly should still be expected, even if less strong, as shown by Brennan [BRE 93] and Brennan *et al.* [BRE 12].

A related explanation of the low-risk anomaly was proposed by Haugen and Baker [HAU 12]. They focus on the typical compensation structure of professional active portfolio managers and show that the incentive structures resemble a call option. The value of call options increases with volatility and thus, assuming that active portfolio managers seek to maximize the expected value of the call options upon which their compensation is based, they are incentivized to take risks and should prefer to invest in high-risk stocks rather than low-risk stocks. Falkenstein [FAL 09] goes further, arguing that since rewards are typically much larger for top quintile portfolio managers than for second quintile portfolio managers, the incentive to take risk and invest in risky stocks is heightened.

Haugen and Baker [HAU 12] also argue that the investment teams responsible for selecting the stocks for actively-managed funds are usually incentivized to focus on high-risk stocks, mainly due to career pressure. It is those who select stocks with stellar performances that are more likely to be promoted, and stocks with stellar performances can be more likely found in the universe of riskier stocks, even if the average return of the universe of all riskier stocks is shown to be poor. They are also under pressure to focus on stocks which are in the spotlight and receive above median coverage, the “hottest” stocks in the market, which are typically risky stocks. Discussions with lead portfolio managers and clients are much easier when it comes to explaining the decision to invest in a given stock if they are also familiar with that particular stock. Finally, privately-owned asset management firms selling actively-managed funds have an incentive to generate more volatile fund performances, as discussed by Chevalier and Ellison [CHE 97] and Sirri and Tufano [SIR 98]. This is because the funds with the top performance relative to peers, in particular following periods of good market performance, tend to receive the largest inflows. The relationship between fund flows and performance supports the idea that asset management firms should concentrate their efforts on high-beta funds to maximize their profits.

In conclusion, there is strong evidence that the way in which the active management industry operates creates strong demand for riskier stocks. However, none of the authors above explores the practicalities of managing active funds. In particular, they do not take into account that, in most asset management firms managing active funds based on fundamental approaches, the stock selection is typically made by sector specialists who pick the stocks with the highest expected returns from their sector. There are reasons for this. Stocks from any given sector tend to be exposed to a number of common factors and are thus easier to compare. The decision behind stock selection is easier when apples are compared with apples. Analysts can also specialize and focus only on more manageable universes in terms of the number of stocks to cover.

Analysts involved in stock selection at asset management firms are nearly always organized by sector, while analysts involved in stock research in brokerage firms, providing company research to asset management firms, are also almost invariably organized by sector. We asked seven heads of research at large international brokerage firms³ with bases in the United States, Europe and Asia, how many of their clients operate on this basis and the answers suggested that the vast majority do. They also confirmed that the equity analysts at their brokerage firms are indeed also organized by sectors, much in line with their client base. When asked about the most commonly used sector definition used to delineate sector coverage, we were told that even if the 10 sector GICS⁴ definition is not always strictly used, for the most part, some relatively similar definition is employed with occasionally one or another sector broken into some of its constituent industries. Only one brokerage house highlighted that some clients tend to go down to the 24 industry GICS definition when managing portfolios benchmarked against broader indices.

Active portfolio managers tend to invest in a limited number of selected stocks from the investment universe to which they are assigned. Sector active weights in portfolios are often constrained as a crude way of managing tracking-error risk. When asked about how many of their clients tend to keep tight-to-moderate sector constraints, the brokerage firms gave essentially the same answer. When it comes to portfolio construction, quantitative active managers, those who rely on quantitative systematic approaches for stock-picking and which have represented a large portion of the actively managed funds market in the past, seem to invariably use tighter sector constraints than fundamental managers, who follow the process described above. When asked to put a number behind their answer, we were given results with some level of dispersion. In terms of average, the brokerage firms put at about 40% the percentage of fundamental active managers who impose strong-to-moderate

3 The persons contacted kindly provided the information on their behalf and based on their own experience. The views provided are not based on a rigorous statistical analysis. The views expressed do not, by any means, reflect an official view of the firms employing the persons contacted and they were never intended to represent official firm views.

4 The Global Industry Classification Standard (GICS®) is an industry taxonomy developed by MSCI and Standard & Poor's (S&P). The GICS structure consists of 10 sectors, 24 industry groups, 68 industries and 154 subindustries into which S&P has categorized all major public companies. The system is similar to Industry Classification Benchmark (ICB), a classification structure maintained by Dow Jones Indexes and FTSE Group. GICS® is a registered trademark of McGraw-Hill and MSCI Inc. Due to licensing constraints, we have replicated as much as possible the GICS classification prior to August 2006 using the publicly available information on the methodology. We believe that differences between the actual GICS classification and our classification should be minor and have no relevant impact on the results of this chapter.

controls on active sector exposures, while for quantitative active managers this figure rises to about 70%. Moreover, quantitative managers often seem to add constraints on the beta of their portfolios, restricting it to be above one for benchmarked funds and above zero for long-short portfolios.

We believe this evidence is supportive of the results of Baker *et al.* [BAK 14] and Asness *et al.* [ASN 14] and probably explains why their results are not in line with what should have been expected from the reasoning advanced by Samuelson [SAM 98], i.e. stocks are priced more efficiently than sectors or industries. The fact that the stocks are almost invariably picked using sector-based approaches and that a large percentage of portfolio managers apply some level of sector control when building their portfolios is consistent with a stronger risk anomaly in the cross-section of stock returns within each sector rather than in the cross-section of sector returns. The evidence collected from heads of research at brokerage firms points toward a more widespread use of the 10 sector GICS definition than the more granular industry definition used by either Baker *et al.* [BAK 14] or the subindustries definition used by Asness *et al.* [ASN 14]. For this reason, we concentrate our research on sectors rather than industries or subindustries.

11.3.2. Universality of the low-risk anomaly in equity sectors

In this section, we present results from historical simulations designed to compare the return and risk of systematic strategies invested in the lowest volatility stocks of each sector with those from a similar strategy invested in the riskier stocks of the same sector. We run the analysis through a number of developed and emerging markets. We used the following list of indices:

- developed countries: MSCI World Index (MSCI Inc.). From 1995¹;
- U.S.: S&P 500 Index (U.S. stock exchanges). From 1990;
- Europe: Stoxx Europe 600 Index (18 countries of the European region which today are Austria, Belgium, Czech Republic, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, the Netherlands, Norway, Portugal, Spain, Sweden, Switzerland and the United Kingdom). From 1991;
- Japan: Topix 500 Index (Tokyo stock exchange). From 1993;
- Canada: S&P/TSX Composite Index (Toronto stock exchange). From 2004;
- Emerging Markets: MSCI Emerging Markets Index (MSCI Inc.). From 2002¹;
- China: CSI 300 Index (Shanghai and Shenzhen stock exchanges). From 2005;
- Brazil: IBrX Index (São Paulo stock exchange). From 2001;

- Taiwan: TWSE Index (Taiwan stock exchange). From 1993;
- South Korea: Kospi Index (Stock Market Division of South Korea exchange). From 2001.

For each universe of stocks defined by these indices, we used the longest history available. Since not all indices have the same starting dates, the results cover different periods varying from 9 to 24 years. All data were collected using FactSet and the original data providers are indicated adjacent to each index.

In the historical simulations for each index above, we started by estimating the historical volatility of each stock in the index universe at the end of each month from the past 2 years² of total returns in local currencies. The stocks in each sector were then ranked by their historical volatility² into three portfolios with the same number of stocks. Stocks in each of these three portfolios were then equally weighted. We used the 10 sector definition of GICS⁴; in cases where the GICS classification was missing for a given stock, the FactSet industry classification was used instead. A small number of stocks that had neither GICS nor FactSet classification were excluded. Only sectors with at least 15 stocks were considered at each point in time, i.e. a minimum of five stocks in each tercile portfolio was required. Over the period of the simulation, the portfolios were rebalanced once every month at the start of each month to take into account changes in the historical volatility.

In Table 11.2, we show the results from these historical simulations for developed markets and emerging markets, respectively. In these tables we include the beta of the portfolio strategy invested in the lowest volatility stocks of each sector i , $\beta_{Lowest Risk}^i$, and the beta of the portfolio strategy invested in the highest volatility stocks of each sector i , $\beta_{Highest Risk}^i$. These two metrics were calculated from a regression over the entire period of the monthly returns, in excess of cash, of each portfolio strategy against the monthly returns, in excess of cash, of the underlying benchmark index which includes all sectors. The alpha generated from the lowest risk portfolio strategy for a given sector i can be estimated from the same regression:

$$\alpha_{Lowest Risk}^i = \left(R_{Lowest Risk}^i - R_{Cash} \right) - \beta_{Lowest Risk}^i \left(R_{Benchmark Index} - R_{Cash} \right) \quad [11.1]$$

with $R_{Lowest Risk}^i$ the annualized performance of the lowest risk portfolio strategy, $R_{Benchmark Index}$ the annualized performance of the market capitalization-weighted benchmark index and R_{Cash} the annualized return of money market instruments in the currency used. A similar equation can be used to estimate the alpha from the

highest risk portfolio strategies, $\alpha^i_{Highest Risk}$. The alpha in each sector, α^i , shown in these tables, is given by:

$$\alpha^i = \frac{1}{\lambda} \left(\frac{\alpha^i_{Lowest Risk}}{\beta^i_{Lowest Risk}} - \frac{\alpha^i_{Highest Risk}}{\beta^i_{Highest Risk}} \right) \tag{11.2}$$

Here, λ is the constant that is required for the volatility of the returns to be exactly 5% annualized over the entire period of the simulations:

$$r_t^i = \frac{1}{\lambda} \left(\frac{r_{t,Lowest Risk}^i - r_{t,Cash}}{\beta^i_{Lowest Risk}} - \frac{r_{t,Highest Risk}^i - r_{t,Cash}}{\beta^i_{Highest Risk}} \right) \tag{11.3}$$

r_t^i is the time series of monthly returns to a long-short portfolio, long portfolio strategy with the lowest risk stocks and monthly returns $r_{t,Lowest Risk}^i$, with a weight $1/\beta^i_{Lowest Risk}$, and short portfolio strategy with the highest risk stocks and monthly returns $r_{t,Highest Risk}^i$, with weight $1/\beta^i_{Highest Risk}$. The weights are such that the final beta of the long-short portfolio is exactly zero and the strategy has zero exposure to the benchmark index in the period⁵. We call this long-short portfolio strategy low volatility minus high volatility (LVMHV).

The results in Table 11.2 show that the lowest volatility stocks of each sector in developed countries tend to have a beta below one with the exception of those in the information technology sector for which the beta is close to one or even higher, as is the case for the US and Europe. The highest volatility stocks tend to have a beta above one with the exception of those from the defensive sectors, i.e. consumer staples, health care and utilities. In Canada, defensive sectors did not have enough stock representation for the analysis to be carried out. Here, the lowest risk stocks from the material sectors have a beta above one.

5 Asness *et al.* [ASN 14] use a relatively similar approach, which they call betting-against-beta (BAB). The key difference is that these authors apply the beta neutralization and risk adjustment every month using *ex ante* beta and *ex ante* volatility. The returns to this strategy are not exactly beta neutral as discussed by De Carvalho *et al.* [DEC 12] since the *ex post* beta for the lowest risk portfolio strategy tends to be higher than the *ex ante* beta and the *ex post* beta for the highest risk portfolio strategy tends to be lower than the *ex ante* beta. The returns to the BAB strategy are thus positively exposed to the benchmark index and cannot be associated with pure alpha.

a)

	Developed Markets MSCI World Index 1995-2014			U.S. S&P 500 Index 1990-2014			Europe Stoxx Europe 600 Index 1991-2014			Japan Topix 500 Index 1993-2014			Canada S&P/TSX Composite Index 2004-2014		
	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)
Consumer Discretionary	2.6% (2.22)	0.83 (30.7)	1.34 (28.5)	1.1% (1.00)	0.89 (2.47)	1.42 (21.0)	2.0% (1.75)	0.87 (28.1)	1.38 (25.2)	1.4% (1.27)	0.71 (24.7)	1.17 (27.6)	1.3% (0.91)	0.50 (8.35)	1.20 (10.3)
Consumer Staples	4.5% (3.89)	0.47 (13.4)	0.79 (21.1)	1.9% (1.81)	0.49 (12.2)	0.80 (20.2)	1.5% (1.33)	0.50 (13.7)	0.87 (20.4)	3.0% (2.65)	0.40 (12.0)	0.76 (16.8)	-	-	-
Energy	2.0% (1.72)	0.79 (14.0)	1.40 (13.5)	1.9% (1.77)	0.74 (11.8)	1.18 (11.8)	-0.6% (-0.5)	0.86 (17.7)	1.19 (12.1)	-	-	-	3.9% (2.71)	0.83 (12.3)	1.74 (15.7)
Financials	1.8% (1.49)	0.82 (24.3)	1.51 (24.9)	-0.1% (-0.0)	0.88 (19.4)	1.58 (22.7)	2.3% (2.04)	0.80 (28.4)	1.59 (26.5)	-1.0% (-0.8)	0.57 (13.5)	1.54 (12.4)	2.9% (2.03)	0.63 (12.9)	1.05 (14.4)
Health Care	2.0% (1.70)	0.55 (15.3)	1.02 (15.0)	1.7% (1.56)	0.68 (15.7)	0.96 (17.1)	3.0% (2.79)	0.52 (13.6)	0.94 (16.2)	1.5% (1.36)	0.48 (10.0)	0.69 (13.3)	-	-	-
Industrials	3.2% (2.73)	0.75 (26.3)	1.31 (27.9)	2.7% (2.60)	0.86 (24.4)	1.37 (24.7)	2.0% (1.82)	0.78 (25.3)	1.37 (28.1)	0.6% (0.53)	0.64 (23.1)	1.24 (24.6)	4.9% (3.43)	0.76 (11.9)	1.30 (10.5)
Information Technology	1.0% (0.80)	1.05 (26.4)	1.89 (16.9)	0.6% (0.49)	1.24 (28.2)	1.94 (18.1)	-0.3% (-0.2)	1.23 (21.0)	1.76 (17.8)	2.8% (2.54)	0.92 (25.8)	1.37 (14.0)	-	-	-
Materials	1.7% (1.42)	0.91 (23.8)	1.30 (18.0)	0.9% (0.78)	0.89 (21.5)	1.31 (16.3)	0.7% (0.51)	0.88 (24.5)	1.31 (19.0)	0.7% (0.60)	0.85 (22.7)	1.19 (22.0)	-0.9% (-0.5)	1.16 (14.5)	1.68 (9.72)
Telecom Services	2.4% (2.06)	0.73 (19.0)	1.44 (15.8)	-	-	-	2.8% (2.09)	0.73 (11.9)	1.46 (12.1)	-	-	-	-	-	-
Utilities	2.7% (2.34)	0.26 (5.89)	0.87 (14.5)	2.3% (2.22)	0.29 (5.59)	0.72 (8.92)	2.4% (2.19)	0.41 (10.3)	0.86 (16.2)	-	-	-	-	-	-

b)

	Emerging Markets MSCI Emerging Market Index 2002-2014			China CSI 300 Index 2005-2014			Brazil IBRX Index 2001-2014			South Korea Kospi Index 1994-2014			Taiwan TWSE Index 2001-2014		
	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)	α (t-stat)	β Lowest Risk (t-stat)	β Highest Risk (t-stat)
Consumer Discretionary	3.0% (2.01)	0.77 (22.5)	1.27 (23.7)	-0.2% (-0.1)	0.94 (18.4)	1.13 (21.1)	-	-	-	0.9% (0.74)	0.78 (21.5)	0.91 (12.5)	4.6% (3.24)	0.76 (18.6)	1.28 (16.9)
Consumer Staples	3.7% (2.49)	0.59 (17.8)	0.90 (24.8)	3.6% (2.08)	0.74 (11.9)	0.80 (13.3)	-	-	-	0.2% (0.14)	0.68 (15.7)	0.82 (8.70)	5.8% (4.01)	0.69 (12.8)	1.21 (12.6)
Energy	3.2% (2.16)	0.93 (23.8)	1.21 (23.5)	-0.2% (-0.0)	0.95 (21.0)	1.39 (14.5)	-	-	-	-	-	-	-	-	-
Financials	2.9% (1.97)	0.79 (28.3)	1.30 (31.0)	1.2% (0.70)	0.91 (22.2)	1.19 (22.9)	-	-	-	-0.3% (-0.1)	0.79 (15.7)	1.31 (13.2)	-0.6% (-0.4)	0.82 (14.4)	1.58 (11.6)
Health Care	3.1% (1.88)	0.57 (9.32)	0.74 (8.64)	-	-	-	-	-	-	2.4% (2.02)	0.75 (14.5)	0.81 (10.4)	-	-	-
Industrials	1.3% (0.89)	0.77 (26.5)	1.32 (23.3)	-2.3% (-1.3)	0.85 (22.6)	1.09 (24.2)	-	-	-	1.8% (1.50)	0.81 (20.2)	0.96 (12.0)	5.4% (3.72)	0.84 (17.0)	1.23 (16.1)
Information Technology	3.4% (2.30)	0.85 (18.1)	1.10 (15.8)	-	-	-	-	-	-	1.8% (1.51)	0.99 (23.5)	1.08 (11.4)	4.4% (3.02)	0.99 (19.1)	1.37 (16.3)
Materials	2.1% (1.43)	0.87 (30.2)	1.40 (28.8)	-1.4% (-0.7)	1.07 (23.9)	1.30 (23.7)	-0.1% (-0.0)	0.76 (11.4)	1.15 (12.7)	1.2% (0.96)	0.80 (21.4)	0.94 (12.4)	4.8% (3.28)	0.83 (14.5)	1.37 (12.1)
Telecom Services	1.9% (1.31)	0.52 (16.5)	1.01 (16.4)	-	-	-	-	-	-	-	-	-	-	-	-
Utilities	1.6% (1.07)	0.53 (16.1)	1.06 (16.8)	1.1% (0.47)	0.88 (15.4)	1.10 (12.9)	1.6% (0.94)	0.60 (9.94)	0.73 (6.82)	-	-	-	-	-	-

Table 11.2. Alpha from LVMHV for different sectors and countries or regions. The beta of the long portfolio, invested in the lowest volatility stocks, and the short portfolio, with the highest volatility stocks, are also shown. In a) developed countries and in b) emerging countries. T-stat is estimated at 5% significance level. January 1995–December 2014

The alpha from the LVMHV strategy is positive for all sectors in the MSCI World index, the index with the largest number of stocks. In the other universes, with smaller number of stocks, the alpha is positive with a few exceptions such as financials in the US and Japan, energy and information technology in Europe and materials in Canada. All these levels of alpha are for exactly 5% annualized volatility. They are significant more often than not.

The results for the MSCI Emerging Markets index in Table 11.2(b) are comparable to those for the MSCI World index in Table 11.2(a), with the alpha also positive for all sectors in emerging markets. However, we find that the riskier stocks in each sector are more likely to have a beta above one than in developed markets, since now only high-risk consumer staples have a beta below one. And similarly, low-risk stocks from all sectors seem more likely to have a beta below one.

Carrying out the analysis in each individual emerging market country was not as easy as for developed countries because of the smaller number of stocks in each sector, in particular in Brazil, and because of shorter history of returns, in particular in China. The evidence of a low-risk anomaly seems stronger for South Korea and Taiwan than for China or Brazil. In the latter, only two sectors had enough stocks to perform the analysis and only in utilities is there evidence of positive alpha, despite the fact that the beta of high-risk utilities is below one. In China, stronger evidence of a positive alpha is found only in consumer staples, along with some weak evidence in financials and utilities. However, the history of returns is relatively short. In Taiwan, the evidence is stronger and only financials do not have a strong positive alpha. Evidence is less strong for South Korea than for Taiwan, with two sectors in seven not showing significant alpha.

11.3.3. Diversification in sector-neutral low-volatility investing

In Table 11.3, we show the pair-wise correlation of the time series of return for the LVMHV strategies defined in [11.3] for any two pairs of sectors, for the MSCI World Index and the MSCI Emerging Markets Index.

The correlation of LVMHV returns for any two sectors is always positive with the exception of the correlation between the LVMHV returns for the energy sector and the LVMHV returns for the health care sector in emerging markets. Nevertheless, the average correlation of LVMHV returns from sectors in the MSCI World Index is low, at 34%, and from the MSCI Emerging Markets Index is only 20%. These results show a potential diversification gain from investing in low-volatility stocks from different sectors. We will discuss this point later.

a)									
Developed countries									
	Consumer Staples	Energy	Financials	Health Care	Industrials	Information Technology	Materials	Telecom. Services	Utilities
Consumer Discretionary	45%	31%	42%	41%	58%	43%	31%	37%	39%
Consumer Staples		28%	45%	39%	51%	20%	20%	35%	41%
Energy			31%	33%	23%	33%	11%	45%	26%
Financials				20%	38%	22%	15%	31%	47%
Health Care					45%	52%	8%	51%	26%
Industrials						34%	48%	42%	46%
Information Technology							13%	40%	25%
Materials								21%	17%
Telecom. Services									33%

b)									
Emerging countries									
	Consumer Staples	Energy	Financials	Health Care	Industrials	Information Technology	Materials	Telecom. Services	Utilities
Consumer Discretionary	41%	20%	50%	8%	51%	17%	34%	14%	19%
Consumer Staples		9%	38%	16%	17%	-3%	22%	10%	16%
Energy			12%	-10%	23%	1%	18%	7%	10%
Financials				17%	52%	13%	52%	22%	39%
Health Care					4%	4%	12%	4%	21%
Industrials						20%	52%	11%	24%
Information Technology							11%	12%	11%
Materials								15%	21%
Telecom. Services									24%

Table 11.3. Correlation of LVMHV returns for any two sectors from a) the MSCI World Index and b) the MSCI Emerging Markets Index. January 1995–December 2014

In Table 11.4, we show the correlation of the returns to LVMHV strategies applied to the different sectors of the MSCI World Index and MSCI Emerging Markets Index with the returns to equivalent strategies positively exposed to small capitalization, value and momentum. What we call small minus big (SMB) is a long-short strategy that invests in the one-third of stocks in the universe with the smallest capitalization in the MSCI indices and sells short the one-third of stocks with the largest capitalization. The stocks are equally weighted in the long and short legs of the portfolio. The beta is neutralized as before by allocating a weight $1/\beta_{\text{Smallest Market Cap}}^i$ to the long leg and $1/\beta_{\text{Largest Market Cap}}^i$ to the short leg, fully neutralizing the beta, with $\beta_{\text{Smallest Market Cap}}^i$ and $\beta_{\text{Largest Market Cap}}^i$ the *ex post* beta for each leg over the entire period. The final leverage is adjusted so that the *ex post* volatility is exactly 5% over the period. A similar strategy is built, this time ranking stocks every month by price-to-book and investing in the stocks with the lowest price-to-book while selling short the stocks with the largest price-to-book. We call this strategy high minus low (HML) in analogy to the HML strategy as defined by

Fama and French [FAM 92], although we follow a somewhat different approach. Finally, we construct another similar strategy but with stocks now ranked by momentum defined as the past 11-month return of each stock measure done month before portfolio formation. We call this portfolio Mom in analogy to what was defined by Carhart [CAR 97], although again our strategy is not exactly the same.

a)

Developed countries												
		LVMHV										
HML	Mom	Consumer Disc.	Consumer Staples	Energy	Financials	Health Care	Industrials	Information Technology	Materials	Telecom. Services	Utilities	
SMB	50%	-45%	-4%	-13%	-13%	-12%	-7%	-3%	-7%	-2%	-7%	2%
HML		-69%	11%	12%	20%	-7%	43%	21%	28%	1%	21%	19%
Mom			16%	3%	10%	18%	-6%	0%	1%	3%	11%	5%

b)

Emerging countries												
		LVMHV										
HML	Mom	Consumer Disc.	Consumer Staples	Energy	Financials	Health Care	Industrials	Information Technology	Materials	Telecom. Services	Utilities	
SMB	51%	-57%	-26%	-9%	-20%	-6%	-6%	1%	-5%	8%	-1%	11%
HML		-57%	-19%	-18%	-22%	-15%	1%	-18%	3%	-7%	-11%	-2%
Mom			22%	1%	16%	12%	15%	4%	7%	-13%	13%	-5%

Table 11.4. *Correlations between the returns to LVMHV strategies applied to the sectors in (a) MSCI World Index and (b) MSCI Emerging Markets Index with the returns to SMB, HML and Mom returns applied to the same universes, respectively. In (a), the period is from January 1995 to August 2014 and in (b) the period is from January 2002 to August 2014 markets. Monthly USD total returns*

The average correlation of the returns to LVMHV strategies with the returns to small minus big (SMB), high minus low (HML) and Mom is only 5% when formed using stocks from the MSCI World Index and -3% when formed with stocks from the MSCI Emerging Markets Index. This shows clearly that the returns to LVMHV strategies are uncorrelated from the returns of these other strategies such as SMB, HML and Mom.

11.3.4. Tail risk in sector-neutral low-volatility investing

We will now show that low-risk investing has only a small or no exposure to stocks with future poor performances. In a simple exercise, each month we ranked stocks in each sector of the MSCI World Index¹ by historical volatility² and formed decile portfolios in each sector. We then put together the corresponding decile portfolios from each sector to form 10 portfolios, from 1, the lowest volatility in each sector, to 10, the highest volatility in each sector. We then checked the future returns of the stocks in each of these portfolios and asked the question of how many had monthly returns below -50%, or less than -70%, in subsequent months. The results can be found in Table 11.5, where we show the probability that a stock with a

monthly return less than -50% in (a) or less than -70% in (b) was found in a given decile portfolio up to 3 months before that month and up to 3 months after that month.

The period of the analysis is January 1995–March 2014. This corresponds to 231 months with a total of 390,380 monthly stock returns observed, i.e. 1,689 stocks on average per month. Of these, we find 53 monthly returns observed to be less than -70% from 46 unique stocks and 356 observations less than -50% with 275 unique stocks. In Table 11.5, we show the results of our analysis: we found no stock with a monthly return less than -70% in a given month ranking in the lowest volatility decile in the preceding month, or in the preceding 2 or 3 months. These stocks are found with an increasing frequency in most volatility deciles. Only 10% of these observations come from stocks ranking in the half of the universe with the lowest volatility stocks in the preceding month, 16% 2 months before and 17% 3 months before. If we put the threshold at -50% , then there is the largest percentage found in the lowest volatility universe but most stocks with the poorest performances still come from the riskier half of the universe. Only 18, 21 and 23% of these observations were from stocks ranking in the lowest volatility half of the universe 1, 2 and 3 months before the event, respectively.

a)

Probability that a stock with monthly return < -50% is observed in one given decile											
Volatility decile	Lowest volatility									Highest volatility	
	1	2	3	4	5	6	7	8	9		10
Months before	3	2%	2%	6%	5%	8%	6%	6%	17%	17%	31%
	2	2%	1%	6%	5%	7%	4%	8%	15%	16%	36%
	1	1%	2%	4%	5%	6%	5%	6%	16%	19%	36%
Month of observation	1%	1%	5%	4%	5%	5%	7%	16%	17%	39%	
Months after	1	1%	0%	3%	5%	5%	5%	5%	13%	22%	41%
	2	0%	0%	1%	3%	3%	4%	2%	11%	23%	53%
	3	0%	0%	1%	3%	2%	3%	2%	10%	19%	61%

b)

Probability that a stock with monthly return < -70% is observed in one given decile											
Volatility decile	Low volatility									High volatility	
	1	2	3	4	5	6	7	8	9		10
Months before	3	0%	2%	5%	5%	5%	2%	2%	14%	14%	51%
	2	0%	0%	7%	2%	7%	0%	5%	9%	16%	55%
	1	0%	2%	2%	4%	2%	7%	2%	9%	17%	54%
Month of observation	0%	0%	4%	2%	2%	4%	0%	11%	20%	57%	
Months after	1	0%	0%	2%	0%	2%	2%	2%	5%	21%	65%
	2	0%	0%	0%	0%	0%	0%	0%	0%	9%	91%
	3	0%	0%	0%	0%	0%	0%	0%	0%	4%	96%

Table 11.5. Percentage of the stocks with an absolute monthly return less than -50% a) or less than -70% b) found in each decile portfolio before and after that event. USD returns. Stocks from the MSCI World Index universe¹. January 1995–March 2014

11.4. Sector-neutral versus non-sector neutral low-risk investing

In this section, we compare traditional low-risk investing based on investing in the lowest risk stocks and strongly biased toward defensive sectors with sector-neutral low-risk investing. We focus only on the MSCI World index universe for developed countries with the larger number of stocks and sufficiently long history, from January 1995 to May 2013¹.

11.4.1. Performance and sector exposures

In Table 11.6, we compare the alpha of two beta-neutral strategies, both with exactly 5% annualized volatility. The first strategy, which we call sector neutral, is an aggregation of LVMHV sector long-short portfolios, one for each sector, as defined before but using deciles instead of terciles. Each sector of LVMHV is allocated an equal weight over the entire period, and the leverage of aggregation of these 10 sector LVMHVs is such that the ex-post volatility is exactly 5%.

The second strategy, which we call non-sector neutral, is based on an LVHVMV long-short portfolio which does not take into account sectors. The stocks are ranked by historical volatility once a month and the portfolio is rebalanced once at the start of each month. Stocks are equally weighted just as before. However, this portfolio invests in the decile of stocks with the lowest historical volatility and short sells the decile of stocks with the highest historical volatility, irrespective of their sectors. As before, the weight of the long and short legs is equal to the inverse of each observed beta, $1/\beta_{LowestRisk}$ and $1/\beta_{HighestRisk}$, respectively, and the allocation is rescaled by λ as in [11.2] so that the *ex post* volatility is 5%. In Table 11.6, we also consider the same strategies but now implemented with a 6-month lag, i.e. the portfolio is implemented 6 month after formation.

MSCI World Index Jan-1995 - May-2013				
	Sector neutral		Non-sector neutral	
	1 month	6 months	1 month	6 months
Alpha	3.7%	3.8%	3.2%	3.2%
Information ratio	0.73	0.77	0.65	0.64

Table 11.6. Alpha and information ratio for two LVMHV strategies, one based on equally weighting individual LVMHV sector strategies, which we call sector-neutral, and one applying the LVMHV across the entire stock universe ignoring sectors, which we call non-sector neutral. The beta of both strategies is zero and the volatility is 5% by construction. We also consider the same strategies implemented with 6 months lag

The results in Table 11.6 show an improvement of 14% in the information ratio of the sector-neutral strategy when compared to that of the non-sector neutral. It is also interesting to see that the strategies with lower turnover reach the same levels of alpha as those rebalanced more frequently. This seems to indicate that the rotation of stocks in the portfolios is low, something we will investigate in the next section.

In Figure 11.1, we show the average net sector weights in each of these two strategies. The net sector weight is the sum of the weights allocated to each stock in a given sector. The sector-neutral strategy has a positive weight in all sectors. This is because, in order to neutralize the market exposure and reach a beta equal to zero, the strategy allocates a larger weight to the low-volatility stocks it buys than to the high-volatility stocks it sells short. The size of the net sector weights is a function of the dispersion of beta, i.e. the larger the difference between the beta of low-volatility stocks and high-volatility stocks, the larger the net sector weight. The main difference between the sector-neutral strategy and non-sector neutral strategy is the much larger weight allocated to consumer staples, financials and utilities and the much smaller weight allocated to information technology, consumer discretionary and energy found in the non-sector neutral strategy. The non-sector neutral strategy has always been strongly biased toward financials except for a few months in the aftermath of the 2008 crisis.

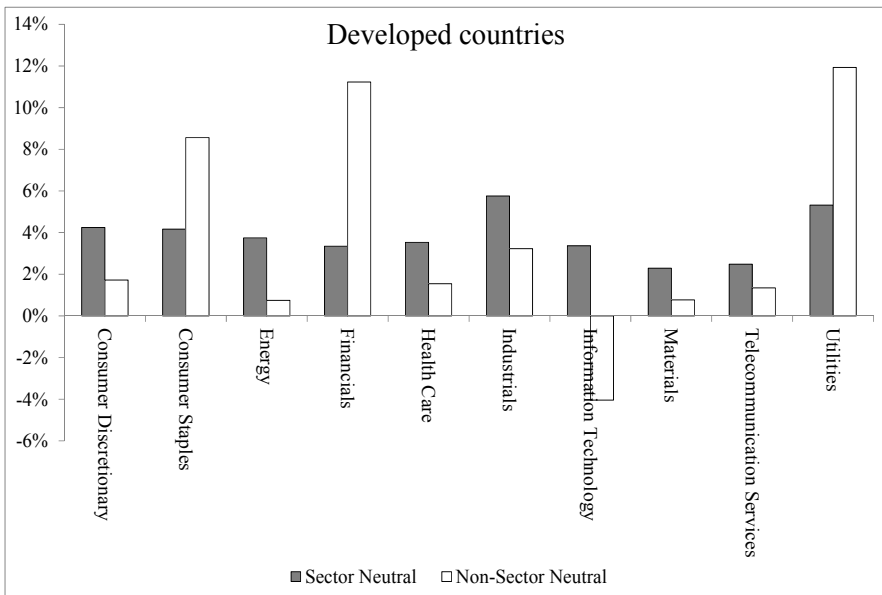


Figure 11.1. Average net sector weights for sector-neutral and non-sector neutral LVMHV strategies for stocks from the MSCI World index¹. January 1995–May 2014

11.4.2. Persistence of volatility

We now focus on the reasons why, in Table 11.6, the turnover could be reduced significantly without a reduction in the alpha of the LVHVMV strategies. This is in fact due to the persistence of volatility. Perchet *et al.* [PER 14] have recently investigated the persistence of volatility at aggregate market level, for different asset classes, and Perchet *et al.* [PER 14] have done the same for value and momentum factor premium. They found that the persistency of volatility is strong at the aggregate level, which explains why volatility can to some extent be predicted at an aggregate level.

a)

MSCI World Index Jan-1995 - May-2014									
Sector Neutral					Non-Sector neutral				
1-month transition probability matrix									
Volatility tercile next month					Volatility tercile next month				
		Low	Mid	High	Out	Low	Mid	High	Out
Volatility tercile today	Low	95%	5%	0%	0%	96%	4%	0%	0%
	Mid	5%	90%	4%	1%	4%	92%	4%	1%
	High	0%	4%	94%	1%	0%	4%	95%	1%
6-month transition probability matrix									
Volatility tercile next month					Volatility tercile next month				
		Low	Mid	High	Out	Low	Mid	High	Out
Volatility tercile today	Low	83%	14%	1%	2%	85%	12%	1%	2%
	Mid	14%	69%	14%	3%	12%	72%	13%	3%
	High	0%	13%	80%	6%	0%	12%	82%	6%

b)

MSCI Emerging Markets Index Jan-2002 - May-2014									
Sector Neutral					Non-Sector neutral				
1-month transition probability matrix									
Volatility tercile next month					Volatility tercile next month				
		Low	Mid	High	Out	Low	Mid	High	Out
Volatility tercile today	Low	94%	5%	0%	2%	95%	4%	0%	2%
	Mid	5%	89%	5%	1%	4%	91%	4%	1%
	High	0%	5%	93%	2%	0%	4%	94%	2%
6-month transition probability matrix									
Volatility tercile next month					Volatility tercile next month				
		Low	Mid	High	Out	Low	Mid	High	Out
Volatility tercile today	Low	80%	13%	1%	6%	81%	12%	0%	6%
	Mid	14%	67%	13%	6%	13%	69%	13%	6%
	High	1%	15%	77%	8%	0%	15%	78%	8%

Table 11.7. Probabilities that a stock ranking in a given tercile of volatility in a given month is still found in the same tercile of volatility in the following month. The table also shows the probability of that stock being found in the same tercile of volatility in the following 6 months. Results for sector-neutral and non-sector neutral portfolios are included. In a), the universe is defined from the MSCI World index¹ and the period is from January 1995 to May 2014. In b), the universe is defined from the MSCI Emerging Markets index¹ and the period is from January 2002 to May 2014

In Table 11.7, we show transition probability matrices for stocks in the MSCI World Index and MSCI Emerging Markets Index, respectively. With the sector-neutral portfolio, stocks in each sector are ranked by historical volatility² every month and then, in each sector, the universe of stocks is ranked by historical volatility and divided into terciles. For the non-sector neutral portfolio, this is done across the universe defined by the index rather than on a sector-by-sector basis.

This exercise was repeated each month. We then calculated the average number of stocks which ranked as being of low volatility in a given month and also in the immediately following month. We repeated the exercise for stocks that ranked as mid-volatility and high-volatility. The probabilities are the average percentage of stocks staying in the same tercile of volatility from one month to another. Similarly, we also estimated the probability that a stock remains in the same tercile of volatility 6 months after being ranked. We also included the probability that a stock leaves the index in the following month or within the following 6 months; these are indicated as *out*.

The results in Table 11.7 show a strong persistency in the volatility of stocks. For the sector-neutral strategy, 95% of the stocks ranked the lowest volatility in the MSCI World Index in a given month then remained ranked the lowest volatility in the following month and the other 5% ranked mid-volatility. The results are comparable for the non-sector neutral strategy, with 96 and 4%, respectively. The results are also comparable for the stocks in the MSCI Emerging Markets Index, with 94% of stocks ranked the lowest volatility still remaining the lowest volatility 1 month later for the sector-neutral approach and 95% for the non-sector neutral approach. Six months after being ranked, 83% of the lowest volatility stocks in the MSCI World Index still rank the lowest volatility for the sector-neutral approach and 85% for the non-sector neutral approach. For the MSCI Emerging Markets Index, this is just slightly lower at 80 and 81%, respectively. It is also interesting to note that the probability of stocks leaving the index is higher for the highest volatility stocks than for the lowest volatility stocks.

11.4.3. Liquidity of low-volatility strategies

Low-volatility investing is an active strategy that invests away from the market capitalization portfolio and requires rebalancing. Thus, liquidity is an important issue. Here, we give some crude idea of the liquidity of simple low-volatility strategies and compare this liquidity with other simple style strategies for small capitalization, value and momentum. We consider both sector-neutral and non-sector neutral low-volatility strategies.

In Table 11.8, we show the average number of days needed to liquidate a 100 million USD portfolio invested in the decile of stocks with the lowest volatility,

sector-neutral and non-sector neutral, and compare this with the average number of days to liquidate a portfolio of a similar size invested in the decile of stocks with the smallest market capitalization, the stocks with the lowest price-to-book ratio and momentum stocks with the highest 11-month return measured 1 month before portfolio construction. The averages are based on the allocation at the start of each month. The number of days required to liquidate the portfolio assumes that a maximum of 30% of the monthly volume of each stock can be traded every day. We ran the analysis between 2008 and 2013. Stock volume data are provided from MSCI. The results are for stocks in the MSCI World Index.

Not surprisingly, the market capitalization index has the greatest liquidity and can be liquidated with the least difficulty. There is not a large difference between the sector-neutral and the non-sector neutral low-volatility portfolios, with perhaps no significant advantage for the non-sector neutral portfolio seen at the level of full liquidation only. Liquidity of the low-volatility portfolios is large at this level and comparable to that of momentum portfolios. Not surprisingly, the small capitalization portfolio has the poorest liquidity. Value also shows poor liquidity because we did not remove the small capitalization bias that a selection based on the lowest price-to-book ratios tends to create.

At 50 and 70% levels of liquidation of the portfolio, low volatility fares better than the other strategies and even in 2008 the portfolio could still be liquidated with less difficulty than both the sector-neutral and the non-sector neutral low-volatility portfolios.

	50%						70%						100%					
	2008	2009	2010	2011	2012	2013	2008	2009	2010	2011	2012	2013	2008	2009	2010	2011	2012	2013
Low volatility non-sector neutral	3	4	6	4	4	4	6	14	14	10	10	9	52	94	70	49	74	92
Low volatility sector neutral	3	4	5	8	10	4	7	12	14	63	109	8	165	139	97	9	10	67
Small capitalization	32	44	37	33	37	40	49	71	59	52	60	64	4890	779	1070	791	280	239
Value	10	13	15	13	12	14	21	26	30	26	25	31	1222	451	118	645	280	99
Momentum	5	9	7	6	7	7	12	16	15	15	13	15	55	104	61	65	67	134
Market Capitalization	0	0	0	0	0	0	0	1	1	1	1	1	41	111	15	15	11	4

Table 11.8. Average number of days in each year needed to liquidate 50, 70 and 100% of a USD 100 million sector-neutral and non-sector neutral low-volatility portfolios compared to equally weighted, non-sector neutral portfolios invested in the 10% top ranked stocks by the lowest market capitalization, value as measured by the lowest price-to-earnings ratio and momentum as measured by the highest 11-month returns measured 1 month before portfolio formation. Stocks are from the MSCI World index

11.5. Conclusions

In this chapter, we give empirical evidence of risk anomalies in the sector of activity at a global level, in developed and emerging markets. Positive returns to

beta sector-neutral long-short portfolios invested in the lowest-volatility stocks of a given sector and short the highest volatility stocks of the same sector cannot be explained by market exposure. Portfolios invested in the lowest volatility stocks of a given sector have been returning more than expected from their level of risk, whereas portfolios invested in the highest volatility stocks of the same sector have been returning less than expected from their level of risk. This risk anomaly had been reported in the cross-section of stock returns of almost all countries and regions in the world by Haugen and Baker [HAU 12] and also in the cross-section of country returns by Baker *et al.* [BAK 14]. However, evidence of such a risk anomaly is much weaker in the cross-section of industry returns as shown by Baker *et al.* [BAK 14] and Asness *et al.* [ASN 14], with the latter suggesting that it is in fact more efficient to capture low-risk alpha using industry-neutral approaches. Indeed, they give evidence of the risk anomaly in the cross-section of stock returns in industries without advancing any explanation or analyzing each industry in detail.

To our knowledge, this chapter is the first to provide evidence of the risk anomaly in the cross-section of stock returns in each sector of activity using the 10 sector GICS definitions and to put forward an explanation for why there are good reasons to expect the anomaly to be stronger in the cross-section of stock returns in sectors than in the cross-section of sector returns. Indeed, we believe that active managers benchmarked against market capitalization indices are most likely behind the anomaly in the cross-section of stock returns in sectors. Evidence that active managers have a preference for risky stocks was given by Falkenstein [FAL 09], Brennan [BRE 93], Brennan *et al.* [BRE 12], Haugen and Baker [HAU 12], Chevalier and Ellison [CHE 97] and Sirri and Tufano [SIR 98]. However, we argue that because equity analysts and fund managers select stocks almost invariably from within sectors and because a number of these fund managers, in particular quantitative active managers, tend to impose constraints on the level of sector deviation of their portfolios against the market capitalization index, it is then reasonable to expect the risk anomaly to be stronger in sectors and to show in all sectors. Our empirical results for stocks of developed countries at global level do suggest that the risk anomaly is stronger when some level of sector neutrality is imposed, thus corroborating the results from Asness *et al.* [ASN 14], who reached a similar conclusion when imposing industry neutrality. For the period considered, we found 14% more risk-adjusted alpha in the sector-neutral strategy than in the non-sector neutral strategy. Imposing sector neutrality in the portfolios tilted in favor of low-volatility stocks leads to much smaller exposures in particular to the financials, utilities and consumer staples sectors and to a much larger exposure to the information technology, consumer discretionary and energy sectors than when sector neutrality is not imposed.

The higher risk-adjusted alpha found in the sector-neutral strategy is explained by the diversification gain arising from the low correlation of the returns generated from beta-neutral long-short portfolios invested in the lowest volatility stocks in a given sector and selling short the highest volatility stocks from the same sector. We also found a low correlation of the returns to these portfolios with the returns to beta-neutral long-short portfolios invested in value stocks and selling short expensive stocks, to beta-neutral long-short portfolios invested in the smaller capitalization stocks and selling short the largest capitalization stocks and to beta-neutral long-short portfolios invested in the stocks with the strongest momentum and selling short the stocks with the poorest momentum.

Finally, we have shown that low-volatility investing offers a level of liquidity higher than that found in other styles such as momentum, value and in particular small capitalization. We have also shown that the level of turnover required for low-volatility investing can be reduced without a significant impact on the risk-adjusted alpha due to the persistency of the volatility of individual stocks. As demonstrated, stocks which ranked among the lowest volatility show a very low probability of becoming higher volatility in the near future. A consequence of this low probability is the fact that in the history used in our analysis, we also find that low-risk investing naturally filters out the stocks more likely to deliver extremely poor performances in the near future.

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The Low Volatility Anomaly and the Preference for Gambling

Previous literature has established that low volatility stocks outperform high volatility stocks on a risk-adjusted basis. In this Chapter, we show that this low volatility anomaly is linked to the preference for lottery stocks, which have experienced a high daily return in the previous month. Not all high volatility stocks underperform. The underperformance of high volatility stocks arises largely from the underperformance of high volatility lottery stocks. High volatility lottery stocks underperform high volatility stocks in 31 out of 32 countries.

12.1. Introduction

Low volatility strategies have been one of the most successful and popular smart beta and quant active strategies in the recent years. The risk reduction for low volatility strategies is easily understood and has been highly desirable post the global financial crisis where risk awareness has become high. Low volatility equity strategies generally run a beta of 0.7 with respect to a cap-weighted market factor. Unsurprisingly, these strategies usually have 30% less volatility than a traditional cap-weighted market index. However, the driver of excess return over the market benchmark is a not well understood, given the traditional CAPM-type asset pricing intuition. To understand the excess return more clearly, we perform a simple factor attribution exercise (see Table 12.1). We show that representative variants of the low volatility strategy have earned their historical excess return primarily from exposure to low price stocks (positive loading on the value high-minus-low (HML) factor)

and low beta stocks (positive loading on the low beta betting against beta (BAB) factor).

	alpha	alpha (T-stat)	Mkt-RF	SMB	HML	WML	BAB	DUR	R ²
U.S. (1967-2012)									
Minimum Variance (PCA)	0.42%	0.50	0.61*	0.22*	0.12*	0.01%	0.27*	0.08*	78%
Low Volatility (1/Vol)	0.06%	0.09	0.69*	0.13*	0.16*	-0.04*	0.35*	0.15*	88%
Low Beta (1/β)	-0.22%	-0.29	0.66*	0.31*	0.14*	-0.03*	0.39*	0.12*	85%
Factor Sharpe Ratio			0.35	0.26	0.44	0.54	0.41	0.30	

*Significant at the 95% confidence level

Table 12.1. Return attribution of low volatility strategies

The value exposure is a well-known source for excess return. We simply note that low volatility (beta) stocks have historically been low price as well—exploring the plausible reasons is outside the scope of this chapter. Moreover, this chapter does not explore the robustness and persistence of the value premium as a source for low volatility strategy outperformance. Instead, we focus on the excess return derived from overweighting low volatility (beta) stocks and underweighting high volatility (beta) stocks, or what is now more commonly referred to as exposure to BAB¹. The outperformance of low volatility stocks versus high volatility stocks is one of the most persistent and perhaps most counterintuitive issues in finance. While a number of rational and quasi-rational hypotheses have been put forth as potential explanations for the anomaly, we explore the behavioral channel in this chapter.

Using global equity data, we find strong evidence that the outperformance of low volatility stocks over high volatility stocks is driven meaningfully by investors' willingness to pay high prices for "lottery stocks" or stocks with high positive skew. In essence, high volatility serves as a proxy for high positive skew². Empirically, investors' willingness to overpay is so high that expected returns for these lottery stocks can be extremely low relative to other stocks. It thus seems entirely appropriate to refer to these stocks with high skew and high prices as "lotteries" as their expected premiums are generally negative but can at times produce large upside returns. It also seems reasonable to label the investors' demand for these lottery stocks with negative premiums as a preference for gambling instead of rational investment behavior. In section 12.6, we illustrate that the preference for positive skew appears to be an irrational demand rather than rational risk sharing.

Note that in the above "mechanism", the variable of primary importance is positive skew rather than beta or volatility. What this means is that sorting stocks by volatility or beta would not fully capture the anomalous return associated with this

1 See [FRA 14].

2 See [BAL 14].

preference for gambling. This has significant implications for the construction of low volatility portfolios, which have traditionally focused only on the second moments but not on the higher moments.

Empirically, on average, high volatility stocks (highest quintile) underperform the market by roughly 5.2% per annum using historical U.S. data. However, when we examine the quintile of stocks with the highest past positive skew in the cross-section, we find that the high skew + high volatility stocks underperform the market by 15.3%. When we examine the international data, we find that the high skew + high volatility stocks underperform the high volatility stocks substantially in 31 out of 32 countries. Interestingly, within the cohort of stocks with low skew (roughly 40% of all stocks), on average, the low volatility stocks underperform the high volatility stocks. That is the low volatility anomaly appears to live only within high positive-skewed stocks. The above empirical observations suggest that skewness has an almost dominant impact on the performance of low volatility strategies and should be incorporated into any low volatility portfolio construction.

12.2. A brief review of the literature

Contrary to modern portfolio theory, low beta stocks tend to outperform high beta stocks on a risk-adjusted basis and, often, on an absolute basis. Merton's consumption CAPM [MER 73] argues that expected excess return ought to scale linearly with the beta exposure to the equity market³. The fact that the opposite holds at all is an enigma. A long and deep literature has emerged on the anomaly starting with Bob Haugen and his co-authors in the early 1970s.

First, as it turns out, this paradoxical risk-return relationship appears to exist in many other asset classes in addition to equities [FRA 14]. Chow *et al.* [CHO 14] showed that this effect is remarkably robust to definitions of risk and portfolio construction methodologies. These two papers suggest that lowly correlated low volatility portfolios all appear to benefit from the same anomaly. This suggests that covariance with some hidden macrorisk factor(s) cannot explain the puzzle. This then argues in favor of an interpretation that is more behavioral in nature as opposed to the one that originates from a more classical risk-based framework.

Several competing theories have arisen to explain this unexpected phenomenon. Black [BLA 72] and Asness *et al.* [ASN 12] posited that the outperformance of low risk securities is driven by leverage aversion or constraints. They argue that, in practice, many institutions are barred from taking leverage. Additionally, many investors, especially on the retail side, have a very high cost of borrowing. Still

³ We contrast this to Sharpe's CAPM where the beta is measured relative to the unobservable global portfolio of all risk assets.

others may simply have psychological aversion to leverage. Under this theory, investors who desire higher expected returns do not borrow to lever up their risky bets; instead, they buy higher beta securities, bidding them up and ultimately pushing down their expected return.

Without leverage, the only avenue for earning greater returns is to load up on higher expected return securities. According to Merton [MER 73], these securities should be higher market beta securities. This causes a higher demand for higher beta securities, pushing their prices up and their expected returns down. For example, if the market has an expected excess return over the risk-free rate of 5%, then a security with a beta of 2 might have an expected excess return of 8% instead of the 10% as predicted by Merton's consumption CAPM.

However, leverage aversion theory cannot explain the outperformance of low beta stocks in absolute terms, which is what is documented in the data. It can only partially account for the higher than "rational" price paid for high beta stocks. If investors buy high risk stocks to gain a higher expected return, they cannot bid up the stocks to such an extent that the high beta stocks have a lower return than low beta stocks. Thus, to fully explain the anomaly, a theory must ascribe to high beta stocks some other characteristics for which the investors overpay.

One interesting explanation is that sell-side analysts tend to hype high beta stocks more aggressively. Using I/B/E/S data, Hsu *et al.* [HSU 13] found that analysts' bias earnings estimates and price targets upward more aggressively for high volatility stocks. Potentially, unsuspecting investors do not realize this analyst behavior, are fooled and consequently overpay for volatile stocks.

We will argue in this chapter that high skewness in payoff is valued by many investors, who are not the standard power utility optimizers but who have a preference for lottery-like payoffs – a small probability of a very high return⁴. We argue our theory by demonstrating that the low beta stocks' return advantage over high beta stocks is present mostly among lottery stocks only; we illustrate that this is robust across a large number of developed and emerging markets. Contrary to the standard low volatility literature, we find that low volatility stocks with high positive skew can have lower returns relative to high volatility stocks.

Investors' preference for lottery is not a new or previously undocumented phenomenon, though much of the literature has emerged more recently. Kraus and Litzenberger [KRA 76] first demonstrated this preference for skewness empirically. They showed that incorporating this preference into the capital asset pricing model creates a security market line that is more in line with data. Selling out-of-the-money

⁴ This could alternatively be interpreted as someone who has an unreasonably large subjective probability attached to positive outlier events.

options is well known for producing high-risk-adjusted excess returns; this is due in part to their high positive skewness (see [HE 15] for a summary); Boyer and Vorkink [BOY 14] also find that options with high expected skew experience lower expected returns. Other papers looked directly at stocks with high expected skewness – that is they have a chance of a large payoff⁵. Bali *et al.* [BAL 11] found, using U.S. equities data, a high maximum daily return in the previous month – a proxy for high positive skewness or lottery-like payoff – predicts lower return in the subsequent month, presumably as investors overpay for the positive skew. Blau *et al.* [BLA 15] extended these results internationally and found that in countries that are more accepting of gambling, the phenomenon is stronger. In line with this theory, Eraker and Ready [ERA 15] documented a microcap anomaly, where very risky over-the-counter stocks, which they argue are more similar to lotteries, experience low mean returns (relative to liquid large cap stocks). Bali *et al.* [BAL 14] are the first to argue that in the U.S. equity market, lottery stocks drive much of the measured low volatility effect. In this chapter, we extend Bali, Brown, Murray and Tang’s work and demonstrate that this phenomenon is robust in most developed and emerging markets. The implication is that the low volatility effect is perhaps primarily a lottery gambling phenomenon manifested in the equity market. However, we also find that the past skew information does not fully subsume the volatility information, suggesting that both variables should be used in portfolio construction.

There are several justifications for using a high previous month’s maximum daily return as a proxy for high expected skewness or lottery-like behavior. The first involves the inability to predict future skewness with historical skewness. Expected skewness is difficult to measure directly since skewness is a measure of rare jumps in price in a particular direction. It is this rareness that makes using historical skewness a weak measure of future skewness. More importantly, it may be the case that a high maximum daily return creates the perception of high skewness, whether or not such a perception is justified. A large jump in prices may create the belief that such a jump can happen again. This latter point cannot be directly tested. Instead, the growing literature on the lottery effect suggests that having a high previous month’s maximum daily return suggests that the firm will be overpriced. The conjecture that this anomaly arises from a desire for lottery-like payoffs is ultimately a behavioral hypothesis.

To argue for persistency of the low volatility/lottery stock anomaly, we invoke Brennan, Cheng, and Li [BRE 12] and Baker *et al.* [BAK 11], who argued that the low volatility anomaly persists due to a limit to arbitrage. Namely institutional investors are incentivized to maximize information ratio – and not Sharpe ratio. Overweighting low beta and underweighting high beta stocks creates very large tracking errors thus making low volatility strategies inherently low information ratio

5 We will often use the terms “high skewness” and “lottery” interchangeably.

strategies (despite their high Sharpe ratios); as a result, most institutional investors are unwilling to invest in low beta-based strategies.

12.3. Lottery and volatility double sort

First, we motivate our definition of lottery stocks. Lottery stocks are stocks with a high expected skewness. As with Bali *et al.* [BAL 11] and Bali *et al.* [BAL 14], we sort stocks based on the previous month's maximum daily return, which we will simply call MaxRet, and define lottery stocks as those with a high maximum return in the previous month. The rationale for this definition is that investors may perceive stocks that have experienced a high daily return in the previous month as a stock that could again experience such high returns. It is worth noting that whether MaxRet perfectly captures the true skewness of the distribution is less important than whether it captures the investor's perception of "lottery-like" payoff. Indeed, these high MaxRet stocks do tend to experience higher skewness and they also appear to have lower expected returns.

We are not just concerned with the difference in returns between low volatility and high volatility stocks, which we will call the low volatility premium. This has been more than well studied. Nor are we just concerned with the difference in returns between lottery stocks and non-lottery stocks. Instead, we are interested in stocks that are simultaneously high volatility and lottery stocks. We claim that not all high volatility stocks are overvalued. Only high volatility lottery stocks are overvalued. This observation adds to the low volatility literature.

We capture this effect by sorting first on 5-year trailing volatility. We break these stocks into quintiles ensuring that stocks within each quintile roughly share the same volatility. Within each volatility quintile, we sort stocks on *ex ante* skewness – as measured by the maximum daily return in the previous month – and break those stocks into quintiles based on skewness. This is the oft-used double sort to determine interactions between factors. While we do not exhaustively reject all competing hypotheses to argue explicitly that the irrational pursuit of lottery-like returns causes the documented low volatility effect, our empirical result does provide support for this hypothesis.

To reiterate, we first sort stocks on volatility and break those stocks into quintiles. Within the volatility quintiles, we sort stocks on previous month's maximum return and break these stocks into subquintiles. We use Center for Research in Security Prices (CRSP) data for U.S. stocks and Datastream data for non-U.S. data. We first restrict stocks to those that are higher than median market capitalization to ensure liquidity and investability. We calculate volatility using daily data. We weigh constituents within each portfolio by market capitalization and reconstitute portfolios every month; we note that equally weighing the constituents results in qualitatively similar outcome.

We only use countries for which we can compute at least a decade of returns ending December 2014. We report our results in Table 12.2.

Volatile Quintiles	Lottery Quintiles				
	Non-Lottery	2	3	4	Lottery
Stable	12.4%	10.7%	11.1%	10.5%	11.4%
2	12.8%	11.9%	11.9%	10.4%	9.7%
3	14.2%	12.2%	10.3%	11.0%	6.8%
4	12.3%	10.2%	11.9%	8.7%	7.7%
Volatile	9.6%	12.5%	5.9%	3.4%	-4.7%

Table 12.2. Annualized return of monthly rebalanced United States volatility-lottery double-sorted quintiles (January 1967–December 2014)

In Table 12.2, the columns show the lottery quintiles from the lowest maximum return on the left to highest maximum return on the right. The rows show the volatility quintiles from the lowest volatility on the top to highest volatility on the bottom. We will refer to the highest maximum return quintile as the lottery quintile and the lowest maximum return quintile as the non-lottery quintile. Note this is a conditional sort where we sort on volatility first; we cannot compare across the rows in Table 12.2 meaningfully. What we can conclude is that for stocks with comparable volatility (in the same volatility quintile) the ones with lottery characteristics have meaningfully poorer returns. In fact, we see that many high volatility stocks with low lottery characteristics have high returns; screening out these high volatility stocks would not improve portfolio returns but could meaningfully reduce portfolio diversity. The return patterns observed in Table 12.2 suggest that there is expected return information that is captured by the lottery characteristic which is not captured by the traditional volatility measure.

In Table 12.3, we perform the opposite sort. First, we sort by the previous month's maximum return (lottery characteristic). Then, we sort by volatility. Returns increase with volatility for the low lottery characteristic quintiles (1 and 2), corresponding to stocks with low positive skewness. This means that for stocks with comparable skew or lottery characteristics, the intuitive relationship between volatility (and thus beta) and return is generally observed. There is no low volatility anomaly for non-lottery stocks. However, for the high lottery quintiles, especially quintile 5, high volatility stocks significantly underperform low volatility stocks. This suggests that there is also information contained in volatility that is not captured by the MaxRet variable, which proxies lottery characteristic. This contrasts against Bali *et al.* [BAL 14] where they argue that volatility is subsumed by MaxRet for the purpose of capturing the traditional low volatility anomaly. Our evidence

suggests that the preference for lottery is a meaningful part of the explanation but not the complete explanation. This should not be surprising as low volatility (beta) does also capture leverage aversion and analyst bias as reviewed before.

Lottery Quintiles	Volatility Quintile				
	Stable	2	3	4	Volatile
Non-Lottery	11.4%	9.1%	11.4%	13.3%	13.8%
2	11.5%	10.9%	11.9%	11.4%	12.2%
3	11.1%	10.8%	9.4%	13.2%	7.8%
4	9.3%	9.7%	10.6%	9.6%	7.2%
Lottery	8.0%	6.6%	8.3%	1.9%	-5.4%

Table 12.3. Annualized return of United States lottery-volatility double-sorted quintiles (January 1967–Dec 2014)

Our primary findings are twofold: (1) using MaxRet to capture lottery characteristic, we find support that high lottery characteristic stocks tend to have poor returns adjusted for volatility – this suggests that the traditional low volatility strategy does not fully exploit the preference for lottery behavioral bias; (2) there is a meaningful intersection between MaxRet and volatility as portfolio sort signals, however they do not subsume each other. There is an interaction that makes the intersection of high volatility and high MaxRet extremely undesirable. Our results are useful for portfolio construction where investors can eliminate high volatility + high MaxRet stocks from their benchmarks to meaningfully improve returns. This approach is likely to create a more efficient portfolio which captures the various behavioral biases targeted by the traditional low volatility strategies.

12.4. International evidence

In this section, we use international data to verify our claims in the previous section. However, showing two-by-two double sorts for every country is both distracting and intractable from a space management perspective. Instead, we compare the annualized return for the high volatility + lottery intersection quintile with the annualized return of the high volatility quintile⁶. Our goal is to demonstrate that high volatility lottery stocks perform substantially worse than plain vanilla high volatility stocks. The high volatility + lottery intersection is constructed by first sorting on volatility and then on lottery. The reverse sort gives similar results.

⁶ In this case, the return of the high volatility quintile is measured as the average of all five lottery subquintiles within the volatility quintile. This method ensures that all lottery quintiles are given equal weight.

	High volatility lottery	High volatility	t-stat of difference	Start date
Australia	-13.3%	-2.1%	-2.50	01-1987
Belgium	-23.9%	-4.8%	-2.38	07-1999
Brazil	-6.3%	14.7%	-2.16	01-1999
Canada	-14.7%	-0.9%	-3.49	01-1987
Chile	5.6%	15.7%	-1.23	01-1999
Denmark	-10.7%	-1.4%	-1.24	08-1990
Finland	-16.2%	-0.4%	-2.02	06-1998
France	0.7%	1.6%	-0.25	01-1987
Germany	-12.3%	0.3%	-2.80	01-1987
Greece	-32.7%	-11.4%	-2.95	07-1993
Hong Kong	-16.8%	-4.7%	-2.48	04-1989
India	0.0%	13.5%	-2.01	01-1999
Indonesia	-22.9%	6.1%	-3.30	01-1999
Italy	-10.3%	-1.1%	-2.08	01-1987
Japan	-9.1%	-3.4%	-2.10	01-1987
Malaysia	-6.7%	2.3%	-1.97	01-1999
Netherlands	-4.2%	6.8%	-2.38	04-1987
New Zealand	8.9%	15.0%	-0.75	01-2002
Norway	-9.1%	2.9%	-1.37	07-1995
Philippines	-10.0%	7.0%	-1.43	01-1999
Poland	12.3%	5.7%	0.53	09-2001
Singapore	0.5%	6.3%	-0.92	03-1989
South Africa	-2.9%	11.3%	-1.99	01-1999
South Korea	-26.0%	-9.0%	-2.75	01-1999
Spain	-19.2%	-4.1%	-2.56	10-1999
Sweden	-1.9%	4.4%	-0.85	04-1992
Switzerland	1.5%	7.2%	-1.08	01-1987
Taiwan	-1.7%	-1.0%	-0.16	01-1999
Thailand	-1.2%	9.7%	-1.43	01-1999
Turkey	-2.1%	24.6%	-2.75	01-1999
UK	-4.7%	3.1%	-2.24	01-1987
US	-4.7%	5.7%	-5.97	01-1967

Table 12.4. Monthly rebalanced high volatility lottery stock returns versus monthly rebalanced high volatility stocks returns

Table 12.4 shows that in 31 out of 32 countries, high volatility lottery stocks underperformed the high volatility stocks in that country. The differences are significant at a 0.01 level in seven countries and a 0.05 level in an additional 13 countries. This provides support to our assertion that the high volatility lottery stocks are meaningfully more overpriced relative to other high volatility stocks. Indeed, the puzzling historical observation that high volatility (high beta) stocks have low returns is not observed when controlling for positive skewness.

12.5. Conclusion

In this chapter, we argue in favor of the preference for lottery as the significant explanation for the low volatility anomaly. While the low volatility (beta) effect may also be driven by investor aversion to leverage and higher analyst bias for high volatility stocks, the preference for positive skew seems to dominate the low volatility effect empirically. Using global country data, we find that high volatility (beta) stocks only underperform substantially if they are also lottery stocks – that is they appear to have high past positive skew. Since large recent positive jumps in returns can also meaningfully increase the volatility measurement, volatility can often be a noisy proxy for positive skewness. Thus, the low volatility effect and the preference for lottery effect are intimately related. Disjoining the effect, we find that observed return advantage for low volatility stocks over high volatility stocks is substantially, though not entirely, driven by the overpricing of high positive skew stocks – or lottery stocks. Indeed, when controlling for skewness, we find many high volatility stocks to offer high returns, which confirm the traditional CAPM intuition. All of these have meaningful portfolio construction implications. The results suggest that traditional minimum variance and low volatility portfolio constructions may eliminate too many of the reasonably priced high volatility stocks, which hurts portfolio returns and portfolio diversity.

12.6. Appendix

Differences in skewness are hard to interpret. Comparatively, mean and volatility are ubiquitous and intuitive. The skewness of a strategy is rarely mentioned and describes the asymmetry of a distribution, a rather abstract concept.

We attempt to reframe the notion of skewness in a way that is more concrete. We want to answer the following question: how does skewness affect the highest returning month in 20 months? In other words, given a particular skewness, what is the mean of over 95th percentile monthly returns?

In order to analyze the effect of skewness, we must first fix the mean and volatility of the distribution. We assume a monthly mean return of 0.80% with a monthly volatility of 1.67%, both similar to the S&P 500 since 1926. We use a skew normal distribution and adjust a parameter that shifts both the skew and the excess kurtosis. Due to the nature of the skew normal distribution, we cannot fix the excess kurtosis. Note that these skewness values use the traditional calculation of skewness (Pearson's moment coefficient of skewness).

Skewness	Excess kurtosis	>95%ile monthly mean
-0.72	0.56	3.50%
-0.67	0.52	3.57%
-0.62	0.45	3.63%
-0.56	0.41	3.71%
-0.49	0.33	3.78%
-0.42	0.28	3.85%
-0.34	0.21	3.93%
-0.28	0.16	4.00%
-0.21	0.11	4.06%
-0.14	0.07	4.11%
-0.09	0.03	4.15%
-0.04	0.01	4.20%
0.00	0.00	4.24%
0.03	0.01	4.27%
0.10	0.05	4.34%
0.14	0.06	4.38%
0.20	0.11	4.44%
0.27	0.15	4.50%
0.35	0.21	4.57%
0.41	0.27	4.63%
0.50	0.36	4.70%
0.55	0.39	4.74%
0.61	0.44	4.79%
0.67	0.52	4.84%
0.72	0.58	4.87%

Table 12.5. 95%ile Monthly mean return conditional on skew and kurtosis

Increasing skewness from -0.72 to 0.72 only increases the mean return of the best month in 20 from 3.50 to 4.87%. In this particular distribution and for this range of skewness and kurtosis, an increase in 1.00 of skewness translates to a 94 bp increase in 95th percentile mean monthly returns.

To apply this knowledge, we consider the skewness of high volatility lottery stocks and low volatility non-lottery stocks. High volatility, high *ex ante* skewness portfolios experience a 1.26 higher skewness and a 0.16% lower monthly return than low volatility, low *ex ante* skewness portfolios. A 1.26 higher skewness roughly translates to a 1.18% increase in 95th percentile returns but at a 0.16% lower average monthly return (2.0% lower annualized return). It would require an implausibly high desire for lottery-like payoffs to justify the lower return and higher volatility for such a small increase in skewness.

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The Low Beta Anomaly and Interest Rates

The reasons for outperformance in smart beta portfolios remains a mystery. We extend previous literature on the link between portfolio performance and macroeconomic factors by exploring the response of a low beta portfolio to interest rate movements. The implications for fund managers heavily invested in low-risk strategies where the immediate risk lies in the future rise in interest rates are worth considering. In particular, low beta funds appear to go up when interest rates fall more than when interest rates rise. We focus on the case of US equity investment based on the capital asset pricing model (CAPM). We find that the anomaly is partially explained by interest sign changes due to macroeconomic events, and observe heterogeneous impacts for low and high beta portfolios.

One of the observations over the cross-section of stocks is that the historical risk-return trade-off is flat or inverted: within the CAPM, we would expect that stocks with high systemic risk would outperform their low risk counterparts, but results have shown otherwise. It is an empirical fact that interest rates have been declining over the recent decades, and there is evidence that interest rate movements affect portfolio choice. The question then arises whether there are heterogeneous impacts to the interest rate for high and low beta portfolios, as the anomaly arises from the observation that low beta portfolios outperform their high beta counterparts. We want to find the origin of this so-called “anomaly”, which we believe is linked to the behavior of portfolios to interest rates.

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There is some evidence that in the context of Sharpe's market model [SHA 64] the differing exposures to interest rate movements are not captured by systematic risk, but by an alpha effect that is heterogeneous over portfolios. We observe that low beta portfolios outperform high beta portfolios at times of low interest rates: we saw a steady decrease in interest rates over 1980–2010, which matches the period of low beta outperformance. However, a model that estimates the interest rate effect as a structural break would fail to take the one period nature of the CAPM into account, and the resulting effect on the *ex ante* expectations set by the model. This relates directly to the setting of the interest rate target by the Federal Open Market Committee (FOMC); movements in the target rate are gradual, almost constant in magnitude, and highly persistent.

Hence, we propose a method where we use sign changes in interest rates to capture the underlying macroeconomic policy implications for actual reactions of investors. The heterogeneous impact can be quantified through the effect on the intercept of the CAPM, indicating a violation of the CAPM assumptions and suggesting a change in behavior around a zero change. We validate the threshold with a grid search along the likelihood function of our data, and link the asymmetry in the portfolio returns to the persistence of interest rate sign changes.

At the source is the trade-off between being implicitly long or short bonds in times of interest rate changes, and the mismeasurement that occurs if we do not account for the term structure. This is at the heart of the argument in this chapter, which is that the type of interest rate used is dependent on the composition of investors in the market. Investors differ in their degree of risk aversion, and we argue that this is pronounced through either a spread between a borrowing and lending rate, or investing on different parts of the yield curve. The argument follows from the observation of inverted yield curves in times of recession, and suggests that the anomaly arises from exogenous macroeconomic influences.

There are two lines of argument as to why low and high beta portfolios react differently: first, the opportunity cost when the interest rate decreases makes safer investments more attractive, and second, the interest rate is a reflection of real economic conditions and economic health, which particularly impacts firms that have more gearing. We do not see a similar switch in high beta portfolios as of the heterogeneous gearing across firms in a high beta portfolio: firms that are riskier are generally more equity financed in absolute terms rather than leveraged on debt.

As firms with a lower market beta usually have a higher gearing ratio, we expect that increases in the interest rate affect their performance more than firms with a

higher market beta; low beta portfolios will have a lower return when interest rates increase, but see a higher return when the interest rate is decreasing. Thus, interest rate changes affect low beta portfolios asymmetrically because of the underlying composition of debt.

We combine the literature on leverage constraints with macroeconomic factors and studies relating to the term structure of interest rates (see [EST 96] and [BAL 10]), where we distinguish portfolios as heterogeneous investors as in [BRE 93]. We argue that the term structure of interest rates and the impact of heterogeneous risk aversion across investors lead to a discrepancy between portfolio returns, and that the anomaly arises for a failure to account for this effect. The chapter focuses on two potential explanations of the low beta anomaly, namely interest rate sign changes and failure to account for the interest rate term structure.

13.1. Literature review

The anomaly has been recognized empirically in many applications (see, for instance, [BLA 72a, BLA 72b, FAM 92, HAU 75]). Baker and Wurgler [BAK 11] provide an extensive review in favor of the low beta anomaly. Also, see Ang *et al.* [ANG 06], who find that stocks with higher idiosyncratic risk earn lower returns in all cases considered.

The causes of the anomaly and how to quantify them are at the heart of the literature: for instance, approaches using mismeasurement and volatility premiums on high risk stocks [DIB 12, KLE 13], the impact of unobservables and leverage on the returns [FAM 96, COC 13, FRA 11], approaches using cumulative prospect theory from [KAH 92] to model lottery preferences and different preferences in the loss domain [COR 08, BAR 08, BHO 11, LEV 12, KUM 09] and manager behavior perspectives [CHE 97, SRI 98, ASN 12].

We focus on the literature relating to unobservables and underlying leverage, and combine it with macroeconomic factors and studies relating to the term structure of interest rates [EST 96, BAE 10], where we distinguish portfolios as heterogeneous investors as in [BRE 93]. We argue that the different portfolio return distributions for interest sign changes lead to a discrepancy between low and high beta portfolio returns.

Di Bartolomeo [DIB 12] and Klepfish [KLE 13] argue that high-frequency arithmetic rates of returns are mistakenly compared to the geometric rates of return over longer period, leading to a volatility premium. For instance, we can show that a discrete return adjusted for a volatility premium can be expanded as a Taylor series:

$$E_t\left(\frac{P(t+1)}{P(t)} - 1\right) - E_t(\ln(P(t+1)) - \ln(P(t))) = \frac{1}{2}(\mu^2 - \sigma^2) + o(\mu^3)$$

The symbol $o(\mu^3)$ means that the remainder is of order three in the instantaneous mean. We note too that under these assumptions, as long as the instantaneous mean is small, we require that μ be greater than σ in absolute value for arithmetic expected returns to be greater than geometric ones. Not accounting for this factor causes substantial differences between arithmetic and geometric returns, particularly in their average volatility. Hence, portfolios with a higher beta would underestimate the expected return if the volatility bias is not taken into account. Related is the work by Haugen and Wichem [HAU 74] who explore the impact of holding duration of risky versus riskless assets on their relative price volatility.

Mispricing can also occur through the effect of unobservable factors, as in the three factor model by Fama and French [FAM 96]. This model uses three stock specific factors that offer potentially orthogonal dimensions of risk and a return [SCH 11] premium for investors willing to take the risk with these factors [COC 13]. The factor premiums capture effects formerly incorporated in a CAPM intercept, which implies that the higher low risk return is not an anomaly but a mismeasurement of missing factors.

This is related to leverage constraints on portfolio choice. Frazzini and Pedersen's [FRA 11] explanation of this phenomenon follows from the preference of investors to carry more risk than the market can provide, but leverage is costly to obtain. In turn, these investors buy high beta stocks instead of leveraging, driving up the cost for high beta stocks relative to low beta counterparts. An extension using option theory is provided by Cowan and Wilderman [COW 11]. In the context of our simple model, explicitly leveraging low beta simply gives the high beta portfolio due to two fund money separation so we will not pursue this explanation.

The riskiness of leverage strategies is determined by the underlying risk-free rate: interest rates can affect the portfolios through the effect of maturity premia and the borrowing constraints of investors. The yield curve shows the range of interest rates across bonds of the same risk and liquidity but with differing maturities. It is argued in previous work by Estrella and Mishkin [EST 96] that the slope of the yield curve is a good predictor of recessions in the US as the sign gives an indication of whether the economy is slowing down or the money supply is tightening. In economic turmoil, it is possible that the yield inverts: as the long-term interest rate represents the risk-adjusted average of the expected future short-term

interest rates and the long-term interest rates will fall, but by a smaller amount than the short-term interest rates. Others confirming this result are Adrian *et al.* [ADR 10], Bernanke and Blinder [BER 92], Bernard and Gerlach [BER 98] and Rudebusch and Wu [RUD 04], who find evidence in favor of the prediction power of the term structure.

Furthermore, the magnitude of changes in the target interest rate has been remarkably constant, regardless of the sign of the respective change (see, for instance, [COI 11, GOO 05, GUR 05]). Also, Coibion and Gorodnichenko show that there is substantial persistence in the target rate set by the FOMC, which implies that there are cumulative, non-independent expectations of interest rate changes. The leverage argument provides substantial insight as to how portfolio returns may differ with regard to their interest sensitivity, with more importance to the gearing on debt of the firms underlying the portfolio that causes the anomaly. As the gearing ratio is an indicator of the debt structure of a firm, there are heterogeneous responses to interest rate movements over high and low beta firms. We reconcile the above approaches to argue that failure to account for interest rate movements leads to substantial mispricing which causes the low beta anomaly.

13.2. The anomaly and interest rates

Let μ_i, μ_m be the expected arithmetic rates of return on asset i and the market m , respectively. Let β_i, r_f be the population beta of asset i with respect to the market m and the riskless rate of return, respectively. The CAPM states:

$$\mu_i - r_f = \beta_i(\mu_m - r_f) \quad [13.1]$$

We will look at this relationship to see how changing conditions influence the price of the asset. We can conceive of this as being the following things within the model framework: (1) multiple changes, (2) changes in the risk premium, (3) changes in expectations of future earnings and (4) changes in aggregate risk aversion.

Noting that at time t , $\mu_i = \frac{E_t(P_{i,t+1})}{P_{i,t}} - 1$, where $E_t(P_{i,t+1})$ is the expectation held at time t of the price of asset i at time $t+1$, an amount that would take into account expected capital gains and dividends:

$$P_{i,t} = \frac{E_t(P_{i,t+1})}{1+r_f+\beta_i(\mu_m-r_f)} \quad [13.2]$$

Suppose, we were to consider a change in the market expected rate of return and a simultaneous change in the riskless rate of return. We denote these changes by $d\mu_m$ and dr_f , respectively. Let the change in the price be dp_{it} . Thus:

$$dP_{it} = \frac{dP_{it}}{dr_f} dr_f + \frac{dP_{it}}{d\mu_m} d\mu_m$$

$$dP_{it} = \frac{-E_t(P_{i,t+1})}{(1+r_f+\beta_i(\mu_m-r_f))^2} (dr_f + \beta_i(d\mu_m - dr_f)) \quad [13.3]$$

Since the terms to the left of the brackets are unambiguously negative, we can see that a total change in the risk premium ($d\mu_m - dr_f$) that is positive, say 2% with an asset with a beta of 0.5 will decrease prices as long as the associated interest rate fall is less than 1%. There is a difference in the response across portfolio types: as high beta portfolios are linked to being short bonds while low beta ones are long bonds, the latter carry a different sensitivity to the interest rate. By going long on the riskless bond, low beta portfolios see an increase in their relative return in times of interest decreases, while high beta portfolios see a decrease under similar conditions.

Ross [ROS 71, ROS 76] developed a theory of asset pricing following the attack on the conclusions reached by the CAPM as equity returns are not normally distributed and the model is not empirically validated. Arbitrage pricing theory (APT) follows from the notion that, for any financial asset, there is no single systematic risk factor but rather a combination of many. One of the main implications of the APT is the principle of diversification, meaning that idiosyncratic risk is not present for well-diversified portfolios.

$$\mu_i - r_f = \beta_{i1}(\mu_{m1} - r_f) + \dots + \beta_{ik}(\mu_{mk} - r_f)$$

Burmeister *et al.* [BUR 03] provide an overview of the methods in which risk factors can be included in the empirical justification of the APT. Again, empirical specifications of the APT are subject to the critique of Fama and French [FAM 96] as we can think of an infinite set of factors that might have an influence on the expected returns: hence, there is a need for a proper theoretical foundation of the factors. For instance, interest rate risk is identified as a strong potential risk factor. We write the CAPM as follows:

$$\mu_i = \beta_i \mu_m + (1 - \beta_i) r_f$$

Inspecting the CAPM above, it is clear that, if we are in equilibrium, a fall in the interest rate will lower the expected rate of return for a low beta asset and raise the expected rate of return for a high beta asset. A possible explanation of a failure of modeling this in the CAPM lies in the difficulties of using a one period model with a time series of data, and the failure to provide insights into disequilibria.

By decomposing the CAPM to incorporate the risk-free rate directly, we see that macroeconomic interest rate movements have a direct impact on the portfolio returns. Our contribution is empirical but has a theoretical basis: interest rate

movements follow from the CAPM as a subcase of the APT and we estimate the potential difference in impacts for low and high beta portfolios. Following from the observation that the magnitude of interest rate changes is fairly constant, we argue that interest rate sensitivity is captured by the sign changes and cumulative persistence of the target rate.

A rise in the interest rate is equivalent to a fall in the price of “cash” and shorting such an asset will increase the value of the portfolio, the high beta stock. We argue that the cost of taking on gearing is related to interest rate movements: when the relative cost of borrowing increases, firms underlying a low beta portfolio which generally take on more debt are more affected than firms that are mostly equity financed: investment moves toward (away) high (low) beta portfolios, driving up (down) the price and return of these products.

13.3. Model specification

In keeping with an APT interpretation, we extend the traditional CAPM analysis by including a term that captures the relative leverage of portfolios to the risk-free rate. In order to test for heterogeneous impacts for high and low beta portfolios, we study two portfolios with differing beta exposures.

$$r_t = \alpha + \beta r_{mt} + V_t \quad [13.4]$$

For a time series regression on a single portfolio, the ordinary least squares estimator (OLS) will be unbiased and efficient if the characteristics of our error term and estimator follow the Gauss–Markov assumptions. Under a correct CAPM specification, we should find that the intercept term α is insignificant in the specification. However, many attempts at CAPM modeling have concluded that this is not the case, particularly for low beta stocks. To capture why we would see a non-zero intercept, we estimate the CAPM again but model the changes in the interest rate directly as an extra factor:

$$r_t = \alpha + \beta r_{mt} + \gamma \Delta r_{ft} + V_t \quad [13.5]$$

We expect that portfolios with different degrees of systematic risk are affected asymmetrically: low beta portfolios are expected to be negatively affected by the positive changes in the interest rate, while high beta returns are expected to increase.

Rather than modeling the magnitude of interest rate changes, we are more interested in the effect of interest sign changes on the portfolio intercept and market beta as the magnitudes of changes in the rate are constant over time. A structural

break analysis at the point of major change in interest rate movements only gives us information on the effect on different samples rather than the actual change in expectations. We propose a threshold analysis where we estimate the CAPM based on the sign of the interest rate change around a reference point c :

$$i_t = \begin{cases} 1 & \text{if } \Delta r_{ft} > c \\ 0 & \text{if } \Delta r_{ft} \leq c \end{cases}$$

The reference point takes a natural value of zero when we are interested in the sign of interest rate changes. We estimate the threshold using a grid search upon the likelihood function with refined tolerances as a robustness check. We estimate the model with interaction terms with the market premium to test whether interest rate changes also affect systemic risk of a portfolio.

$$r_t = \alpha_1 + \beta_1 r_{mt} + \alpha_2 i_t + \beta_2 i_t * r_{mt} + V_t \quad [13.6]$$

13.4. Empirical analysis and results

As the CAPM is a one period theory of portfolio choice of a representative agent, we need to be clear on which interest rate would correspond to the dominating factor. We estimate the model using the 10-year bond rate as well as a mixed equilibrium rate. We argue that there is no distinct difference between the monthly T-bill rate and the 10-year bond rate when it comes to their general movements over the time period, but in terms of changes and volatility there is a major difference. The short-term rate is much less volatile than the long-term rate, which can have substantial differences in a one period model such as the CAPM. Hence, even though interest rates in general may have been declining over the recent decades, what matters is the change over the time frequency which explains our preference for a sign change indicator rather than a structural break analysis.

We use long run industry level data to analyze beta effects. The source of the data is the monthly industry level Fama–French industry level returns from Kenneth French’s Website. We use 43 industry groupings from 1953.01 to 2012.12 to calculate full sample betas. Some initial rolling calculations on the data found five industries that had betas less than 1 (defensive) and nine with betas greater than 1 (aggressive). The defensive industries are food products, tobacco, oil, utilities and telecoms. The aggressive industries are building materials, fun and entertainment, construction, steel, machinery, electrical equipment, chips, lab equipment and financials. Then, we build market capitalization-weighted portfolios of the high beta and low beta industries.

The rationale for this methodology could also be construed in Bayesian terms. We could argue that we have prior beliefs about the nature of certain sectors, for example, we think of utilities as defensive and computers as aggressive. The reason for taking this approach is that it avoids the high degrees of uncertainty in estimated beta. Our empirical approach simply supports what could be justified by prior beliefs. Summary statistics are available upon request, where the numbers reported show noticeable differences between arithmetic and geometric returns. We also report the medians and standard deviations of geometric returns. In all periods, and overall, the standard deviations of high-beta portfolios are higher than those of low-beta portfolios.

We estimated the CAPM by regressing portfolio excess returns on an intercept and market excess returns, and present our results in the first panel of Table 13.2. We would expect the intercept to be zero if the CAPM holds; interestingly, the low beta portfolio has a positive intercept, while the high beta portfolio does not. This demonstration shows the returns to low risk portfolios based on a CAPM theory of risk. Investing in low beta portfolios gives us an extra 3.68% per annum relative to what the CAPM suggests.

Panel 1	<i>Interest</i> ↑	<i>Interest</i> ↓	<i>HIB</i> ↑	<i>HIB</i> ↓	<i>LOB</i> ↑	<i>LOB</i> ↓
<i>Mean</i>	0.016	-0.016	0.568	0.803	0.003	1.354
<i>Standard Dev</i>	0.016	0.018	5.891	5.829	3.514	3.630
<i>Skewness</i>	2.504	-3.224	-0.592	-0.340	-0.533	0.021
<i>Kurtosis</i>	10.903	15.475	3.407	0.880	1.384	1.259

Panel 2	<i>Pre</i>	<i>Post</i>	<i>HIBPre</i>	<i>HIBPost</i>	<i>LOBPre</i>	<i>LOBPost</i>
<i>Mean</i>	6.088	6.232	0.666	0.708	0.547	0.825
<i>Standard Dev</i>	2.948	2.553	5.467	6.230	3.445	3.813
<i>Skewness</i>	1.146	0.581	-0.048	-0.747	0.150	-0.495
<i>Kurtosis</i>	0.935	-0.048	0.739	2.883	1.456	1.414

Table 13.1. Moments of 10-year rate, HIB and LOB conditional on interest changes

Panel 1		α	$t(\alpha)$	β	$t(\beta)$	γ	$t(\gamma)$	R^2	-
<i>Equation (4)</i>									
	<i>HIB</i>	-0.007	-0.102	1.274	81.138	-	-	0.902	-
	<i>LOB</i>	0.307	4.096	0.696	40.857	-	-	0.699	-
<i>Equation (5)</i>									
	<i>HIB</i>	-0.011	-0.154	1.282	81.324	10.740	3.569	0.903	-
	<i>LOB</i>	0.313	4.254	0.681	40.267	-17.585	-5.448	0.711	-
Panel 2		α	$t(\alpha)$	β	$t(\beta)$	δ	$t(\delta)$	R^2	-
<i>Sample Break</i>									
	<i>HIB</i>	-0.061	-0.612	1.273	80.863	0.113	0.792	0.903	-
	<i>LOB</i>	0.404	3.911	0.702	40.982	-0.201	-1.332	0.711	-
Panel 3		α_2	$t(\alpha_2)$	β	$t(\beta)$	$\alpha_1 - \alpha_2$	$t(\alpha_1 - \alpha_2)$	$\beta_1 - \beta_2$	$t(\beta_1 - \beta_2)$
<i>Equation (6)</i>									
	<i>HIB</i> $\beta\alpha$	-0.270	-2.852	1.273	57.730	0.545	3.977	0.011	0.347
	<i>HIB</i> α	-0.280	-2.928	1.278	82.087	0.551	4.053	-	-
<i>Equation (6)</i>									
	<i>LOB</i> $\beta\alpha$	0.766	7.381	0.693	29.466	-0.923	-6.321	-0.010	-0.297
	<i>LOB</i> α	0.770	7.496	0.688	41.424	-0.929	-6.402	-	-
Panel 4		α_1	$t(\alpha_1)$	α_2	$t(\alpha_2)$	β	$t(\beta)$	R^2	-
<i>Double Alpha</i>									
	<i>HIB</i>	0.269	2.789	-0.282	-2.928	1.278	82.087	0.904	-
	<i>LOB</i>	-0.159	-1.540	0.770	7.496	0.688	41.424	0.716	-
Panel 5		α_1	$t(\alpha_1)$	β	$t(\beta)$	$\alpha_1 - \alpha_2$	$t(\alpha_1 - \alpha_2)$	-	-
<i>Estimated (6)</i>									
	<i>HIB</i> c1	-0.282	-2.928	1.278	82.087	0.551	4.053	-	-
	<i>HIB</i> c2	-0.307	-3.033	1.279	82.001	0.549	4.018	-	-
	<i>HIB</i> c3	-0.255	-2.907	1.280	82.267	0.624	4.486	-	-
<i>Estimated (6)</i>									
	<i>LOB</i> c1	0.770	7.496	0.688	41.424	-0.929	-6.402	-	-
	<i>LOB</i> c2	0.648	6.870	0.687	41.028	-0.860	-5.745	-	-
	<i>LOB</i> c3	0.750	7.520	0.686	41.294	-0.947	-6.506	-	-

Table 13.2. CAPM results per specification¹

By including the change in the interest rate (10-year bond rate) as in equation [13.5], we see an increase in the explanatory power for the low beta model and significance for both specifications. The estimates are of opposite signs, which

¹ Numbers are estimates of coefficients of variables, including constant and market risk, and the 10-year rate, respectively. Values of the t-statistic above 1.645 indicate significance above the 5% level, while values above 2.326 indicate significance above the 1% level.

confirms our hypothesis that the low beta portfolio is negatively affected by positive changes in the risk-free rate, while the opposite holds for high beta portfolios. We expect alpha to be significantly different from zero and negative for a portfolio with low beta, and insignificant for high beta portfolios. Table 13.2 shows that alpha is a significant factor for low beta portfolios, albeit not negative. The negative sign is captured by the estimate on the changes in the interest rate.

In the second panel of Table 13.2, we allow for a structural break in 1983 when interest rates started to decrease. From the results, we see that there is no significant difference in the two samples², and no evidence of an increase in systematic risk for either portfolios in the different interest rate regimes. There is some significance on the 15% level for low beta portfolios, but this only results in a double alpha effect rather than a systematic change. Table 13.1. shows the moments for the interest sign changes (Panel 1) and interest magnitude changes (Panel 2); clearly, a structural break is unable to pick up the asymmetry in mean returns in the way a sign change does. The moments for interest rate changes are remarkably similar across positive and negative times for both the sign and magnitude specification. The abnormal return for low beta portfolios for negative interest changes is more visible with sign changes, where we see a positive return of 1.354 for negative changes and 0.003 for positive changes.

Next, we turn to the model specification in equation [13.6]. Using the indicator setup, we are able to pinpoint the impact of the sign of interest rate changes from a reference point, and absorb these changes in double alpha and/or double beta effects. It is clear from Table 13.1 that the distribution of positive and negative changes is quite different over the two sample periods: before 1983, there were significantly more positive interest rate changes than from 1984 onward, which can be explained by the rise of monetarism and the focus on inflation fighting by the chairman of the Federal Reserve under Reagan's administration. Furthermore, the number of "ups" and "downs" is remarkably similar in structure in that the proportion of decreases prior to 1983 is approximately the same as the proportions of increases post-1983. This also suggests that our data set represents a fairly complete epoch of history as the overall proportion of ups, taken over both periods, is very close to 50%. Together with the relatively constant magnitude of the changes in the target interest rate set by the FOMC, this provides evidence that the sign of the change is more important for the expected return.

Panel 3 of Table 13.2 presents our results for interest sign changes. We observe that the impact of the sign of interest rate changes is not captured by a two beta model for both low and high beta portfolios. Instead, the impact is captured by a two

2 Structural break within equation [13.5] around 1983–1. Estimates are the coefficients of the constant, market risk and changes in the 10-year rate, respectively.

alpha model for both portfolios³. Focusing on the alpha effect-only models, we see that alpha becomes a significant factor in the high beta portfolio if we include the sign of interest rate changes (see Panel 4). The estimates for beta in both portfolios hardly change when we include the sign changes, suggesting that systemic risk itself is not affected.

Continuing with our discussion of Panel 4, we demonstrate the impact on low beta portfolios as an example of the total effect on the intercept when including sign changes. We see that the intercept is positive (0.770) whenever we have a negative change in the interest rate as alpha is positive and the indicator takes value zero. This result is in line with the observation that returns of low beta portfolios are positively affected by negative changes in the interest rate. Whenever we see positive changes in the rate (and the indicator takes value unity), alpha for low beta portfolios is negative (-0.159) which confirms our hypothesis that low beta portfolios are asymmetrically affected.

The opposite mechanism holds for high beta portfolios: a decrease in the rate leads to a decrease in the return (-0.282) and an increase leads to a positive change (0.269). Again, this confirms our hypothesis that low beta portfolios outperform high beta counterparts in times of interest rate declines. We found some evidence for a lower alpha in the period leading up to 1983 for low beta than in the period after 1983 and thus gives support to the argument listing interest rates as a factor in low beta outperformance. Interest rates are a significant factor in low beta outperformance and extend the result to the one period CAPM model.

We checked our estimates of our preferred model, the double alpha specification from equation [13.6], for robustness by estimating the reference point using a refined grid search over the likelihood function to find the global minimum. We find the possible minimum and maximum value of the threshold and start by estimating the model for each step starting from the minimum, and computing the sum of squared residuals at each point. Then, we find the optimal threshold by minimizing the Residual Sum of Squares (RSS) function⁴.

3 Estimates of the double alpha model in equation [13.6], rewritten to reveal the underlying significance of the alpha parameters. We test whether alpha 1 and alpha 2 are statistically different, and we find that a Wald test rejects equality for both high (15.394) and low (39.967) beta portfolios.

4 The distribution of the estimates is non-standard and can be estimated using bootstrapping methods. We use three refinement scales (steps of 0.01, 0.001 and 0.0001 which we denote by c1, c2 and c3). We find that the points are not significantly different from zero for the largest refinement scales, but they are for the finest scale (minimum at 0.0034, 95% confidence interval of 0.0648, 0.1021). But, this distance is so close to zero that we do not change our results.

The results are presented in Panel 5. We see that there is not a significant difference from zero, and the estimates are robust to the refinement level. The results support our original model. We find that there is strong evidence that the alphas are for both portfolios, but with opposite signs depending on the interest rate changes. Figure 13.1 shows the behavior of the RSS function for the low beta portfolio, and shows a clear minimum at, or very near, the reference point. We obtain a similar result for the high beta portfolio.

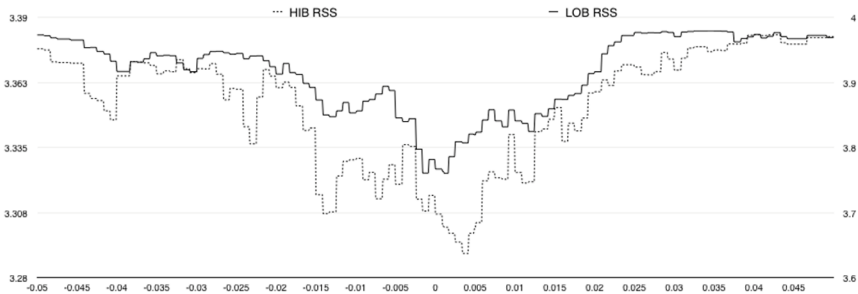


Figure 13.1. Residual sum of squares (RSS) to the threshold level for the low-beta portfolio

Hence, the preferred specification is equation [13.6] where we only allow for a double alpha effect. We see that the sign of the interest rate change is the most significant in distinguishing the effects for both portfolios: whenever we see an increase in the interest rate from the reference point, low beta portfolios will be negatively affected while the opposite holds for high beta portfolios. The strategy with low beta portfolios means being implicitly long on the riskless asset. Empirically, whenever we see a shift in the risk-free asset, low beta portfolios are more affected than high beta portfolios. The estimates show that there is a significant alpha impact in this specification for high beta portfolios, which can be explained by portfolio rebalancing after underlying interest rate movements.

As a robustness check, we estimate model [13.6] including positive changes in the interest rate (dyield+) and negative changes in the rate (dyield-). Positive changes are collected in dyield+ as their actual value, where negative or null values are set to zero (similarly for negative changes). A Wald test testing for the equivalence of the effects and bringing us back to [13.5] shows that the parsimonious model is equivalent, suggesting that there is no difference in upward or downward movements of interest rates for this particular model.

Time-varying estimates are computed over a rolling window without overlap (Figure 13.2). We see, unsurprisingly given the construction methodology, that the result for beta is stable and shows that low beta portfolios indeed see a lower systematic risk than high beta portfolios, except for very specific periods. We see that low betas spiked above high beta portfolios in 1994 during the bond price crash: after a long recession with falling inflation, the cycle turned aggressively in this year after economic recovery and a rise in the federal interest rate. The estimates for alpha are less consistent, but show a clear distinction between high beta and low beta portfolios and mean changes over specific periods. We see that the behavior of the low beta alpha mirrors that of high beta and observe similar implications for the interest indicator variable.

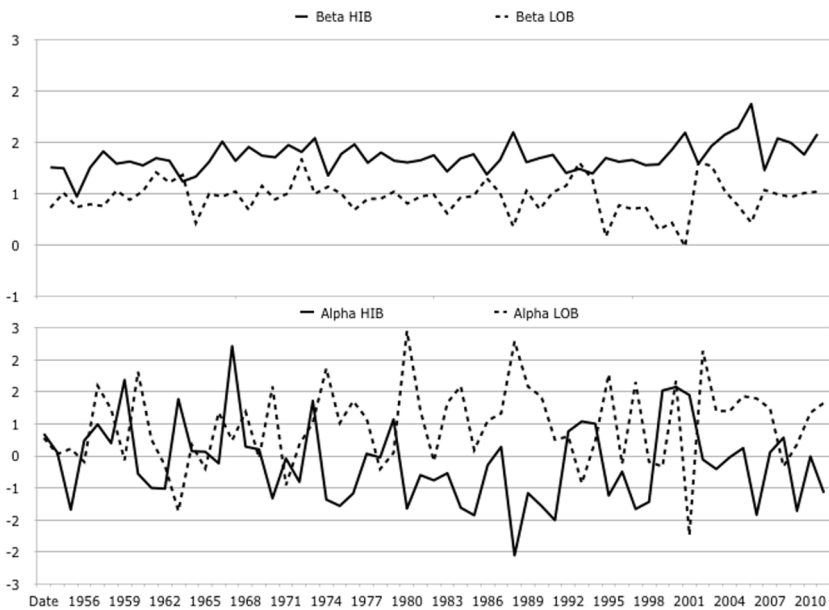


Figure 13.2. Time-varying estimates of equation [13.6]

13.5. The anomaly and interest maturity mismatch

One immediate difficulty with the CAPM is that, in its static form, it is not especially informative about what interest rate we should be using. The CAPM, as discussed above, is a one period theory and the interest rate used would correspond to the holding period of the representative agent. This chapter presents an explanation as to why we might find significant interest rate returns within a CAPM framework. The motivation comes from a traditional theory of interest rate demand

often known as the preferred habitat hypothesis. In this model, investors have at their disposal a bond of a particular maturity, reflecting, perhaps the duration of their liabilities or other considerations. We will capture this by building a CAPM-type model which assumes that there are two agents, both of whom are mean variance optimizers, both confronted by the same set of risky assets, both believing in the same asset price distribution with identical means and variances but having as choice of riskless asset a short-rate bond in one instance and a long-rate bond in the other.

The optimization problem they face is:

$$U = \omega' E(r) - \frac{\lambda}{2} \omega' \Sigma \omega - \theta (\omega' i - 1)$$

where θ is the Lagrange multiplier and λ is the coefficient of absolute risk aversion, ω is a vector of portfolio weights chosen to maximize [13.1] and i is a vector of ones. The vector of expected rate of return of the risky assets is $E(r) = \mu$, and the covariance matrix of returns is given by Σ . We note the following result. The optimal mean-variance weights in the presence of a budget constraint with known parameters are given by:

$$\omega = \frac{1}{\lambda} \Sigma^{-1} \mu - \frac{(\beta - \lambda)}{\lambda \gamma} \Sigma^{-1} i$$

We define $\alpha = \mu' \Sigma^{-1} \mu$, $\beta = \mu' \Sigma^{-1} i$, $\gamma = i' \Sigma^{-1} i$. The expected utility associated with this case is given by, substituting the first into the second equation and simplifying. The maximized value, V , is given by:

$$V = \frac{\alpha \gamma - (\beta - \lambda)^2}{2 \lambda \gamma}$$

If we ignore the budget constraint in the optimization, then the optimal portfolio becomes $\omega = \frac{1}{\lambda} \Sigma^{-1} \mu$ and $E(r) = \frac{\alpha}{2 \lambda}$. Formally, the optimal portfolios where $i = 1, 2$ for short and long rates (r_{1f} and r_{2f} are the short and long rates, respectively) are given by:

$$w_i = \frac{1}{w_0^i} \lambda_i^{-1} \Sigma^{-1} (E(r) - r_{if}).$$

This is the same result as given above except that individuals differ in terms of initial wealth, absolute risk aversion and riskless rates of return. Defining societal wealth as W_{m0} :

$$W_{m0} = W_0^1 + W_0^2$$

Thus, societal investment in the different assets (i.e. the market portfolio) is equal to w , and where $\lambda = ((\lambda_1)^{-1} + (\lambda_2)^{-1})^{-1}$ is societal risk aversion.

We can distinguish two types of agents, with a different risk aversion λ_i . Both agents would choose the same market portfolio, but have a different slope of the riskless rate. The interest rate for investor (1) is lower than the rate for investor (2): in normal economic conditions, this would imply that investor (2) invests on the longer part of the yield curve. Therefore, we can see that it is far from trivial which interest rate we should be using when we depart from the assumption of homogeneous risk aversion.

Therefore, the optimal portfolio weights are:

$$w = \frac{1}{w_{m0}} \Sigma^{-1} E(r) \left(\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \right) - \frac{1}{w_{m0}} \Sigma^{-1} (i) \left(\left(\frac{r_{1f}}{\lambda_1} + \frac{r_{2f}}{\lambda_2} \right) \right)$$

Now,

$$\text{Cov}(r, w'r) = \Sigma w = \frac{1}{w_{m0}} \left(E(r) \left(\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \right) - (i) \left(\left(\frac{r_{1f}}{\lambda_1} + \frac{r_{2f}}{\lambda_2} \right) \right) \right) = aE(r) + bi$$

$$\text{Var}(w'r) = w' \Sigma w = a\mu_m + b.$$

$$\text{Therefore, } \beta = \frac{aE(r) + bi}{a\mu_m + b}.$$

$$\text{Thus, } aE(r) + bi = \beta(a\mu_m + b).$$

Dividing both sides by a , we arrive at:

$$E(r) - \frac{\left(\frac{r_{1f}}{\lambda_1} + \frac{r_{2f}}{\lambda_2} \right)}{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right)} i = \beta \left(\mu_m - \frac{\left(\frac{r_{1f}}{\lambda_1} + \frac{r_{2f}}{\lambda_2} \right)}{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right)} \right)$$

This we call the heterogeneous interest rate CAPM. Defining the relative risk tolerance of the short-rate investors as δ_1 with the relative risk tolerance of long-rate investors being δ_2 , it follows immediately that $\delta_1 + \delta_2 = 1$. The interest rate term in the heterogeneous interest rate CAPM now becomes $\delta_1 r_{1f} + \delta_2 r_{2f}$ and we can write our CAPM as:

$$E(r) - (\delta_1 r_{1f} + \delta_2 r_{2f}) i = \beta (\mu_m - (\delta_1 r_{1f} + \delta_2 r_{2f}))$$

The question that naturally arises is: would we expect the long-rate investors to be more risk averse than the short-rate investors? We would think this to be the case so that the short-rate investors would tend to dominate; that is $\delta_1 > 50\%$.

Suppose, we now run a conventional short-rate CAPM. We would assume the constraint:

$$E(r) - r_{1f}i = \beta(\mu_m - r_{1f})$$

instead of the true model [13.1]. This misspecification would lead to additional terms:

$$E(r) - r_{1f}i = \beta(\mu_m - r_{1f}) + \beta(r_{1f} - (\delta_1 r_{1f} + \delta_2 r_{2f})) - r_{1f}i + i(\delta_1 r_{1f} + \delta_2 r_{2f})$$

$$E(r) - r_{1f}i = \beta(\mu_m - r_{1f}) + \beta((1 - \delta_1)r_{1f} + \delta_2 r_{2f}) + i((\delta_1 - 1)r_{1f} + \delta_2 r_{2f})$$

$$E(r) - r_{1f}i = \beta(\mu_m - r_{1f}) + (\beta - i)(1 - \delta_1)r_{1f} + (\beta + i)\delta_2 r_{2f}.$$

This equation gives us the misspecified CAPM and shows how interest rates can occur as a result of the misspecification. If δ_1 is near 1, and β is near i , we might expect the short rate to have a coefficient close to zero, while the long rate should be typically much larger.

Of course, reality is much more complex and the precise nature of the misspecification could involve almost any point in the term structure. It is worth noting that the equilibrium discussed above generalizes to K different rates where each one will be weighted by the relative risk tolerance of the investors who use the particular discount factor. Furthermore, these relatives' weights will add to 1.

13.6. Model specification

While the interest rate impact will be very difficult to estimate with any degree of conviction, we can consider two polar cases, the monthly T-bill rate and the 10-year bond rate. In a world of nominal prices, rather than real prices, these correspond to holding periods of 1 month, consistent with the rebalancing interval of institutional investors, and a holding period of 10 years which would correspond to medium-to-long-term investment. Incorrectly assuming one rate or the other to be correct throws up additional terms in the regression. In our analysis, we make use of the mixed equilibrium riskless rate, which is in line with the use of possible bond yield curve effects.

To further analyze the impact of specific investing horizons, we use a weighted average of the rates based on the ratio of a particular type of investor to the total investors. It is an established fact in the literature that large institutional investors

and professionally managed funds trade on higher frequency (the short end of the yield curve) than smaller, independent investors [SHA 00, DIA 91, COH 75]).

$$\begin{aligned}\mu_t - (\delta_2 r_{1,ft} + (1 - \delta_2) r_{2,ft}) \\ &= \alpha + \beta (\mu_{mt} - (\delta_2 r_{1,ft} + (1 - \delta_2) r_{2,ft})) \\ &+ \gamma (\delta_2 r_{1,ft} + (1 - \delta_2) r_{2,ft}) + u_t\end{aligned}$$

Here, δ_2 represents the share of smaller investors who invest mostly on the long rate $r_{2,ft}$ as described earlier. We assume that 70% of the agents invest on the short rate, representing a large share of professional traders. We test whether there are changes to the market beta in this specification, as well as to the interest rate exposure γ .

Next, we go deeper into the potential misspecification. The case is as follows: if we estimate the traditional CAPM using the short rate, we create a misspecification that could lead to potential bias. We want to test whether this bias is indeed present, and whether wrongly assuming a short rate is a potential cause of the low beta anomaly. We can rewrite the misspecification as:

$$E(r) - r_{1f}i = \beta(\mu_m - r_{1f}) + \delta_2((\beta - i)r_{1f} + (\beta + i)r_{2f})$$

$$E(r) - r_{1f}i = \beta\mu_m + (\delta_2 - 1)\beta r_{1f} - \delta_2 r_{1f} + \delta_2(\beta + 1)r_{2f}$$

Now, imagine the case for a low beta portfolio with $\beta = 0.67$ and a high beta portfolio with $\beta = 1.25$. The equations are formed as follows:

$$E(r_{\text{low}}) - r_{1f}i = 0.67\mu_m + (\delta_2 - 1)0.67r_{1f} - \delta_2 r_{1f} + \delta_2(1.67)r_{2f}$$

$$E(r_{\text{high}}) - r_{1f}i = 1.25\mu_m + (\delta_2 - 1)1.25r_{1f} - \delta_2 r_{1f} + \delta_2(2.25)r_{2f}$$

The effect on the returns is dependent on the composition of traders. If δ_2 is small, e.g. the composition of short rate, traders is high and the long rate effect is negligible. When δ_2 is large and the proportion of long rate investors is high, the long rate is significantly affecting the portfolio returns and creating the misspecification effect. Denoting $\theta_1 = (\delta_2 - 1)\beta - \delta_2$ and $\theta_2 = (\beta + 1)\delta_2$, we test whether θ_2 is insignificant to ensure no misspecification is present. We estimate the model as follows:

$$r_r - r_{1ft} = \alpha + \beta\mu_{mt} + \theta_1 r_{1ft} + \theta_2 r_{2ft} + v_t \quad [13.7]$$

We expect that larger investors tend to be more sensitive to the short-term rate, while smaller investors are more affected by the long-term rate. The result for portfolios is dependent on the main investors in the market: given also that large investors are generally less risk averse, their presence in the market would lead to a

higher demand for portfolios with a higher beta. In our specification, we assume that 70% of the market is dominated by large institutional investors.

13.7. Results

First, we estimate the model for the yield curve specification for both the low and high beta portfolios. After that, we turn to the estimates of the misspecification equation [13.7]. As a robustness check, we allow for multiple values for δ_2 to see whether the misspecification occurs for other investor proportions.

We find that using the 1 month T-bill rate gives insignificant results for the change in the interest rate, which is explained by the frequency of rebalancing of portfolios by institutional investors. In a correct specification of the CAPM, the market premium is the only risk factor. In Panel 1 of Table 13.3, we use the contemporaneous slope of the yield curve as our interest rate variable. In the first specification, we test for the change in the yield curve directly, and we show that there is no significant impact on either the low beta or high beta portfolios. Given that 70% of the investors are assumed to invest on the short rate, this result is not remarkable. When we include the sign change specification, we observe that the sign changes are not as significant any more for high beta portfolios, but still very significant for the low beta set. This confirms our hypothesis that portfolios with low beta are negatively affected by positive interest rate movements, but the impact does not reverse when interest rates decline as we saw with the long rate estimations in the previous section.

Panel 1	α	$t(\alpha)$	β	$t(\beta)$	γ	$t(\gamma)$	ϕ	$t(\phi)$
<i>Yield Curve</i>								
<i>HIB</i>	-0.110	-1.221	1.273	80.954	0.972	0.891	-	-
<i>LOB</i>	0.310	-4.092	0.701	40.840	-1.062	-0.900	-	-
<i>HIB</i>	-0.283	-1.252	1.272	81.231	-	-	0.243	1.732
<i>LOB</i>	0.481	-4.762	0.702	41.402	-	-	-0.381	-2.560
Panel 2	α	$t(\alpha)$	β	$t(\beta)$	θ_1	$t(\theta_1)$	θ_2	$t(\theta_2)$
<i>Equation (7)</i>								
<i>HIB</i>	-0.147	-2.096	1.281	80.540	10.580	3.467	-0.390	-0.353
<i>LOB</i>	0.467	6.210	0.682	39.965	-17.286	-5.280	0.142	0.119

Table 13.3. CAPM results for different interest rate specifications

Panel 2 of Table 13.3 presents results for the mixed interest rate return case. To remind the readers, the excess returns and market premium are based on the mixed interest rate as in equation [13.7], and we use the 1 month rate (coefficient θ_1) and the 10-year bond (θ_2). To explain the anomaly, we would require the long rate to have a significant impact, while the short rate should have a coefficient close to zero (particularly when the share of high-frequency investors is high and beta is near 1). We observe this is false and that the expected return of low beta portfolios is actually smaller when we include the long rate: for this period and market, the anomaly cannot be explained by the misspecification of interest rates at least under the assumption that we have made for investor relative shares.

It is possible that the share of short rate investors is not close to 70%, but actually higher or lower. As a robustness check, we repeat the analysis of the misspecification for different values of δ_2 . We can rewrite θ_1 and θ_2 of Panel 2 of Table 13.3 into the direct “impacts” of the rates:

$$\frac{\theta_1 + \delta_2}{\delta_2 - 1} = \gamma_1 \text{ and } \frac{\theta_2 - \delta_2}{\delta_2} = \gamma_2$$

In our case, θ_2 is insignificant so we focus on the effect of θ_1 instead. In our specification of a share of 30% of investors on the long rate, we see that γ_1 is 27.16 for low beta and -17.06 for high beta. When we assume that the proportion of investors on the short rate is zero, we see a sign change in the interest rate effect: low beta portfolios are positively affected by the short rate (17.28), while high beta is negatively affected (-10.58). When we increase the share to 50%, low beta is negatively affected (-55.32) while high beta is positively affected (33.13). The misspecification is not present for any value of δ_2 , as it is fully dependent on the significance of θ_2 , the coefficient on the long rate. In our case, the long rate is not significant and therefore the effect diminishes.

In this section, we explored the alternative explanation that the low beta anomaly is caused by the composition of interest rate maturities. We find that there is not enough evidence of misspecification in the CAPM to suggest that the anomaly is caused by the proportion of investors on different parts of the yield curve, and that the yield curve specification actually removes the double alpha effect we observed in the previous section. Additionally, the outperformance of low beta is no longer observed in the sign change specification. However, agents only differ in the risk-free rate and still invest in the same market portfolio in the framework presented in this section. The next section provides an extension where we allow for different lending and borrowing portfolios, and where the agents are distributed as combinations of these two portfolios. The implication of this framework is that we do not have a single defined market, which might provide additional insights to the investor composition effect.

13.8. Concluding remarks

This chapter compares different specifications with macroeconomic factors by allowing for threshold CAPMs driven by interest rate movements. From the structural break results, we see that the differing exposures to interest rate movements are not captured by a heterogeneous beta model, but by a double alpha effect for low beta portfolios. However, this method fails to find the impact of actual interest rate changes on the slope and intercept of the two models when there are different changes in the same period.

In our proposed specification, using the sign of the interest rate change (validated by a reference point check using a grid search upon the likelihood function of our specification) rather than the actual change, we find that alpha is negative for low beta portfolios whenever the interest rate is rising and that it is positive whenever the rate is decreasing. In line with the previous results, we find significant evidence of outperformance of low beta portfolios based on interest rate movements and underperformance of high beta portfolios. There is no systematic effect of the interest rate on beta itself. This is evidence that the outperformance of low beta portfolios is not related to their systematic market risk but to interest rate factors that influence the intercept of the CAPM.

We show that the opaque nature of the definition of the riskless asset is a complicating factor. We find evidence that the slope of the yield curve has a significant and differentiating impact on low and high beta portfolios by using a simple general equilibrium model. We consider 1 month, 10-year rates and an equilibrium combination of the two based on an estimated relative share of investors. We might expect that the appropriate rate for the CAPM is the 1-month rate as this would reflect the rebalancing period of institutional investors. What we find empirically is that we see similar results for the slope of the yield curve and the long-term rate.

When we test a misspecified version of the CAPM based on a mismatch in maturity levels and investor preferences, we observe that the short-term interest rate does not have a significant impact on the excess returns of the portfolios, in line with theory. However, we expect the sign of the long-term rate to be positive in both cases. We find that the coefficient for the low beta portfolio is of the opposite sign, resulting in a rejection of the hypothesis that the anomaly arises from this particular form of mismeasurement. However, the analysis might differ if we include more securities of different maturities.

The main force behind the anomaly is likely to be attributed to exogenous macroeconomic factors influencing the risk-free rate. Monetary policy over the last

30 years has favored low beta strategies by increasing the price of bonds and it is fair to say that these macroeconomic factors shape our results, and are the main drivers behind off-equilibrium movements of returns. Hence, our model provides a link between macroeconomic (yield curve related) factors and the origin of the low beta anomaly. It seems that the underlying exposure to the risk-free asset has to be considered for a model consistent with the CAPM implications. To call out of equilibrium movements, an anomaly in the social sciences seems unwarranted.

13.9. Bibliography

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Factoring Profitability¹

Recent studies in financial economics posit a connection between a gross-profitability strategy and quality investing. We explore this connection with two widely used factor models. The first is the four-factor Fama–French–Carhart model, which is a mainstay of empirical research in academia. The second is the Barra USE4 multi-factor model, which is a standard for practitioners. Our findings are:

- consistent with results reported by other researchers, the Fama–French–Carhart model does not provide a satisfactory replication of the gross-profitability strategy over the period July 1995–December 2012;

- over the same period, the Barra USE4 multi-factor model, which is a standard for practitioners, replicates a substantial portion of the gross-profitability strategy with quality and momentum factors;

- the book-to-market factor, which is one of the three value factors in the Barra USE4 model and the only value factor in the Fama–French–Carhart model, does not make a significant contribution to the gross-profitability strategy;

- however, the Barra USE4 earnings-yield factor, which is another measure of value, *does* make a significant contribution to the gross-profitability strategy.

It is important to note that our results rely on relatively short data histories. We will not be able to determine the long-term efficacy until the data history itself is longer.

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14.1. Quality is an active investment strategy with a long and distinguished history

Quality, such as value, momentum and size, is a popular investment style or factor tilt that draws some investors away from market capitalization-weighted indexes². The idea of quality investing is generally attributed to Graham (1949)³, who characterized quality firms in terms of attractive features such as positive and stable earnings, low volatility and low leverage. However, there is no single definition of quality just as there is no single definition of value.

Recent studies in financial economics connect quality investing to gross profitability, which is the difference between revenue and cost of goods divided by firm assets. The relationship between gross profitability and quality investing is developed in [NOV 13a]:

Gross profit is the cleanest accounting measure of true economic profitability. The farther down the income statement one goes, the more polluted profitability measures become, and the less related they are to true economic profitability.

The relationship is further developed in [NOV 13b]. Using a technique that is well established in the empirical finance literature, Novy-Marx [NOV 13a] builds an idealized strategy based on the ranking of firms by gross profitability⁴. The word “idealized” is important here: turnover, impediments to short selling and other market frictions may render the gross-profitability strategy uninvestable. Nevertheless, the gross-profitability strategy can provide insight into the drivers of risk and return, and it serves as an element in the construction of the investable factor-tilted strategies mentioned above^{5,6}.

According to Novy-Marx [NOV 13a], the gross-profitability strategy earned a significant positive excess return between July 1963 and December 2010 over and

2 A factor tilt is an active bet that requires frequent rebalancing versus a capitalization-weighted index.

3 [GRA 49] has been reprinted numerous times, most recently in 2006.

4 The idealized strategy is long top-quintile profitably firms and short bottom-quintile firms. In practice, these quintiles are often constructed within size or industry cohorts and then averaged.

5 For this reason, idealized strategies of this type are often called “factors” in the literature.

6 Table 1 in [NOV 13b] reports the performance of the long and short sides of the factor separately over the period from July 1963 to December 2012. The long side, which corresponds to high-gross-profitability securities, had a Sharpe ratio of 0.47 driven by a net excess return of 6.7% and a volatility of 14.3%. This is inconsistent with quality investing, which typically includes low risk as a distinguishing feature.

above the product of its beta with the excess return of the market. This qualifies the gross-profitability strategy as a capital asset pricing model (CAPM) anomaly⁷. There are numerous candidate explanations for gross-profitability and other CAPM anomalies. For example, it may be that high-gross-profitability stocks were underpriced in July 1963 and overpriced in December 2010. If this is the case, high-gross-profitability stocks might be a poor investment at this point. It is also possible that the abnormal returns to high-gross-profitability stocks were a statistical fluke, in which case there is no compelling reason to buy them or not to buy them. The abnormal returns may be compensation for risk, in which case each investor needs to evaluate the tradeoff between risk and expected return that is represented by a bet on profitable stocks. It seems beyond the reach of current scientific practice to determine with conviction the explanation for gross-profitability or other CAPM anomalies.

14.2. Replicating gross profitability with style factors

Even though we cannot determine the explanation for the abnormal returns delivered by complex or opaque investment strategies, it is sometimes possible to replicate these returns using simpler and more transparent elements. For example, Hasanhodzic and Lo [HAS 07] replicate the returns to a variety of hedge fund strategies with liquid, exchange-traded instruments. They find that a significant fraction of the risk and return of hedge funds can be captured by well-chosen linear combinations of these liquid and low-cost instruments.

	1963 to 2012		1995 to 2012	
R-Squared	0.27		0.47	

	1963 to 2012		1995 to 2012	
Factor Name	Coefficient	t-Statistic	Coefficient	t-Statistic
Intercept	0.00	7.43	0.01	6.21
MKT	-0.08	-6.59	-0.16	-7.79
SMB	-0.12	-7.14	-0.14	-5.23
HML	-0.10	-5.62	0.00	-0.05
MOM	0.08	7.14	0.09	5.09

Table 14.1. *Monthly regression coefficients, t-statistics and R-squared values for the replication of the gross-profitability strategy with the Fama–French–Carhart model. Left panel: long history. Right panel: recent period*

⁷ There is an enormous amount of literature on CAPM anomalies. The best known CAPM anomalies are size and book-to-market, which were popularized in [FAM 92] and [FAM 93], and momentum, which is documented in [CAR 97]. Other important anomalies include accruals, whose investigation is pioneered in [SLO 96]; low risk, which was documented in [BLA 72] and is reviewed in [GOL 14]; and asset growth, which is documented in [LI 13].

We carry out an analogous exercise by replicating gross profitability with factor models, which are important extensions of the CAPM. Factor models provide insight into the risk and return drivers of investment strategies. First, we attempt to replicate the gross-profitability strategy using the Fama–French–Carhart four-factor model, which includes the excess return to the market (MKT), size (SMB) and book-to-market (HML) factors developed in [FAM 92] and [FAM 93], and the momentum (MOM) factor developed in [CAR 97]. Next, we replicate a substantial component of the gross-profitability strategy with the Barra USE4 model⁸.

14.3. The four-factor Fama–French–Carhart model does not explain gross profitability

The replication of the gross-profitability strategy with the four-factor Fama–French–Carhart model was carried out over a long horizon, June 1963–December 2012, and also over a shorter, more recent period, July 1995–December 2012. The model coefficients and their t-statistics are shown in Table 14.1. Over both periods that we examined, the intercept and the market (MKT) made small but significant contributions⁹ to the return of the gross-profitability strategy. The significant intercept indicates that alpha may be present in the strategy or that one or more factors may be missing from the model. The intercept can be interpreted as an estimate of monthly return. This translates to an alpha of 4.6 basis points per year for the long horizon and 6.7 basis points per year for the recent period. The small but significant negative coefficient of MKT suggests that the gross-profitability strategy may be slightly anticorrelated with the market. The size factor, SMB, made a significant negative contribution during both periods, indicating a bias toward large-capitalization companies. The book-to-market factor, HML, made a significant negative contribution over the longer period but not over the more recent period. The difference can be explained by the high volatility of the book-to-market factor around the turn of the millennium: over the shorter period, the high volatility overwhelmed any signal that may have been present in the factor. The gross-profitability strategy exhibited small but significant contributions from momentum (MOM) over both horizons we considered.

An important diagnostic on the replication is the R-squared, which indicates what fraction of the strategy’s variation over time is picked up by the model. In the

⁸ We focus on style factors in both replications since we are using an industry-neutral version of the gross-profitability strategy. A strategy is industry-neutral if its industry exposures match the industry exposures of the market.

⁹ In keeping with standard protocol, we call a replication factor significant if the t-statistic of its coefficient exceeds 1.96 in magnitude. However, the standard connection between the likelihood that a coefficient is different from zero and the t-statistic depends on strict assumptions that are rarely satisfied in practice. A sensational example is in [AND 15].

example under consideration, a relatively small portion of the time variation of the gross-profitability strategy is explained by the Fama–French–Carhart model. The Fama–French–Carhart model explained the time variation in the gross-profitability strategy with an R-squared of 27% over the long horizon, and 47% over the recent period. For this reason, and in light of the incompatibilities described above, we conclude that the Fama–French–Carhart model does not provide a satisfactory representation of the gross-profitability strategy¹⁰. However, there may be a broader collection of style factors that do a better job of replicating the gross-profitability strategy. We explore this below.

14.4. The Barra USE4 model explains a substantial portion of gross profitability over the past two decades

While the Fama–French–Carhart model may not provide a great deal of explanatory insight into gross profitability, a broader set of style factors can explain much more. We replicate a substantial component of the gross-profitability strategy with the Barra USE4 style factors. The full set of Barra USE4 style factors is shown in the first column of Table 14.2, and it includes familiar investment drivers such as size, leverage and liquidity. The Barra USE4 model has three value factors: book-to-market, earnings yield and dividend yield. Only the first of these factors, book-to-market is part of the Fama–French–Carhart model¹¹.

Between July 1995 and December 2012, five significant style factors explained time variation in the gross-profitability strategy with an R-squared of 69%. The strategy had a positive weight on earnings yield, which is one of the three Barra USE4 value factors, and negative weights on beta, residual volatility and leverage factors. This profile is consistent with quality investing. Notably, there is a substantial positive loading on momentum, which is not part of a quality profile. Book-to-market, which is the only value factor in the Fama–French–Carhart model, did not play an important role in explaining the return to the gross-profitability strategy¹¹. Similarly, the contribution of the intercept to the gross-profitability strategy was negligible between July 1995 and December 2012. The model coefficients and their t-statistics are shown in the full model columns of Table 14.2.

10 It is also possible that there is a mismatch between the gross-profitability strategy, which has been adjusted to have the same industry exposures as the market, and the four-factor Fama–French–Carhart model, which is not industry-adjusted.

11 The book-to-price factor in the Barra USE4 model is based on the same accounting ratio as the book-to-market factor (HML) in the Fama–French–Carhart model. However, returns to the two factors differ due to different model estimation processes. For example, the estimation of the Fama–French–Carhart book-to-market factor does not control for industry effects, but the estimation of the Barra USE4 book-to-market factor does.

	Full Model		Even Model	
R-Squared	0.69		0.68	

	Full Model		Even Model	
Factor Name	Coefficient	t-Statistic	Coefficient	t-Statistic
Intercept	0.00	0.92	0.00	1.10
Beta	-0.30	-5.69	-0.27	-3.12
Book to Price	-0.25	-1.70	-0.29	-1.36
Earning Yield	0.38	4.90	0.32	2.41
Leverage	-0.81	-6.36	-0.69	-3.14
Momentum	0.19	4.01	0.31	3.91
Residual Volatility	-0.35	-4.27	-0.30	-2.25
Beta Non-Linear	0.04	0.26	-0.29	-1.27
Dividend Yield	0.16	1.25	0.35	1.75
Growth	0.18	1.40	-0.21	-0.91
Liquidity	-0.18	-1.58	0.11	0.53
Size	0.09	1.00	0.10	0.71
Size Non-Linear	0.01	0.07	0.21	1.41

Table 14.2. Monthly regression coefficients, t-statistics and R-squared values for the replication of the gross-profitability strategy with Barra USE4 style factors. The entire data set is used to fit coefficients. In the even model, only even-numbered months are used to fit the coefficients. Factors with t-statistics of magnitude greater than 1.96 are shaded. July 1995–December 2012

A strategy replication is far more credible if its in-sample characteristics persist out-of-sample¹². To analyze the persistence of the replication of gross profitability by Barra USE4 factors, we re-estimated the replication with half the data: the returns from even-numbered months. The linear combination of significant factors from this exercise is called the even model¹³. The coefficients (betas) and their t-statistics are shown in the even model columns of Table 14.2. Note that while there are differences between the full model and even model coefficients, the same set of factors is significant and the factor coefficients have the same signs. In other words, the two sets of coefficients are qualitatively similar.

12 The full model is in-sample because its goodness of fit to the gross-profitability strategy is evaluated using the same data used to estimate the model. In an out-of-sample test, the evaluation is based on complementary data. Out-of-sample tests can indicate whether an in-sample fit is a statistical fluke. However, an out-of-sample test is not a panacea; see, for example, [MAR 94].

13 Specifically, the even model forecast return for time t is $r_t = \sum_i \hat{\beta}^i r_t^i$ where $\hat{\beta}^i$ is the estimated coefficient of (significant) factor i in the even model fit and r_t^i is the return to factor i at time t .

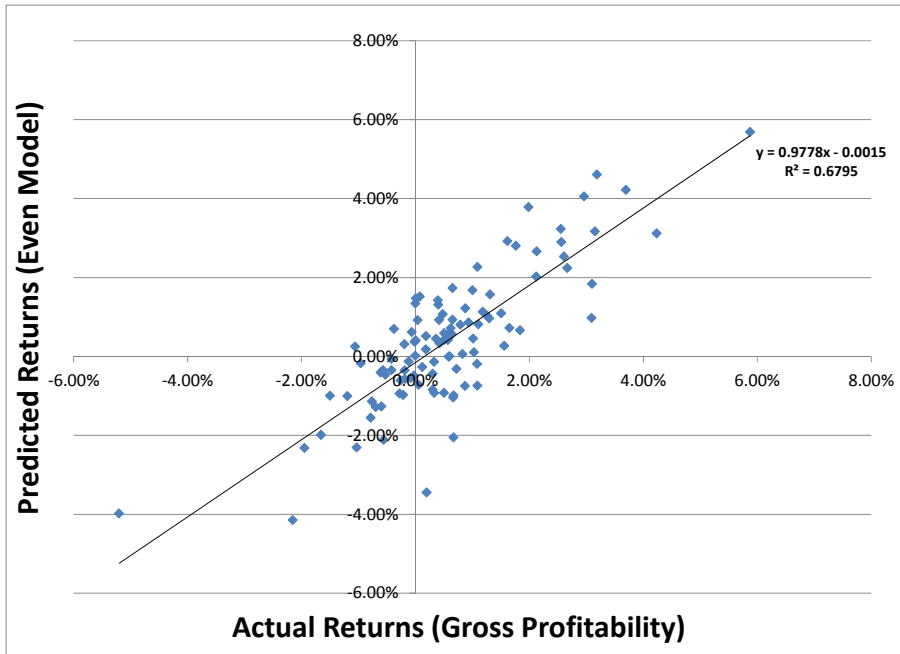


Figure 14.1. Scatterplot of (out-of-sample) odd-numbered month returns to the gross-profitability strategy against even model forecasts. July 1995–December 2012

We use the even model to forecast returns in odd-numbered months, and we compare those forecasts to the returns of the gross-profitability strategy out-of-sample in the odd-numbered months¹⁴. The best out-of-sample line describing the gross-profitability strategy in terms of the even model is shown in Figure 14.1. The line has a significant slope of 0.98 and an intercept that is insignificant. Out-of-sample in odd months, the even model explained the time variation in the gross-profitability strategy with an R-squared of 68%. The remaining 32% is not explained by the even model¹⁵.

Table 14.3 shows the return, risk and Sharpe ratios for gross profitability and its replication with the Barra USE4 model. We consider both the full model and out-of-

14 A disadvantage of this test is that it has look-ahead bias. However, to the extent that the strategy returns are independent, this is not an issue, and an advantage of this test is that it uses data from different economic climates in both the fit and the forecast.

15 This is consistent with a missing factor or a violation of the axioms of OLS regression, which was used to fit the model. A deeper inquiry is required to determine the source of the lost R-squared.

sample performance by the even model in odd months¹⁶. In both examples, the replication outperforms the gross-profitability strategy. Of course, it is often the case that a strategy will outperform its replication.

	Full Model		Even Model		S&P 500 Index
	Gross Profitability	Replication	Gross Profitability	Replication	
Arithmetic Annual Return	5.52%	5.52%	5.24%	7.25%	8.98%
Annual Standard Deviation	6.00%	4.99%	5.82%	4.93%	15.80%
Sharpe Ratio	0.43	0.52	0.40	0.87	0.37

Table 14.3. Performance statistics for the gross-profitability strategy and its replication using Barra USE4 style factors. Statistics for the even model are taken out-of-sample in odd months. July 1995–December 2012

14.5. Conclusion

Recent research has posited a connection between the accounting-ratio-based gross-profitability strategy and quality investing. We found that the Fama–French–Carhart four-factor model does not shed light on this assertion because it does not provide a satisfactory representation of the gross-profitability strategy during the period July 1995 to December 2012. However, during this period, a substantial portion of gross profitability can be explained by five style factors from the Barra USE4 model. Four of the five explanatory factors indicate a quality investment. However, the momentum tilt that was highlighted by our analysis is not part of the standard quality profile and may warrant further investigation. Importantly, these results rely on relatively short data histories¹⁷. More time is required to determine their efficacy in the long term.

14.6. Disclosure

The information contained within this chapter was carefully compiled from sources Aperio that believes to be reliable, but we cannot guarantee accuracy. We provide this information with the understanding that we are not engaged in rendering legal, accounting or tax services. In particular, none of the examples should be considered advice tailored to the needs of any specific investor. We recommend that all investors seek out the services of competent professionals in any of the aforementioned areas.

¹⁶ The out-of-sample results for the even model in odd months are not achievable through investment due to trading costs associated with liquidation at the end of each odd month and reinvestment at the start of each odd month.

¹⁷ Ideally, we would like to assess the factor replication of the gross-profitability strategy in as many different economic climates as possible.

With respect to the description of any investment strategies, simulations or investment recommendations, we cannot provide any assurances that they will perform as expected and as described in our materials. Every investment program has the potential for loss as well as gain.

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Deploying Multi-Factor Index Allocations in Institutional Portfolios

15.1. Introduction

Bender *et al.* [BEN 13] discussed six factors – *value, low size, low volatility, high yield, quality and momentum* – that historically have earned a premium over long periods, represent exposure to systematic sources of risk and have strong theoretical foundations. They also discussed how these factors could be captured through indexation. In this chapter, we turn to the question of how institutional investors interested in factor investing may allocate to and across factors.

In particular, we introduce a new framework for how institutional investors might consider implementing factor allocations through a passive mandate replicating a single multi-factor index. We call this type of allocation a multi-factor index allocation. Multi-factor indexes combine select factor indexes into single mixes created and controlled by the investor. Multi-factor indexes historically have demonstrated four key benefits: diversification, transparency, cost-efficiency via reduced turnover and flexibility.

Most importantly, regarding diversification, combining factors historically could have helped offset the cyclicity in single factor performance. When multiple factor indexes are combined into a single multi-factor index, diversification across factors has historically lead to:

- lower volatility and higher Sharpe ratio;

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- higher information ratios and lower tracking errors;
- less regime dependency over business cycles.

Next, we look at how factor allocations fit in the traditional institutional portfolio setting. Factor allocations have the potential to change the landscape of mandate structures by offering a new way to achieve exposure to systematic factors that formerly could only be captured through active mandates. Factor index-based investing can be viewed as active decisions implemented through passive replication. As such, factor allocations should be tailored to each institution.

The first step is to assess the role of factor investing in the institution's portfolio. The two main dimensions that drive factor investing are the institution's objectives and constraints (governance structure, horizon, risk budget, etc.). For example, those seeking to enhance risk-adjusted returns may be looking for a dynamic allocation (higher return and higher risk), a defensive allocation (moderate return and lower risk) or a balanced allocation (something in between).

Once the institution has established its investment objectives and identified factors that might contribute to these objectives, it must also decide how to structure and implement the factor allocation. The main criteria for deciding which combination of indexes to deploy depend on the institution's assessment of the trade-off between investability and factor exposure (which is tied to performance). Indexes with greater investability generally have lower factor exposure and vice versa. In this implementation phase, there can also be significant turnover reduction benefits to combining multiple factors in a multi-factor index. In particular, "natural crossing" effects may reduce turnover, provided that the allocation is structured around a single passive mandate (or multiple mandates structured to replicate passively the same index) with synchronized rebalancing dates. Since there are different index alternatives with varying levels of exposure versus investability, the appropriate index implementation depends on the institution's objectives and constraints.

15.2. Implementing factors through multi-factor index allocations

15.2.1. Multi-factor indexes: a new approach for institutional mandates

Bender *et al.* [BEN 13] discussed why some institutional investors seek exposure to systematic factors and introduced the notion of factor indexes that represent factor returns. They focused on six factors (value, low size, low volatility, high yield, quality and momentum) that historically have earned a premium over long periods and have strong theoretical foundations. In this chapter, we now discuss a new framework for how institutional investors might consider implementing factor allocations through a passive mandate replicating a single multi-factor index. We

call this type of allocation a multi-factor index allocation. Multi-factor indexes combine select factor indexes into single mixes created and controlled by the investor.

Traditionally, institutional investors structured their allocations around two main sources of return:

– *(Passive) Beta*: based on modern portfolio theory, beta is the return the institution gets from broad exposure to the market, or the full equity investment opportunity set. It is achievable through a portfolio that passively tracks the market, represented typically by a market capitalization-weighted index. For instance, in a global portfolio, global equity beta is represented by a broad market capitalization-weighted index such as the MSCI ACWI Investable Market Index (IMI).

– *(Active) Alpha*: alpha is the additional return that active management can provide. It is excess return (or value-added) over the market capitalization-weighted index. Traditionally, active managers have sought to identify and capture two types of alpha: market inefficiencies and systematic factors associated with excess risk-adjusted returns.

Factor allocations have the potential to change the landscape of mandate structures by offering a new way to achieve exposure to systematic factors that formerly could only be captured through active mandates. Figure 15.1 shows how we can view these allocations as part of a new category in between traditional passive mandates, which replicate market cap-weighted portfolios, and active mandates. Factor index-based investing can be viewed as active decisions implemented through passive replication¹.

Multi-factor index allocations offer a new approach for institutional investors to seek factor returns². Their four key potential benefits are³:

– *Flexibility*: institutions have full control over the selection and the weights of individual factor indexes within a multi-factor index and can adjust the strategic

¹ Note that in Figure 15.1, and throughout the chapter, we generally refer to factor index allocations through a multi-factor index but a factor index allocation could also consist of only one single-factor index. In this case, the benefits of indexation (transparency and simplicity) would apply but not the diversification and natural crossing effects.

² Historically, active managers would have provided institutions with exposure to multiple factors. For instance, quantitative active funds can use optimizers to create portfolios with targeted factor exposures. But, there are significant potential benefits to an index-based approach (transparency, cost-effectiveness and flexibility).

³ Note that the benefits of “Transparency” and “Cost Efficiency” would be potentially applicable for single-factor index allocations as well.

factor allocation dynamically through time. The most appropriate combination of individual factor indexes can be customized to account for institutional constraints (e.g. environmental, social, governance (ESG) policies, plan rules, etc.). Operationally, the multi-factor approach provides flexibility as it can be created and managed easily within the passive mandate and without having to change the structure or the terms of the mandate. Because the multi-factor allocation relies on standardized indexes, it allows for the flexibility of employing existing passive instruments such as exchange traded funds (ETFs) for tactical overlays. We view this as a “building block” approach.

– *Transparency*: multi-factor index allocations provide full transparency regarding the strategy’s underlying building blocks. They allow for easy and consistent analysis not only of the aggregate positions, exposures and risks of the portfolio but also of the individual indexes, all with the same level of granularity.

– *Cost efficiency*: because multi-factor indexes can be replicated passively, multi-factor index allocations can provide a potentially cost-effective alternative to active funds. Moreover, blending multiple factor indexes in a multi-factor index may create natural crossing opportunities, which can reduce turnover and hence potentially reduce transaction costs at rebalancing.

– *Diversification*: factor returns have been highly cyclical historically, with sensitivity to macroeconomic and market forces. They also have underperformed the overall market for long periods. However, they do not all react to the same drivers and, hence, can have low correlations between each other. Consequently, multi-factor index allocations historically have demonstrated similar premiums over the long run to the individual factors but with milder fluctuations.

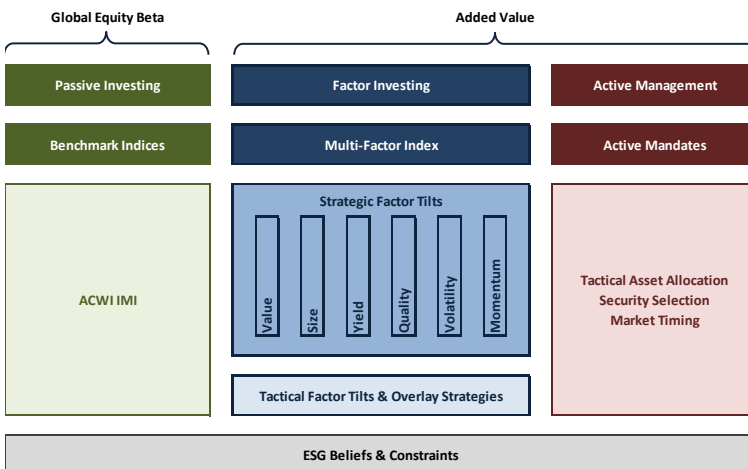


Figure 15.1. Factor allocations within institutional mandates

In Figure 15.1, the category “Factor Investing” contains both “Strategic Factor Tilts” and “Tactical Factor Tilts and Overlay Strategies”. The former refers to strategic static tilts deployed as a long-term strategy, while the latter refers to dynamic allocations in which investors overweight/underweight factor allocations based on their forward-looking expectations.

Also in Figure 15.1, “Pure Alpha” can still be provided by active management, which comprises value-adding activities that are not captured by passive factor allocations to indexes. “Pure Alpha” includes stock selection and sector rotation strategies, as well as top-down asset allocation strategies where factor tilts are not driving excess return.

15.2.2. Deploying factor allocations

Many institutions have struggled to determine the appropriateness of factors for their own plan, what role these allocations might play, which factors should be adopted and how factor indexes can be used⁴.

There are generally three main parts to the process for an institution deploying factor allocations:

- assess the institution’s objectives and constraints;
- select candidate factors;
- decide how to structure the implementation.

In this framework, the institution must first assess the role of factor investing and what it hopes to achieve. This includes setting the investment objectives, assessing the internal governance structure and establishing key constraints such as the risk tolerance. Once the role of factor investing has been established, the institution can then evaluate candidate factors. As discussed in “Foundations of Factor Investing”, certain factors have strong theoretical foundations and have earned a persistent premium over long periods. The institution must form a belief about whether a factor’s long-term historical premium will persist as part of this step.

The third part of the deployment decision process in this framework is the implementation. Among the available options for implementation (including via active managers), we focus here on passive implementation based on indexes. Figure 15.2 illustrates the process for an institution to evaluate its objectives, the

⁴ To add to the difficulty, there has been a rapid proliferation of factor indexes and investment products. Even the breadth of names alone – factor indexes, strategy indexes, smart beta, alternative beta, to name just a few – have challenged even the most sophisticated investors.

relevant candidate factors and the implementation structure. The plan's objectives and constraints inform the combination of the factors chosen and the degree of investability required in the factor allocation. For instance, very large allocations may not be capable of implementation for certain highly concentrated or long short strategies.

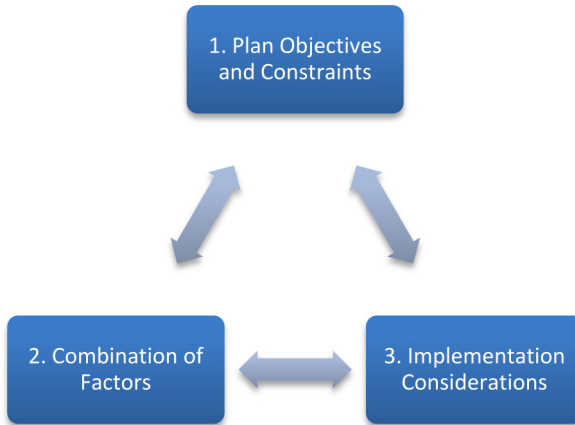


Figure 15.2. *Dimensions for implementing multi-factor index allocations*

Note that the institution's objectives and constraints drive the factor allocation decision, not the indexes themselves, a point that is often lost in the arguments about why one index might be superior to another. Simply focusing on a particular index's rules and construction process leads to the slippery slope of data-mining. There are thousands of options for generating indexes by varying the weights or criteria for selecting stocks. Any given set of index construction rules can lead to outperformance of the market through statistical sampling alone⁵.

Before evaluating any factor indexes, the institution should identify its goals for factor investing and evaluate potential candidate factors based on criteria that follow from their objectives and constraints. Choosing a factor index is an implementation decision that turns the objectives, goals and factor beliefs into actual allocations. Next, in section 15.3, we discuss how an institution's objectives and constraints motivate the appropriate choice and blend of factors and factor indexes. In

⁵ In fact, a recent paper by Arnott *et al.* [ARN 13] argues that any non-price-weighted portfolio will outperform a cap-weighted portfolio because of size and value effects. In our framework, we start with the pure factors first – value and size – and choose the most appropriate index based on key metrics such as factor exposure, investability, tracking error and concentration.

section 15.4, we look at investability, which is a key in determining how to structure the factor index allocation. There are different index alternatives with varying levels of exposure versus investability. Therefore, the appropriate index implementation depends on how the institution prioritizes exposure versus investability, which in turn is based on the institution's objectives and constraints.

15.3. Selecting the right blend of factors

As an institution seeks the right blend of factors, the starting point is the institution's own profile. Factor allocations should be driven first and foremost by the institution's investment objectives and constraints (governance structure, horizon, risk budget, etc.):

– *Objectives*: different investors have different objectives for factor investing, or said another way, different problems for which factor investing is meant to address. One institution may seek to enhance risk-adjusted returns, limit downside risk or improve returns by holding the current level (or market level) of risk or beta constant. Another institution might be trying to replicate the performance of certain style managers, for instance, existing value and small cap managers. Different investors will also have different beliefs regarding the persistence of factors.

– *Constraints*: constraints can also vary among investors. Key constraints are associated with the institution's governance structure which is tied to its investment horizon and risk tolerance. Often, the stronger the governance structure an institution has, the longer the horizon and the higher the risk tolerance it has. Institutions with very strong governance structures and long horizons are better able to withstand long periods of underperformance, and perhaps be compensated for bearing this risk. Funding ratios and the size of assets managed can also affect investability constraints.

Before selecting factors, the institution should begin by screening out any candidate factors which it does not expect to persist in the future. In other words, *all candidate factors should be those the institution believe will persist in the future*. Thus, the *institution's objectives and constraints together drive the choice of factors* among these candidates. For example, an institution seeking to enhance risk-adjusted returns may be looking for a somewhat more aggressive allocation (higher return and higher risk), a defensive allocation (moderate return and lower risk) or a balanced allocation (something in between).

Figure 15.3 shows the historical return and risk characteristics (June 1988–June 2013) of seven MSCI Factor Indexes capturing “risk premia” factors introduced in “Foundations of Factor Investing.” These are factors that have earned a premium over long periods and which have solid theoretical foundations (factor indexes based

on the MSCI World Index are shown). The low volatility factor, represented by the MSCI World Minimum Volatility and World Risk Weighted Indexes, and the quality factor, represented by the MSCI World Quality Index, both have lower risk than the MSCI World Index. The value and yield factors represented by the MSCI World Value Weighted and MSCI High Dividend Yield Indexes had risk levels close to the market. The low size factor and momentum factors, represented by the MSCI World Equal Weighted, and MSCI Momentum Indexes, respectively, have had higher returns. All seven indexes have historically shown higher Sharpe ratios than the MSCI World Index. Determining the appropriate factors to allocate to might depend on the institution's return, risk or Sharpe ratio objectives.

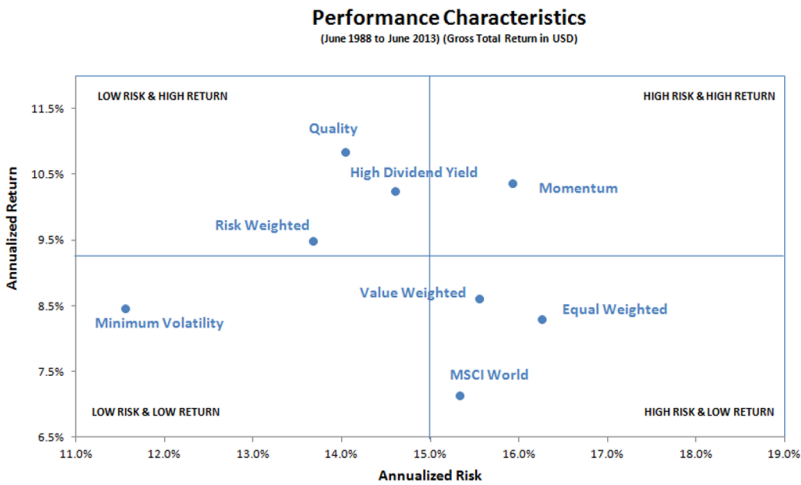


Figure 15.3. Factors have historically exhibited different performance characteristics

15.3.1. Correlations matter when selecting factors: the diversification effects of multi-factor index allocations

Factor selection should also take into account the correlations between factors, which affect portfolio-level risk. Factor returns have historically been highly cyclical. Figure 15.4 shows the cumulative returns relative to the market cap-weighted index (MSCI World Index). Each of the factor indexes shown has undergone at a minimum 2–3 consecutive year periods of underperformance. Some factors historically underwent even longer periods; the small cap or low size factor (captured by the MSCI World Equal Weighted Index in the figure) went through a 6-year period of underperformance in the 1990s.

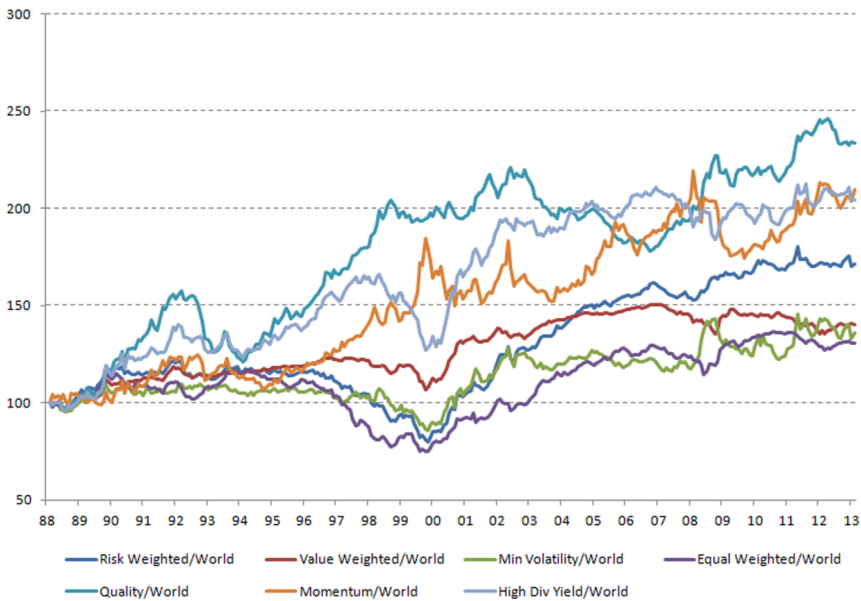


Figure 15.4. All systematic factors are cyclical (cumulative relative returns, June 1988–June 2013). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

But while individual factor returns have all been cyclical, their periods of underperformance have not been identical. Systematic factors have historically been sensitive to macroeconomic and market forces but not in the same way. For instance, during the period between 2001 and 2007, the momentum, value, low volatility and low size factors experienced positive excess returns over the market, but the quality factor experienced negative returns. In contrast, from 2007 onwards, quality fared well while momentum and value did not. Combining quality with momentum and value, for instance, historically achieved smoother returns over time and diversified across multi-year cycles.

There is also strong empirical evidence that factors performed differently over various parts of the business cycle. Some factors such as value, momentum and size have historically been pro-cyclical, performing well when economy growth, inflation and interest rates are rising. Other factors such as quality and low volatility have historically been defensive, performing well when the macroenvironment was falling or weak. Similar to macrobusiness cycles, investors may seek factors that perform well under different types of market cycles such as high/low market

volatility. Measuring the sensitivity of factors to macroeconomic cycles is an area that still requires further research. For recent research in this area, see [WIN 13].

The historical diversification effects can further be seen in the correlations between monthly active returns shown in Table 15.1. Notably, the active returns of the MSCI World Quality and MSCI Momentum Indexes have been very low or negatively correlated with the other factor indexes shown. However, the majority of the correlations range from about 0.30 to -0.30.

	MSCI World Risk Weighted	MSCI World Value Weighted	MSCI World Minimum Volatility	MSCI World Equal Weighted	MSCI World Quality	MSCI World Momentum	MSCI World High Div. Yield
MSCI World Risk Weighted	1.00						
MSCI World Value Weighted	0.61	1.00					
MSCI World Minimum Volatility	0.65	0.14	1.00				
MSCI World Equal Weighted	0.75	0.63	0.12	1.00			
MSCI World Quality	0.07	0.00	0.24	-0.26	1.00		
MSCI World Momentum	0.04	-0.26	0.16	-0.20	0.38	1.00	
MSCI World High Div. Yield	0.62	0.71	0.51	0.26	0.35	0.04	1.00

Table 15.1. Correlations of relative monthly returns (June 1988–June 2013, USD Gross Returns)

When multiple factor indexes are combined into a single multi-factor index, diversification across factors has historically lead to:

- lower volatility and higher Sharpe ratio;
- higher information ratios and lower tracking errors;
- less regime dependency over business cycles.

For illustration, Table 15.2 shows a multi-factor index where four individual indexes are combined: the MSCI World Quality Index, MSCI World Value Weighted Index, MSCI World Momentum Index and MSCI World Risk-Weighted Index⁶. While the returns are a linear combination of the individual indexes, the risk metrics are not. The high information ratio of 0.83, substantially higher than the four individual indexes, reflects how well they diversified each other during this period.

⁶ The multiple-index combination is rebalanced semi-annually at the same time as the underlying indexes in May and November.

	World Standard	MSCI World Quality Index	MSCI World Risk Weighted Index	MSCI World Value Weighted Index	MSCI World Momentum Index	Multi Factor Index
Total Return* (%)	4.2	5.3	8.6	5.5	6.9	6.7
Total Risk* (%)	16.3	14.3	14.6	17.2	16.7	14.9
Sharpe Ratio	0.18	0.26	0.47	0.25	0.33	0.34
Annualized Active Return (bps)		110	440	120	270	250
Tracking Error* (%)		4.5	5.6	3.6	9.0	3.0
Information Ratio		0.25	0.79	0.35	0.30	0.83
Max Rel. Drawdown (Active Returns) (%)		20.5	16.0	10.7	21.6	5.7
Max Rel. Drawdown Period (Active Returns) (in Months)		52	10	9	19	2

* Annualized in USD for the 05/31/1999 to 09/30/2013 period

** Annualized one-way index turnover for the 05/31/1999 to 09/30/2013 period

Table 15.2. Combining multiple factors offers substantial diversification effects (May 1999–September 2013)

Table 15.2 also includes maximum drawdown, relative to the MSCI World Index, and the maximum relative drawdown period in months. Both measures capture prolonged periods of underperformance. This measure of risk is equally if not more important than traditional measures of risk like standard deviation of returns because it arguably captures “career risk.” Even for institutions with long stated horizons, the investment staff will often be forced to reassess allocations if the portfolio underperforms for too long. The multi-factor index historically has significantly lower drawdown measures than the individual indexes.

In sum, historically there are important diversification effects in combining multiple factors. Multi-factor indexes achieved the same historical premium over the long run as the individual factors but with milder fluctuations. Actual use cases include a Canadian pension plan which adopted a combination of MSCI Risk Weighted, MSCI Quality, MSCI Value-Weighted Indexes and a US pension plan which chose a combination of MSCI High-Dividend Yield, MSCI Quality and MSCI Value-Weighted Indexes. These and other use cases presented later in section 15.4 further illustrate the benefits of multi-factor indexes.

15.3.2. Considerations for combining factor indexes

Tying all this together, we arrive at the main considerations for selecting the right blend of factors. It starts with the institution’s objectives and constraints, its beliefs regarding which factors are likely to persist, and in some cases, return expectations for the factors. When choosing an appropriate factor combination, the key criteria are risk, correlations with other factors and performance in different business cycles, as shown in Table 15.3.

Factor	Historical risk	Historical correlation	Historical business cycle
Value	Comparable to market	Low with momentum and quality	Pro-cyclical
Momentum	Comparable to market	Low with value, yield and quality	Pro-cyclical
Low size	Higher than market	Low with min volatility, yield and quality	Pro-cyclical
Quality	Lower than market	Low with value, size, yield and momentum	Defensive
Low volatility	Lower than market	Low with value and momentum	Defensive
Yield	Lower than market	Low with size, quality and momentum	Defensive

Table 15.3. *Considerations for combining factor indexes*

Other criteria that can affect factor selection include sources of return as well as return patterns. For example, an institution may prefer income to capital appreciation or prefer factors which imply higher yields. In addition, an institution may be particularly sensitive to the possibility of a prolonged drawdown and seek factors that are less likely to go through multi-year periods of underperformance or, as illustrated earlier in Table 15.2, blends of factors that minimize prolonged underperformance. Thus, the criteria for choosing factors and combinations of factors could include a variety of characteristics such as *return (including forward-looking expectations)*, *risk*, *Sharpe ratio*, *diversification effects*, *yield levels*, *beta*, *general liquidity characteristics*, *downside risk* and *risk of prolonged periods of underperformance*.

Table 15.4 provides examples of how factor allocations can be tailored by the institution to its objectives⁷.

In sum, there is no universal factor solution, either in the form of a single factor or a combination of factors, which is right for all institutions. Actual use cases are helpful in understanding different types of allocations. Several examples are shown in section 15.4.

⁷ Some institutions may not have explicit performance goals. Rather, they may be seeking ways to make explicit the tilts that the plan's active managers already take.

Sample objective	Example allocation (pure factors)	Example index allocation
Diversified balanced mix	– Value, low volatility, momentum and quality	– MSCI Multi-Factor Index: value weighted, risk weighted, momentum and quality
Diversified dynamic mix	– Low size, momentum and value	– MSCI Multi-Factor Index: equal weighted, momentum and value weighted
Diversified defensive mix	– Low volatility, value and quality	– MSCI Multi-Factor Index (MSCI Quality Mix): minimum volatility, value and quality
De-risking with yield-enhancement	– Low volatility and high dividend yield	– MSCI Multi-Factor Index: minimum volatility and high dividend yield

Table 15.4. *Factor allocations are based on the institution's objectives and constraints*

15.4. Implementation considerations

In this section, we discuss in more detail critical aspects of implementation. We focus, in particular, on a potential framework for how to incorporate the investability dimension in the selection of the individual factor indexes. We also discuss how factor indexes can be combined in a multi-factor index to reduce trading cost by benefitting from potential natural crossing. This last element requires the allocation to be structured around a single passive mandate (or multiple mandates structured to replicate passively the same index) with synchronized rebalancing dates.

15.4.1. Understanding the exposure versus investability trade-off

In selecting the individual factor indexes that make up a multi-factor index or in selecting a single-factor index, the most critical point we stress here is that there is a tradeoff between the strength of the exposure to a factor and the investability of the strategy that reflects it. There is a range of index alternatives that have varying levels

of investability and exposure to a specific factor. Figure 15.5 shows a general framework which visually displays the different index options.

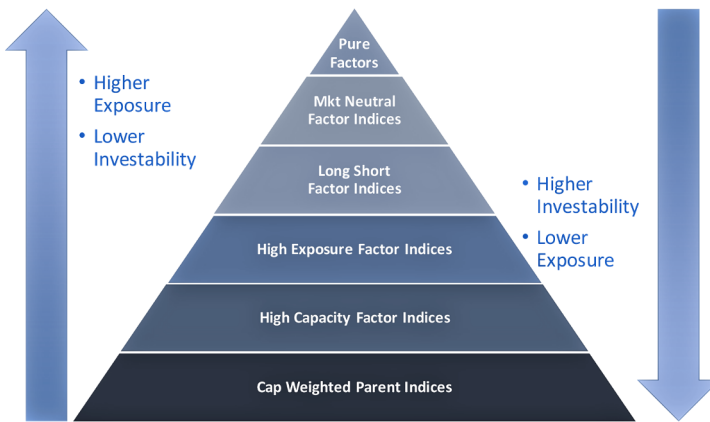


Figure 15.5. *Capturing factors through indexation*

The most investable index, by definition, is the one whose weights are proportional to free float-adjusted market capitalization, the bottom part of the pyramid. The factors at the top (e.g. the Fama–French or Barra factors) are the theoretical or pure factors that the institution may wish to capture, but that are research rather than investability oriented. The closest factor indexes to market capitalization-weighted indexes are high capacity factor indexes. These are indexes that hold all the stocks in the parent index but tilt the market cap weights toward the desired factor. As we move up, high exposure indexes hold a subset of names in the parent index and can employ more aggressive weighting mechanisms. The investor who seeks to control active country or industry weights or exposures to other style factors, or who desires to limit turnover, tracking error or concentration, can use high exposure indexes that employ optimization or systematic stock screening. Next, long/short factor indexes add leverage (e.g. 150/50 and 130/30) primarily to hedge out residual exposure to other factors, and finally market-neutral factor indexes are pure long/short indexes that have zero market exposure⁸. The leveraged index categories typically employ optimization.

Moving up the pyramid yields lower investability and greater exposure to the pure factor.

⁸ Active country and sector weights will be zero and exposures to all other style factors will be zero.

15.4.1.1. Factor exposure

What do we mean by “factor exposure”? Factor exposure captures the degree to which the index captures the pure non-investable factor. To assess the strength of the factor exposure of a particular index, we can use a factor model (which can be used to calculate any portfolio or index’s exposure to the factors in that model). Factor exposure is typically expressed as standard deviations away from the cap-weighted average of the market⁹. Note that for most factor models, which typically employ linear exposures and regressions, the exposure of an index to an underlying factor is just the weighted exposure of the individual stocks’ exposures to the factor in question. (Factor exposure is also often called signal strength in the language of quantitative equity managers.)

As we move up the pyramid, typically higher levels of factor exposure are achieved which translates into higher returns if factor returns scale with exposure and as long as incidental bets are controlled for¹⁰. This last point is important because more concentrated portfolios often have larger sector and country active weights, or even unintended exposures to factors other than the factor of interest. If these are not controlled, they can incidentally negatively affect returns, detracting from the intended factor return.

In Figure 15.6, we illustrate factor exposures using the Barra multi-factor models which estimate factor portfolios using multivariate regressions and have the advantage of specifying factors with little co-linearity. As an example, Figure 15.6 shows the active exposures (relative to the MSCI World Index) of four of the factor indexes. In all cases, the indexes have significant exposure, with the expected sign, to the most relevant pure factors. The MSCI World Value-Weighted Index has an exposure of 0.28 to the Barra GEM2 Value Factor, which is above the usual 0.20–0.25 rule of thumb for statistically significant exposures. In some cases, an index may have significant exposure to factors other than the intended factor. For instance, the MSCI World Risk-Weighted Index has a significant small cap bias as seen by the large negative exposure to the GEM2 Size factor. In this case, the small cap bias contributes to the excess return of the World Risk-Weighted Index. Institutional investors should be aware of these potential secondary exposures and understand/manage them appropriately.

9 An active/relative exposure of 0.25 to the Barra value factor can be interpreted as the portfolio or stock’s value characteristics being 0.25 standard deviations higher than the market cap-weighted benchmark.

10 We can have higher exposure to the desired factor but the positive impact on returns may be negated by other exposures (either to other factors or countries or sectors). Controlling exposures to other factors is possible through optimization. For example, in a value factor index, e might want to neutralize exposures to other factors such as low size and momentum.

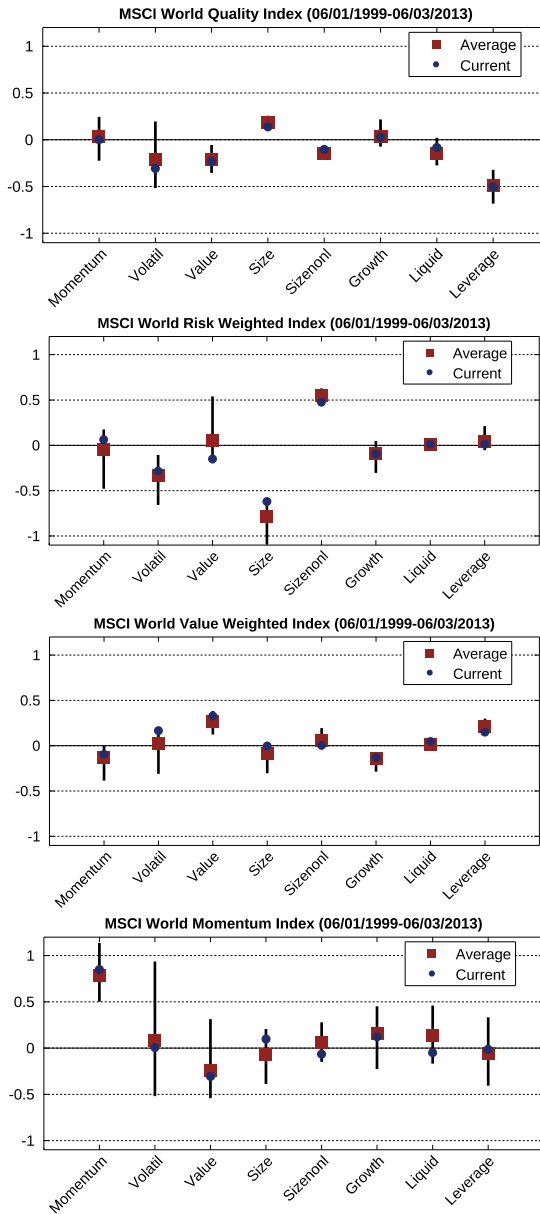


Figure 15.6. Factor exposures (factor exposures for Select World Factor Indexes using the Barra GEM2 Model, Average and Current Exposures, June 1999–June 2013)

15.4.1.2. Investability

What do we mean by “investability”? Investability refers to how liquid and tradable the index is. It also refers to how scalable the allocation to an index replicating vehicle might be. There are multiple dimensions to investability. As shown in Figure 15.7, they include tradability/liquidity, turnover/cost of replication and capacity – for a given degree of active tilt¹¹. Tradability/liquidity quantifies how liquid the stocks are in the index replicating portfolio and how tradable the portfolio is. Metrics include days to trade individual stocks at rebalancings and during the initial construction, and days to trade a certain portion of the portfolio (given a certain size portfolio and a set limit to the amount traded on a single day). Turnover/cost of replication measures the turnover of the index at rebalancing which scales with trading costs. The higher the turnover, generally the higher the cost of trading. Capacity quantifies (for a given size portfolio) the percentage of a stock’s free float or full market capitalization the portfolio would own. The degree to which a portfolio is “active” relative to the index has been traditionally used by many active asset managers to characterize their active strategies’ performance. Metrics like active share and maximum strategy weight capture this.

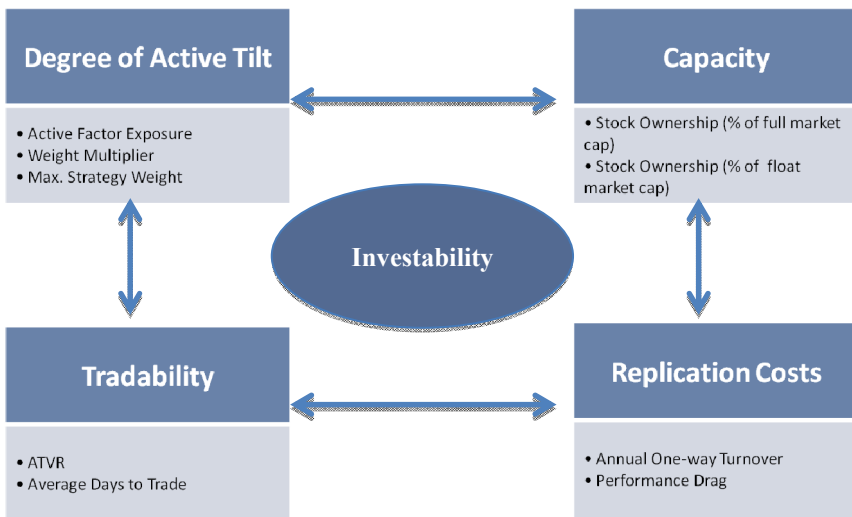


Figure 15.7. Dimensions for investability

Note that some indexes may score well on all four dimensions; the MSCI Value-Weighted Indexes, for instance, historically have exhibited low turnover, high

¹¹ These dimensions were first discussed in [BAM 13].

capacity and good tradability. Others may have good capacity and tradability but incur high turnover (e.g. momentum).

15.4.1.3. *Investability versus exposure*

Since, as we have seen, indexes nearer the top of the pyramid are less investable and less liquid but have greater factor exposure, there is an unavoidable tradeoff between the purity or exposure of a factor index and the investability of a factor index. We can generally only achieve purer factor exposure by sacrificing investability and being willing to take on greater amounts of active risk and complexity. The appropriate index thus depends on the institutional investor's own preferences for factor exposure and investability. Institutions must make a self-assessment of where they desire to be on the pyramid.

It is also important to note that institutions typically care about tracking error, or risk relative to the market cap-weighted parent index. In particular, many plans have active risk budgets at the plan level¹². As we move up the pyramid, tracking error generally increases. Plans with low tracking error targets may want to limit the discussion to the lower end of the pyramid, while those with higher tracking error limits may consider options further up the pyramid.

Table 15.5 shows the characteristics of the MSCI Factor Indexes over the period June 1988–June 2013. Higher capacity indexes typically hold a broad set of names (e.g. all the names in the broad market parent index) and are weighted with investability in mind. As previously discussed in [BAM 13], the MSCI Value-Weighted Indexes effectively employ a weighting scheme that combines a score based on value characteristics and market capitalization, and are an example of a high capacity index.

As illustrated in Table 15.5, the MSCI World Value-Weighted Index has the lowest active risk (tracking error) and very low turnover among the indexes shown. Other “Weighted” indexes (all of which hold all the names in the parent index) also exhibit relatively low tracking errors and turnover. The other indexes (the MSCI Momentum Indexes, MSCI Quality Indexes and MSCI Minimum Volatility Indexes) are more concentrated indexes, holding only a subset of the names in the parent index. These indexes exhibited higher tracking errors and lower levels of investability. (The MSCI Minimum Volatility Indexes are turnover constrained to 20% but other measures of investability are more similar to the MSCI Momentum and MSCI Quality Indexes.)

¹² Many institutional investors have a maximum (or target) level of desired risk, usually in the form of return standard deviation, but sometimes gauged by downside measures such as maximum drawdown or expected shortfall.

Index	Factor exposures*	Total return	Total risk	Active return	Active risk	Annual turnover	Pairwise correlation
MSCI World	–	7.1	15.4	0.0	0.0	3.9	NA
MSCI World Equal Weighted	Size	8.3	16.3	1.2	5.2	31.8	0.22
MSCI World Minimum Volatility	Volatility	8.5	11.6	1.4	6.7	20.0	0.30
MSCI World Value Weighted	Value	8.6	15.6	1.5	3.6	20.3	0.30
MSCI World Risk Weighted	Size, volatility	9.5	13.7	2.4	5.3	27.2	0.46
MSCI World Quality	Growth, leverage	10.9	14.0	3.8	5.9	27.6	0.13
MSCI World Momentum	Momentum	10.4	15.9	3.3	8.5	127.5	0.03
MSCI World HDY	–	10.3	14.6	3.2	6.5	22.0	0.41

* In the column “Factor Exposures”, we show the Barra Global Equity Model (GEM2) factors which are statistically significant on average ($>+/- 0.20$), with the expected sign, since December 1997. Note that there is no “Yield factor” in the GEM2 Model. Instead, yield is a component (with a weight of 10%) in the GEM2 Value factor. Turnover reported is the average annual one-way turnover based on history from June 1988 to June 2013.

Table 15.5. MSCI World Factor Indexes (main characteristics, June 1988–June 2013)

15.4.2. Reducing trading costs by leveraging the benefits of natural crossing

In addition to the investability dimensions we have discussed so far, investors should also consider the potential to reduce trading costs at each rebalancing through operational efficiency. As we discussed in section 15.3, historically there have been significant diversification effects when combining multiple factors. In the implementation phase, there can also be significant investability benefits to combining multiple factors in a multi-factor index.

Combining factor indexes may reduce turnover from “natural crossing” effects. On the index rebalancing dates, the composite index would be rebalanced back to its target weights (e.g. 50/50) and turnover may be reduced as a company deleted from

one factor index might be added as a constituent of another factor index. Take, for example, a stock whose price is falling over time. As the price falls, it may drop out of a momentum index but the lower price could push the stock into a value index. Those shares which overlap the two indexes would be internally crossed. This “natural crossing” leads to lower index turnover and by implication, lower transaction costs in a portfolio replicating the index.

The historical effects of natural crossing are shown in Table 15.6. In this example, we show a blend of the MSCI World Quality Index, MSCI World Risk-Weighted Index, MSCI World Value-Weighted Index and MSCI World Momentum Index. The four indexes are equally weighted and rebalanced semi-annually at the same time the underlying indexes are reconstituted. (Note that the rebalancing for the individual indexes and the rebalancing across indexes needs to be synchronized for the natural crossing to take place). The annual turnover for the individual indexes is 22.98, 22.04, 18.30 and 89.62%, respectively. If these four indexes were replicated separately, their combined turnover would be 40.81%. When they are replicated as a single portfolio in a single mandate, the combined turnover is significantly lower at 31.91%. The turnover declines by 8.9% points. What does this mean in terms of trading costs in index replicating portfolios? If trading costs are 50 basis points (a relatively conservative assumption for global developed market equities), the round-trip trading costs would be 41 basis points for the separately managed portfolio, and 32 basis points for the combined multi-factor index-based portfolio. The latter option saves the investor close to 9 bps in transaction costs.

	MSCI World Quality Index	MSCI World Risk Weighted Index	MSCI World Value Weighted Index	MSCI World Momentum Index	Separate Mandates (A)	Combined Mandates (B)	Reduction (A) - (B)
Turnover(%)	22.98	22.04	18.30	89.62	40.81	31.91	8.90
Performance Drag in bps (at 25 bps)*	11.49	11.02	9.15	44.81	20.40	15.95	4.45
Performance Drag in bps (at 50 bps)*	22.98	22.04	18.30	89.62	40.81	31.91	8.90
Performance Drag in bps (at 75 bps)*	34.47	33.06	27.45	134.42	61.21	47.86	13.35

Annualized for the 05/31/1999 to 9/30/2013 period

* Performance drag aims to represent the total two-way annualized index level transaction cost assuming various levels of security level transaction cost

Table 15.6. *Crossing benefits resulted in lower turnover and lower trading costs (simulated turnover of separate and combined equally weighted allocations to select MSCI Factor Indexes)*

Our conclusion here is that these natural crossing effects may often be overlooked and deserve consideration given the potential additional savings.

15.5. Multi-factor index allocations: examples

The right blend of factors will depend on the institution's preferences for various aspects of performance (return, risk, correlations, etc.), investability and factor exposure, which in turn reflects the institution's objectives and constraints.

Actual use cases can be helpful in understanding how institutions have actually addressed these issues in adopting multi-factor index combinations.

Based on real use cases, in the first example, we show a strategic or long-term static allocation that is designed to be well diversified. Factor indexes in this example are the MSCI Value, MSCI Momentum, MSCI Risk Weighted and MSCI Quality Indexes. The four factors are implemented as a single composite multi-factor index that is rebalanced semi-annually. The index allocation is executed as a passive internal mandate.

The second example focuses on an allocation that provides lower absolute volatility with higher yield. The desire to “de-risk” is driven by the institution's projections of a bearish, low growth market. At the same time, the institution seeks to achieve higher yields while de-risking. This allocation is implemented as a passive external mandate on a multi-factor index combining low volatility via the MSCI Minimum Volatility Indexes and yield via the MSCI High-Dividend Yield Indexes.

One additional use case is an extension of the second example. The “core portfolio” in the second use case (MSCI Minimum Volatility and MSCI High-Dividend Yield Indexes) can also be augmented by tactical factor allocations to factor indexes such as MSCI Momentum, MSCI Quality, MSCI Value Weighted and MSCI Equal Weighted Indexes. These exposures could be adjusted over time based on forward-looking views and deployed via four exchange-traded funds tracking the MSCI indexes. An external consultant or active manager could play a role in advising on the tactical overlay decision.

How do institutions in practice arrive at allocations like these? Institutions must evaluate a number of key dimensions that we have already discussed in this chapter – performance (risk, returns, etc.), factor exposure, investability and the effects of combining multiple indexes. Table 15.7 summarizes a few of the key dimensions that might help the institution form a view on different combinations.

In order to help institutions understand whether their objectives are met given various combinations of factor indexes, MSCI has developed Index Metrics, a structured framework for the analysis of multi-factor blends. Kassam *et al.* [KAS 13] describe this framework in greater detail.

Performance	Exposure	Investability	Combination
<ul style="list-style-type: none"> – Total Returns, Total Risk, Sharpe Ratio – Active Returns, Tracking Error, Information Ratio – VaR, Expected Shortfall – Maximum Drawdown (percentage, length) – Relative Maximum Drawdown – Years of Consecutive Underperformance 	<ul style="list-style-type: none"> – Active Factor, Sector, Region Exposures – Relative Valuation and Fundamental Ratios 	<ul style="list-style-type: none"> – Liquidity – Cost of Replication (Turnover) – Capacity – Concentration 	<ul style="list-style-type: none"> – Diversification of Returns – Turnover Reduction

Table 15.7. Key metrics for evaluating different combinations of factor indexes in structuring a multi-factor index allocation

15.5.1. Example #1: Strategic long-term risk-adjusted return

Strategic allocation:

- MSCI World Value Weighted Index 25%;
- MSCI WorldRisk Weighted Index 25%;
- MSCI World Quality Index 25%;
- MSCI World Momentum Index 25%.

As shown in Figure 15.8, the four indexes exhibited significantly different returns over various sub periods. They historically provided high levels of diversification.

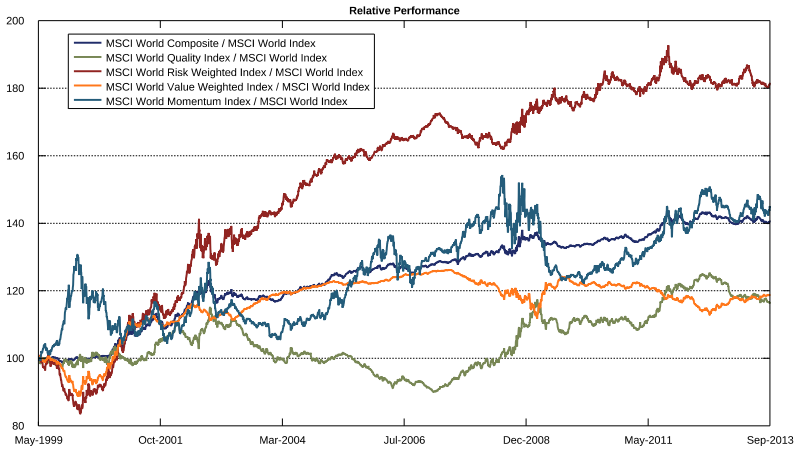


Figure 15.8. Performance using historical returns (May 1999–September 2013). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

The result of combining the four indexes is a balanced portfolio which exhibited return enhancement at lower risk levels than the market historically.

Historical Gross Total Return, USD	MSCI World	MSCI World Quality Index	MSCI World Risk Weighted Index	MSCI World Value Weighted Index	MSCI World Momentum Index	Combined
Total Return Performance						
Total Return* (%)	4.2	5.3	8.6	5.5	6.9	6.7
Total Risk* (%)	16.3	14.3	14.6	17.2	16.7	14.9
Return/Risk	0.26	0.37	0.59	0.32	0.41	0.45
Sharpe Ratio	0.18	0.26	0.47	0.25	0.33	0.34
Active Return Performance						
Active Return* (%)	0.0	1.1	4.4	1.2	2.7	2.5
Tracking error* (%)	0.0	4.5	5.6	3.6	9.0	3.0
Information Ratio	N/A	0.25	0.79	0.35	0.30	0.83
Trading Costs / Investability						
Weighted Average Days to Trade***	0.01	0.04	0.04	0.01	0.18	0.03
Turnover** (%)	3.1	23.0	22.0	18.3	89.6	32.0
Performance Drag in bps (at 50 bps)	3.1	23.0	22.0	18.3	89.6	32.0
* Annualized in USD for the 05/31/1999 to 09/30/2013 period						
** Annualized one-way index turnover for the 05/31/1999 to 09/30/2013 period						
*** Average of last four index reviews ending 09/30/2013. Assuming a fund size of USD 10 bn and a maximum daily trading limit of 20%						

Table 15.8. Performance using historical returns (May 1999–September 2013)

15.5.2. Example #2: De-risking with yield enhancement

Strategic allocation:

- MSCI World Minimum Volatility Index 50%;
- MSCI World High Dividend Yield Index 50%.

Some institutional investors have sought to enhance yield in recent years and at the same time reduce overall volatility. As shown in Figure 15.9, combining the MSCI World High Dividend Yield and MSCI World Minimum Volatility Indexes would have improved the historical performance of the portfolio over the May 1 999 –June 2013 period significantly with overall lower volatility. Meanwhile, the average dividend yield for this period was 3.3% for the multi-factor index compared to 2.2% for the market cap-weighted parent MSCI World Index.

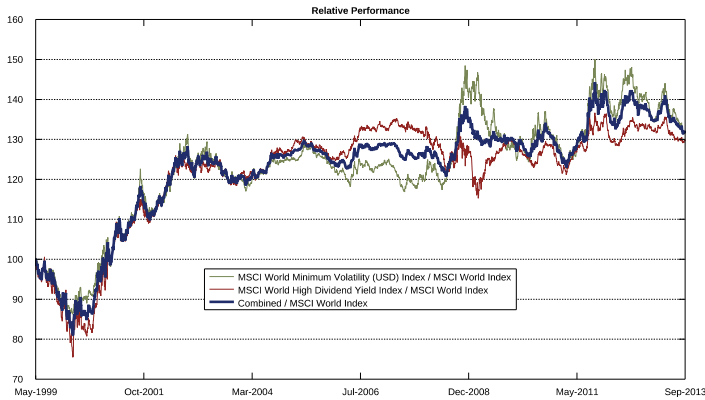


Figure 15.9. Performance using historical returns (May 1999–September 2013). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

The result of combining a high yield factor index with a low volatility index is a portfolio which historically exhibited substantial risk-adjusted return enhancement as seen below.

Historical Gross Total Return, USD	MSCI World Index	MSCI World High Dividend Yield Index	MSCI World Minimum Volatility (USD)	Combined
Total Return Performance				
Total Return* (%)	4.2	6.1	6.3	6.3
Total Risk* (%)	16.3	16.1	11.4	13.5
Return/Risk	0.26	0.38	0.55	0.46
Sharpe Ratio	0.18	0.30	0.37	0.33
Active Return Performance				
Active Return* (%)	0.0	1.9	2.0	2.0
Tracking error* (%)	0.0	6.1	7.9	6.2
Information Ratio	N/A	0.31	0.26	0.33
Trading Costs / Investability				
Weighted Average Days to Trade***	0.01	0.4	1.2	0.5
Turnover** (%)	3.1	20.4	27.1	23.1
Performance Drag in bps (at 50 bps)	3.1	20.4	27.1	23.1
Yield				
Dividend Yield (%)****	2.3	4.0	2.6	3.3

* Annualized in USD for the 05/31/1999 to 09/30/2013 period

** Annualized one-way index turnover for the 05/31/1999 to 09/30/2013 period

*** Average of last four index reviews ending 09/30/2013. Assuming a fund size of USD 10 bn and a maximum daily trading limit of 20%

**** Monthly averages for the 05/31/1999 to 09/30/2013 period

Table 15.9. Performance using historical returns (May 1999–September 2013)

15.6. Conclusion

In this chapter, we discussed a framework for how institutional investors might consider deploying factor allocations based on factor indexes. The framework comprises three key steps. In the first step, the institution assesses the role of factor investing in its portfolio. The second step identifies which factor(s) are appropriate for the institution's portfolio. Finally, the third step implements the factor index allocation. This includes structuring the portfolio to take into account potential diversification effects between factors and the institution's preferences for investability and factor exposure. Factor allocations can play a variety of roles in the investment process, depending on the objectives and constraints of the investor.

Because they reflect systematic factors that respond to macroeconomic and macromarket forces, factor indexes can underperform the overall market for periods of time that may exceed an investment committee's patience. However, many of these factors respond differently to macroeconomic and macromarket forces, so they have historically low correlations which may yield strong diversification effects for combining multiple factors in an allocation. We demonstrated how combining factor indexes in a "Multi-Factor Index" captured these diversification effects as well as additional benefits such as lower turnover as a result of internal crossing.

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Defining the Equity Premium, a Framework

It is noticeable that after 10 years of smart beta and the comeback of risk factors driven investments, we are still left alone with a zoo of heterogeneous approaches and few clues about their fundamental drivers. We develop in this chapter an articulate framework that is a step toward a better understanding of these investment strategies.

We categorize these strategies into two groups: belonging either to a financial or a non-speculative sphere. Endowing investors with a utility function defined over the set of portfolios, in the financial sphere investors take advantage of their ability to distinguish the utilities of assets from one another (or speculate), while in the non-speculative sphere utilities of assets are ex-ante indistinguishable, equal. We then define the equity premium as the return delivered by the portfolio that has maximal utility under the constraint that utilities of assets are equal.

In this framework, and in the particular case of a Sharpe ratio utility function, all smart beta strategies identified in this chapter, but one, bet on some heterogeneity in utilities, and consequently belong to the financial sphere. The strategy that belongs to the non-speculative sphere is the most diversified portfolio (MDP) and as such delivers the equity premium.

The chapter ends with a practical recommendation for long-term investors. In a world where the Sharpe ratio is one of the leading metrics to assess the performance of a portfolio, long-term investors seeking both the equity premium and tactical exposures to market timed factors could adopt a core-satellite approach. The MDP is then arguably a strong candidate to be the core portfolio.

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16.1. Introduction

Over the last 10 years, since Robert Arnott introduced investors to the concept of fundamental indexation [ARN 05], the financial industry has come up with a new set of investment strategies packaged together under the name “Smart Beta”. This set seems at first sight essentially characterized by its lack of unity, ranging from equal weights, fundamental indexation, to the minimum variance or the MDPs. As such, the smart beta initiative opened Pandora’s box and each research program took its own direction.

Despite this methodological heterogeneity, we think these smart beta strategies are grounded on one fact: there exists a non-smart beta portfolio, the market capitalization-weighted portfolio. This non-smart portfolio has been the common spine of most asset allocation recommendations since the publication of seminal papers by W. Sharpe and J. Lintner in the mid-1960s. These papers came up with a very simple result: the cap-weighted portfolio of all financial assets is the portfolio that the representative investor should hold, with the stock market cap-weighted portfolio¹ being used as a proxy for this portfolio.

This result may be beautiful but that does not make it innocuous. History has taught us that the allocation of the cap-weighted portfolio is far from being neutral, even in the long run. Figure 16.1 shows that the sectors weights in the cap-weighted portfolio oscillate reaching their peaks at the top of their prices. In the early 1970s, the cap-weighted portfolio reached its maximum exposure to Consumer Discretionary just before the oil price surge, in the mid-1970s it reached its maximum exposure to the energy sector just before the *great oil price collapse* that started after the Iran and Iraq War in 1981. Similarly, in 2001, it was heavily invested in the information technology (IT) sector, on the eve of its collapse, or in 2007 in financials. This story is compelling: it warns investors against the significant risk concentrations hidden behind passive investments.

As such, even if it may be difficult to find one’s way through all those smart beta strategies, the fact is that there are better candidates to capture the equity premium hidden in the markets than the cap-weighted portfolio.

¹ Hereafter, abbreviated cap-weighted portfolio.

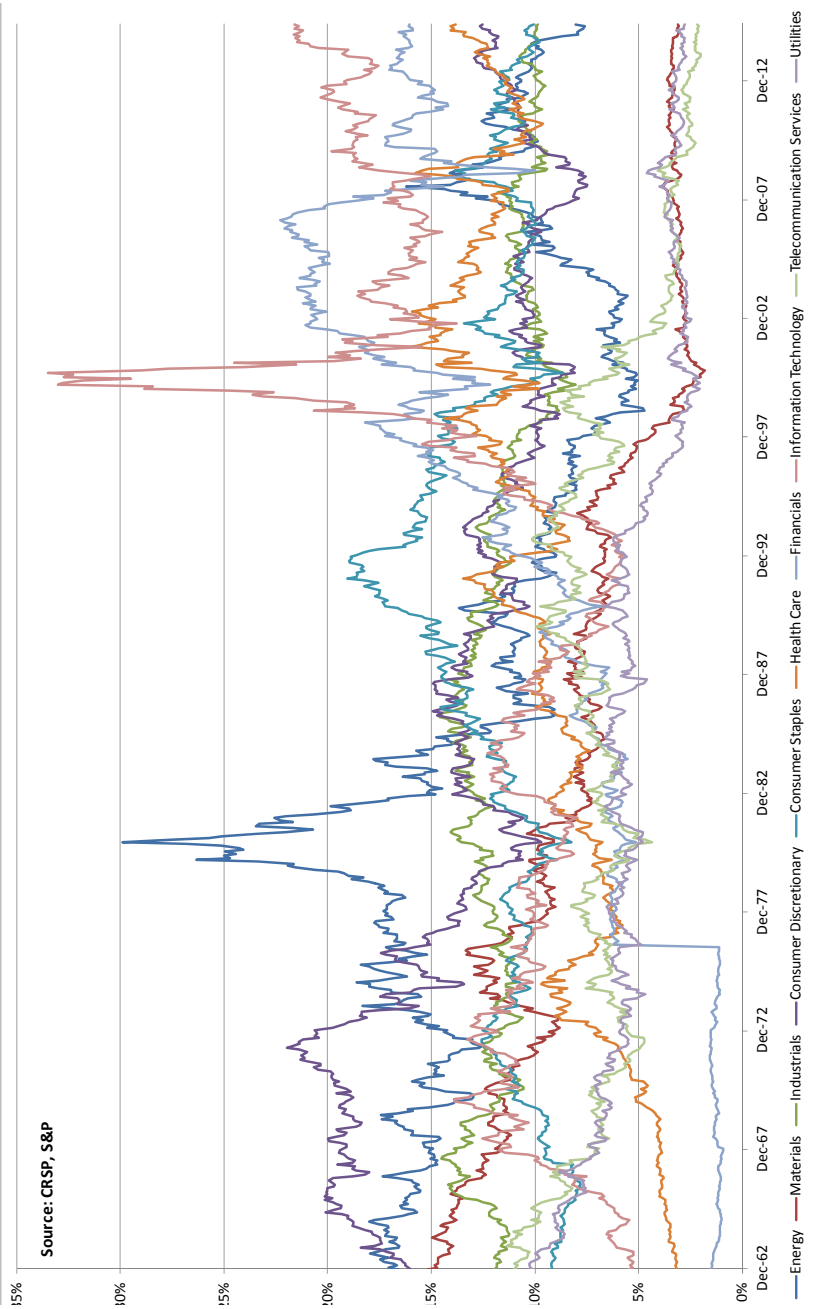


Figure 16.1. S&P 500 Equity Sector Weights. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Building on this conclusion, a new wave of articles explored the replication of the overperformance of smart beta portfolios using well-known risk factors, explicitly controlling the risk allocation to each of them [RON 12]. A non-exhaustive but representative list of such factors would typically feature small caps, value, growth, high dividend, momentum, quality, low volatility, etc. Nowadays, banks and asset managers are proposing a large range of such risk factor-driven investment (RFDI) strategies.

There are, however, some potential pitfalls to address. First, as pointed out by Cochrane [COC 11], significant common asset variations such as industry portfolios need not correspond to any risk premium. Second, factors risk premia are well known to be time-varying and may very well be much less significant in the future. Finally, factor construction can be very challenging in a low turnover, long only, unlevered setting. This takes us back to our starting point, with a zoo of smart beta and RFDI strategies without a framework that could help us discriminate among them. This motivated us to present a formal framework based on a sensible definition of the equity premium as well as a taxonomy of the fundamental drivers behind smart beta and RFDI strategies.

16.2. Defining the equity premium

At the risk of stating the obvious, long-term investors invest in the equity market because they harbor one major investment belief: in the long run, equities should reward the extra risk taken, or in investors' lingo there should exist a positive "Equity Risk Premium". In our framework, there are two potential sources of reward for the risk taken by an investment strategy, depending on whether this strategy belongs to the financial or to the non-speculative sphere.

In the financial sphere, rewards come from taking advantage of an ability to assess mispricings, arbitrage opportunities or to take advantage of forecasting capabilities; in a nutshell being a gifted speculator. As such, the financial sphere is a speculation sphere.

At this stage, it is important to dismiss the belief that the cap-weighted portfolio could be assimilated to a strategy not belonging to the financial sphere. As was shown by Figure 16.1, the cap-weighted portfolio behaves like a dynamic risk allocator and not a neutral risk allocator. This comes from an identity: the cap-weighted portfolio is what remains when all active portfolios have been aggregated together. As such, the cap-weighted portfolio is built upon all speculative bets. It is hence biased toward the aggregated speculative behavior, and is a speculative portfolio. In fact, using the well-known mean-variance optimal portfolio weights formula, the Sharpe ratios of each stock can be implied from their market weights. For

instance, in the MSCI US, the implied Sharpe ratio of Blackrock is 11 times higher compared to Puma Biotechnology². This means that the implied Sharpe ratios of the cap-weighted portfolio are heterogeneous, which virtually amounts to use some forecasting capabilities about the utility of each asset (in this case, the Sharpe ratio).

With this insight, we elaborate on the drivers behind the investment strategies associated with each sphere. The financial sphere is the place where investors use their abilities to identify and forecast ex-ante valuation heterogeneities among stocks while, on the opposite, investors in the non-speculative sphere are not speculating and taking advantage of such abilities. This difference can be restated in terms of investor's preferences. In the financial sphere, the fact that investors use their assessment of stocks heterogeneities implies that they associate heterogeneous utilities to different stocks, whereas in the non-speculative sphere these utilities are identical.

To be more precise, denote $U(\mathbf{w})$ the utility of the portfolio with weights $\mathbf{w} = (w_1, \dots, w_n)$ to the investment universe assets A_1, \dots, A_n . We also denote s_1, \dots, s_n the portfolios of single assets A_1, \dots, A_n . Now, investors belonging to the non-speculative sphere would invest in the portfolio that maximizes $U(\mathbf{w})$ under the constraint $U(s_1) = \dots = U(s_n)$. This motivates us to define the "Equity Premium" as the excess returns (to cash) earned by this portfolio.

DEFINITION.— *the equity premium is the return of the portfolio that has maximum utility, under the constraint that single assets utilities are equal.*

Note that we have deliberately chosen not to use the term "Equity Risk Premium", a different concept. In our view, the equity premium is indeed a return obtained without using any forecasting ability, taking its source in the real economy.

Now, most investors base their decisions on the assessment of the risk-reward characteristics of their portfolios. As such, we will examine the consequences in this framework of considering that investors maximize a Sharpe ratio based utility function.

Under this assumption, we have the following fundamental property of our two spheres: in the financial sphere investors exploit the dispersion of Sharpe ratios among equities, while in the non-speculative one, investors do not. It is important to clarify that we do not mean that non-speculative investors ignore risk-rewards differences among stocks, but that they simply do not believe to have any competitive edge exploiting them.

² Computed using 252 observations to 1 June 2015.

16.3. Risk-rewards homogeneity and the equity premium

Taking this formal framework a step further, we can also draw some conclusions about the link between how expectations are formed and the nature of the optimal portfolio. We first introduce the diversification ratio (DR). The DR of a portfolio is the ratio of the weighted average volatility of the portfolio's stocks to its own volatility [CHO 06, CHO 08]. Formally, the DR of a portfolio with weights $\mathbf{w} = (w_1, \dots, w_n)$ is:

$$DR(\mathbf{w}) = \frac{\sum_i w_i \sigma_i}{\sigma(\mathbf{w})},$$

where σ_i is the volatility of each asset and $\sigma(\mathbf{w})$ is the portfolio volatility. This measure has an interesting property when maximized: the resulting portfolio, the MDP, is the tangency portfolio when all stocks have the same Sharpe ratio [CHO 06]. This can be readily seen rewriting the Sharpe ratio of a portfolio:

$$U(\mathbf{w}) = \frac{\sum_i w_i \sigma_i U(s_i)}{\sigma(\mathbf{w})}$$

with $U(s_i)$ the Sharpe ratios of each asset. Therefore, assuming that investors maximize the Sharpe ratio, we have:

RESULT 16.1.– All portfolios build upon heterogeneous expected risk-rewards are diversifiable according to the DR (i.e. a portfolio that does not achieve maximal DR).

COROLLARY 16.1.– A portfolio belonging to the financial sphere is diversifiable.

Its converse also holds:

RESULT 16.2.– If a portfolio has maximal DR, then it is built upon homogenous expected risk-rewards among stocks.

COROLLARY 16.2.– The MDP delivers the equity premium.

Note that while homogenous expected risk-rewards among stocks are needed for the MDP to be the tangency portfolio ex-ante, they are not needed for the MDP to be more efficient than the market portfolio ex-post.

We now explore the practical implications of our last two results.

16.4. A taxonomy of smart beta and risk factors driven strategies

Equipped with this framework, we are now able to provide a taxonomy of the sources of risk-rewards hidden behind mainstream smart beta strategies. As shown

in Table 16.1, in this framework, all smart beta strategies, but one, bet on some form of risk-rewards heterogeneity and, therefore belong to the financial sphere. Indeed, investing in a minimum-variance portfolio is like betting that low volatility stocks will better reward the risk, while equal weighting and equal risk contribution (ERC) portfolios suppose that risk sources which have many representatives are more rewarded [CHO 13]. The only smart beta strategy that belongs to the non-speculative sphere is the MDP: in this portfolio, all the stocks of the investment universe have been assigned the same Sharpe ratio.

Smart beta strategy	Risk reward heterogeneity	Financial sphere	Non-speculative sphere
Minimum variance	Low volatility stocks reward the risk better	X	
Equal weighting/ERC	Sources of risk that have many representatives reward the risk better	X	
Fundamental indexation	Value stocks reward the risk better	X	
Maximum diversification	None (Result 2)		X
RFDI	Biased by construction	X	

Table 16.1. *Smart beta taxonomy when utility=return/risk*

With this framework, we can also understand a little more about RFDIs. They hinge on the belief that exposures to some source of return variations – the factors – deliver a superior risk-adjusted return. In a long–short setting, RFDI can be compared to an arbitrage strategy which goes long stocks that are cheap according to the factor and short otherwise. Such an investment strategy belongs to the financial sphere. Moreover, in a long only setting, as each factor has its reciprocal – e.g. “value” stocks versus “growth” (or “non-value” stocks) – factor-based portfolios are by construction biased toward one particular source of market (co-)variation. In this sense, each RFDI portfolio is such that there exists at least another portfolio which combined with the former will increase their overall diversification.

16.5. Being practical: a core-satellite portfolio allocation

These results can be turned into practical recommendations about how to build an equity portfolio. Investors believe that equity markets reward risk in the long

term, while common investment practices suggest they also believe that some short-term tactical performance can be drawn from investing in wisely selected funds and strategies. Putting it in the terminology of this chapter, while their main goal is to access the equity premium, long-term investors may still believe that the financial sphere can also be a source of extra returns.

So, beyond the equity premium, long-term investors may also seek short-term exposure that would add market timing attributes to their aggregated portfolios. Here, we are thinking about market timing strategies in a very broad sense, with strategies ranging from style arbitrage, statistical arbitrage (mean-reversion, trend following etc.) to volatility timing and any other types of speculation. The composition of this market timing portfolio will thus reflect how cheap or expensive the factors it is exposed to are perceived. Through time, this composition is likely to change, for example when factor valuations have mean-reverted toward “fair valuations”.

An important question is how to build a portfolio that combines efficiently a long-term exposure to the equity market and market timing strategies. We think that a rational answer is to take a core/satellites approach. In this case, the core portfolio should focus on accessing the equity premium. The formal framework developed in this chapter gives a way to tackle this issue. Indeed, assuming that investors have a Sharpe ratio utility, the MDP gives access to the equity premium and as such is arguably a strong candidate to be the core portfolio.

We present now two empirical analyses illustrating that the cap-weighted benchmark belongs to the financial sphere, and also that the MDP can be a core complement to some of the most popular risk factors:

1) The first plot of Figure 16.2 depicts the 100 days rolling correlation of the MSCI US index to the GICS sectors. We can clearly observe that the cap-weighted portfolio may be biased toward few sectors. For example, from 1997 to 2000, the MSCI US concentrated itself in a few sectors, among which the Consumer Discretionary, Industrials and the IT sectors as shown by the very high correlation of those three sectors with the MSCI US. During the 2001 crisis, that concentration trend lead the MSCI US to essentially emulate the behavior of three sectors only, while achieving a negative correlation to three other sectors. In the second plot of Figure 16.2, we perform the same experiment with a sector MDP computed on the aforementioned sectors. We can observe that it remains almost equally correlated to all sectors at any point in time, even if these are far from perfect proxies for market risk factors. This is expected from a maximally diversified portfolio: it should be the least biased portfolio given its investment constraints. In a

nutshell: the MDP is a “neutral risk allocator, while the Benchmark is a “biased, dynamic risk allocator”.

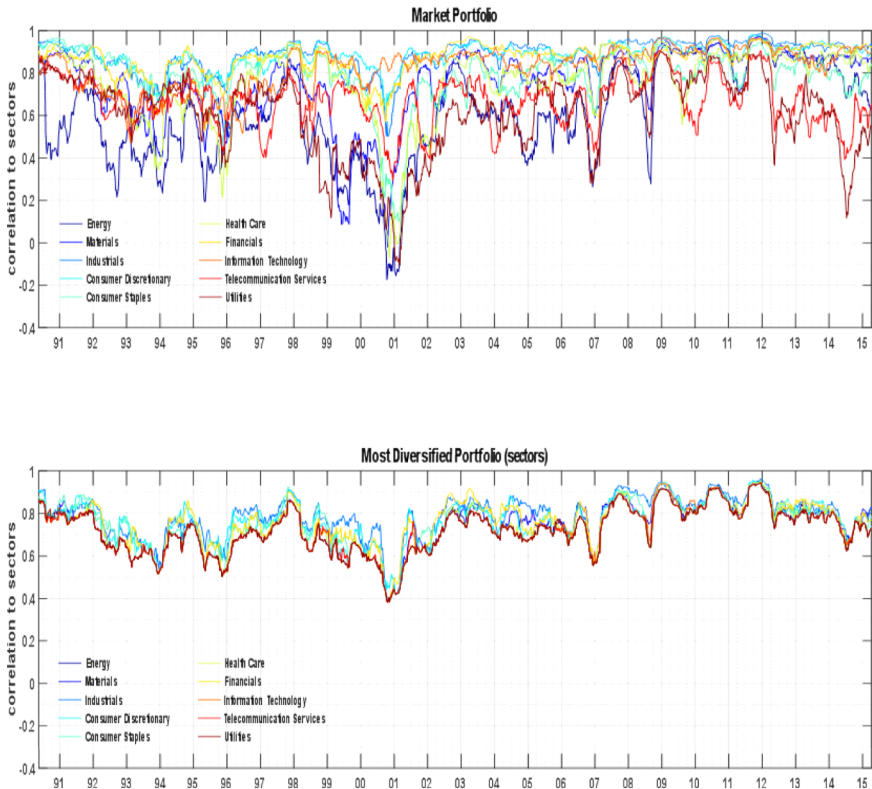


Figure 16.2. Market and MDP portfolios correlation to cap-weighted sectors (source: MSCI, Bloomberg, TOBAM). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

2) We now turn to the correlation between the MDP and standard equity risk factors, shown in Table 16.2.

The striking results are that (1) the MDP has among the weakest correlations to other factors and (2) the correlation between a factor and its complementary (e.g. “value” versus “growth”) is higher than the correlation between that factor and the MDP. In this sense, the MDP behaves like a universal diversifier: this could be expected from a portfolio offering a diversified exposure to the equity market.

Correlation	AB US	Bench US	Small	Large	Value	Growth	Average	Max
AB US		89.7%	86.1%	86.4%	88.2%	87.7%	87.6%	89.7%
Bench US	89.7%		91.5%	98.9%	98.3%	97.9%	95.3%	98.9%
Small	86.1%	91.5%		87.5%	91.4%	88.0%	88.9%	91.5%
Large	86.4%	98.9%	87.5%		96.7%	97.4%	93.4%	98.9%
Value	88.2%	98.3%	91.4%	96.7%		92.5%	93.4%	98.3%
Growth	87.7%	97.9%	88.0%	97.4%	92.5%		92.7%	97.9%

Correlation	AB World	Bench World	Small	Large	Value	Growth	Average	Max
AB World		90.2%	89.8%	88.1%	81.0%	79.2%	85.7%	90.2%
Bench World	90.2%		96.5%	99.1%	89.7%	88.7%	92.8%	99.1%
Small	89.8%	96.5%		92.5%	82.5%	80.6%	88.4%	96.5%
Large	88.1%	99.1%	92.5%		90.8%	90.3%	92.1%	99.1%
Value	81.0%	89.7%	82.5%	90.8%		92.8%	87.4%	92.8%
Growth	79.2%	88.7%	80.6%	90.3%	92.8%		86.3%	92.8%

Correlation	AB EM	Bench EM	Small	Large	Value	Growth	Average	Max
AB EM		89.4%	90.4%	83.4%	88.4%	89.2%	88.2%	90.4%
Bench EM	89.4%		95.9%	97.0%	99.2%	99.2%	96.1%	99.2%
Small	90.4%	95.9%		89.4%	94.5%	95.9%	93.2%	95.9%
Large	83.4%	97.0%	89.4%		96.5%	95.9%	92.4%	97.0%
Value	88.4%	99.2%	94.5%	96.5%		97.0%	95.1%	99.2%
Growth	89.2%	99.2%	95.9%	95.9%	97.0%		95.4%	99.2%

Table 16.2. Correlation between the MDP and a set of standard MSCI risk factors across the USA, World and Emerging Market universes. Period: 2000–2014. (source: Tobam, MSCI, Bloomberg. The MDP is abbreviated AB in the table) For a color version of the table see www.iste.co.uk/jurczenko/risk.zip

16.6. Conclusion

We developed a formal framework in which the equity premium is delivered by the portfolio that maximizes the utility of investors under the constraint that single assets utilities are equal. This portfolio is unbiased in the sense that no forecast is made regarding the heterogeneities of the utilities of single assets.

If we further endow investors with a Sharpe ratio utility function, the MDP delivers the equity premium. As a byproduct, we obtain that all mainstream smart beta strategies, but the MDP, bet on some heterogeneity of risk-rewards, and as such are biased.

This result may be turned into a practical recommendation for long-term investors seeking both the equity risk premium and tactical exposures to some other factors/strategies. In this case, a core-satellite approach is the rational way to allocate strategies between portfolios. Moreover, for investors using the Sharpe ratio as the main metric to assess the performance, the MDP is arguably a serious candidate to be the core portfolio.

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Designing Multi-Factor Equity Portfolios

17.1. Introduction

This chapter reviews efficient index design methods for factor indices referred to as smart factor investing. It then uses such smart factor indices as building blocks to design suitable allocation strategies to address specific risk/return objectives.

In this study, we focus on four well-known rewarded factors – size, value, momentum and low volatility. We review the concept of *smart factor index*, which can be regarded as an efficient investable proxy for a given risk premium. In a nutshell, a risk premium can be thought of as a combination of a *risk* (exposure) and a *premium* (to be earned from the risk exposure). Smart factor indices have been precisely engineered to achieve a pronounced factor tilt emanating from the stock selection procedure (right risk exposure), as well as high Sharpe ratio emanating from the efficient diversification of unrewarded risks related to individual stocks (fair reward for the risk exposure). The access to the fair reward for the given risk exposure is obtained through a well-diversified *smart-weighted* portfolio (as opposed to concentrated cap-weighted portfolio) of the selected stocks so as to ensure that the largest possible fraction of individual stocks' unrewarded risks is eliminated.

We then show that such smart factor indices can be used as attractive building blocks in the design of an efficient allocation to the multiple risk premia to be harvested in the equity universe. In fact, additional value can be added at the allocation stage, where the investor can control for the dollar and risk contributions of various constituents or factors to the absolute (volatility) or relative risk (tracking error) of the

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portfolio. As a result, extremely substantial levels of risk-adjusted out-performance (information ratios) can be achieved even in the absence of views on factor returns.

17.1.1. Designing efficient and investable proxies for risk premia

Current smart beta investment approaches only provide a partial answer to the main shortcomings of capitalization-weighted (cap-weighted) indices. We discuss a new approach to equity investing referred to as *smart factor investing*. It provides an assessment of the benefits of simultaneously addressing the two main shortcomings of cap-weighted indices, namely their undesirable factor exposures and heavy concentration, by constructing factor indices that explicitly seek exposures to rewarded risk factors while diversifying away from unrewarded risks. The results we obtain suggest that such smart factor indices lead to considerable improvements in risk-adjusted performance.

The results in Table 17.1 confirm that the combination of relevant security selection and appropriate weighting schemes in a two-step process leads to substantial improvements in risk-adjusted performance with respect to the use of a standard cap-weighted index, which typically implies an inefficient set of factor exposures and an excess of unrewarded risk.

	Broad CW	Mid Cap		High momentum		Low volatility		Value	
		CW	Diversified multi-strategy	CW	Diversified multi-strategy	CW	Diversified multi-strategy	CW	Diversified multi-strategy
Ann returns	9.74%	12.54%	14.19%	10.85%	13.30%	10.09%	12.64%	11.78%	14.44%
Ann volatility	17.47%	17.83%	16.73%	17.60%	16.30%	15.89%	14.39%	18.02%	16.55%
Sharpe ratio	0.24	0.39	0.52	0.30	0.48	0.29	0.50	0.35	0.54
Historical daily 5% VaR	1.59%	1.60%	1.50%	1.64%	1.50%	1.42%	1.28%	1.59%	1.47%
Max drawdown	54.53%	60.13%	58.11%	48.91%	49.00%	50.50%	50.13%	61.20%	58.41%
Ann excess returns	–	2.80%	4.45%	1.10%	3.56%	0.35%	2.90%	2.04%	4.70%
Ann tracking error	–	5.99%	6.80%	3.50%	4.88%	4.44%	6.17%	4.74%	5.82%
95% Tracking error	–	9.39%	11.56%	6.84%	8.58%	9.20%	11.53%	8.72%	10.14%
Information ratio	–	0.47	0.66	0.32	0.73	0.08	0.47	0.43	0.81

Table 17.1. Performance comparison of USA Cap-Weighted Factor Indices and USA Multi-Strategy Factor Indices. The table shows the absolute performance, relative performance and risk indicators for Cap-Weighted (CW) Factor Indices and Multi-Strategy Factor Indices for four factor tilts – mid cap, high momentum, low volatility and value. The complete stock universe consists of the 500 largest stocks in the USA. The benchmark is the cap-weighted portfolio of the full universe. The yield on secondary market US Treasury Bills (3M) is the risk-free rate. The return-based analysis is based on daily total returns from 31/12/1972 to 31/12/2012 (40 years). All weight-based statistics are average values across 160 quarters (40 years) from 31/12/1972 to 31/12/2012

On the one hand, starting with a focus on the systematic risk exposure, we find that a higher Sharpe ratio can be achieved with the same weighting scheme, here a cap-weighting scheme, for stocks selected on the basis of their loadings on the value, size, momentum and low-volatility factors, compared to the case where the full universe is held in the form of a cap-weighted portfolio.

The results we obtain, reported in Table 17.1, show that while the Sharpe ratio of the broad cap-weighted index is 0.24 on the sample period, it reaches values as high as 0.39 for a cap-weighted strategy using a mid cap stock selection, 0.30 for a high momentum stock selection, 0.29 for a low-volatility stock selection or 0.35 for a value stock selection¹. These results suggest that a systematic attempt to harvest equity risk premia above and beyond broad market exposure leads to additional risk-adjusted performance. It should be noted at this stage that substantially higher levels of max drawdown are incurred for the mid cap and value selections, confirming that the reward harvested through the factor exposure is a compensation for a corresponding increase in risk. In contrast, we note that high momentum and low-volatility selections lead to lower levels of max drawdown compared to the no selection case, suggesting that the excess performance earned on these two factors has, at best, a behavioral explanation and is not necessarily related to an increased riskiness.

On the other hand, shifting to the management of specific risk exposures, we find that even higher levels of Sharpe ratio can be achieved for each selected factor exposure through the use of a well-diversified weighting scheme, which we take to be an equally weighted combination of five popular smart weighting schemes².

1 The cap-weighted tilted strategies are implemented by selecting on a quarterly basis the top 50% of stocks in the reference universe by the relevant factor score (i.e. the 50% of stocks with, respectively, the lowest market cap, highest book-to-market, highest past returns or the lowest volatility) and weighting them in proportion to their free-float-adjusted market cap.

2 Diversified multi-strategy weighting is an equal-weighted combination of the following five weighting schemes – maximum deconcentration, diversified risk-weighted, maximum decorrelation, efficient minimum volatility and efficient maximum Sharpe ratio. Maximum deconcentration consists of maximizing the effective number of stocks subject to turnover and liquidity constraints and thus corresponds to an adjusted version of equal-weighting. Diversified risk-weighted attributes stock's weights inversely proportional to their volatility. Maximum decorrelation constructs a portfolio of stocks that behave differently over time, which is achieved by minimizing portfolio volatility subject to the assumption that volatility is identical across stocks. Efficient minimum volatility consists of a volatility minimization subject to norm constraints. Efficient maximum Sharpe ratio maximizes the Sharpe ratio of the portfolio given the assumption that expected returns are proportional to the median semi-deviation of stocks in the same decile resulting from a sort on stock-level semi-deviation. The three latter strategies require a covariance matrix as an input to the optimization problem. The covariance matrix is estimated using a robust estimation procedure employing a statistical factor model based on principal component analysis where the number of components is selected using a criterion from random matrix theory. For more details on the weighting schemes and the derivation of required input parameters, see www.scientificbeta.com.

Thus, the Sharpe ratio of the so-called *diversified multi-strategy* combination reaches 0.52 for mid cap stocks, 0.48 for high momentum stocks, 0.50 for low-volatility stocks and 0.54 for value stocks.

These results suggest that multi-strategy factor-tilted indices obtain the desired factor tilts without undue concentration, which provides an explanation for their superior risk-adjusted performance with respect to the cap-weighted combination of the same selection of stocks.

Overall, it appears that the combined effects of a rewarded factor exposure ensured by a dedicated proper security selection process and an efficient harvesting of the associated premium through improved portfolio diversification leads to a Sharpe ratio improvement of around 100% compared to the broad cap-weighted index.

17.1.2. Risk allocation with smart factor indices

Once a series of smart factor indices have been developed for various regions of the equity universe, they can be used as attractive building blocks in the design of an efficient allocation to these multiple risk premia.

In an attempt to identify, and analyze the benefits of, the possible approaches to efficient risk allocation across the various smart factor indices, we identify four main dimensions that can be taken into consideration when designing a sophisticated allocation methodology (see Figure 17.1).

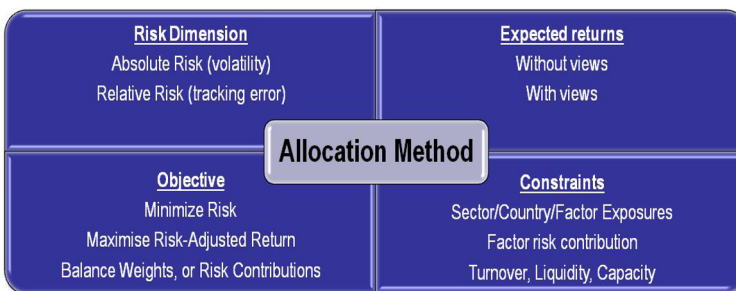


Figure 17.1. *The various dimensions of allocation methodologies across assets or risk factors*

The first, and arguably most important, dimension relates to whether risk is defined by the investor from an absolute perspective in the absence of a benchmark, or whether it is instead defined in relative terms with respect to an existing benchmark, which is more often than not a cap-weighted index. In the former

situation, we would use volatility as a relevant risk measure, while tracking error with respect to the cap-weighted index would instead be used in the latter case.

The second dimension concerns whether we would like to incorporate views regarding factor returns in the optimization process. While additional benefits can be obtained from the introduction of views on factor returns at various points of the business cycle, in the following we focus on approaches that are solely based on risk parameters, which are notoriously easier to estimate with a sufficient degree of robustness and accuracy [MER 80]. The third dimension is related to the objective of the allocation procedure. Indeed, there are several possible targets for the design of a well-diversified portfolio of factor exposure, depending upon whether we would like to use naive approaches (equal dollar allocation or equal risk allocation) or scientific approaches based on minimizing portfolio risk (volatility in the absolute return context or tracking error in the relative return context). The fourth and last dimension is related to the presence of various forms of constraints such as minimum/maximum weight constraints, turnover constraints or factor exposure constraints, which are obviously highly relevant in the context of risk factor allocation.

17.2. Absolute return perspective

In the context of generating a “smart” (meaning *efficient*) allocation to smart factor indices, a natural first, albeit naive, approach consists of forming an equally weighted portfolio of the selected smart factor indices, in this case the indices that serve as proxies for the value, small cap, momentum and low-volatility risk premia.

While an equally weighted scheme is the simplest approach we can use, it is likely that the use of more sophisticated weighting schemes could add additional value, in particular when it comes to the management of the risks relative to the cap-weighted (CW) benchmark. In what follows, we will sequentially consider the absolute return approach both with and without factor risk parity/budgeting constraints. We consider naive approaches to diversification (maximum deconcentration in terms of dollar or risk contributions) and scientific approaches (minimum risk from the absolute return perspective). One of the important aims of this chapter will also be to show that it is possible to perform risk parity in the long-only world, i.e. to have an exposure that is equal in terms of risk factors rewarded over the long term without necessarily having pure or orthogonal factors that are impossible to obtain in the long-only space. This point is all the more important in that often, under the pretext of purity, investors choose excessively concentrated factor indices that contribute neither purity nor diversification and therefore have a fairly low risk-adjusted return. Our argument here is that by using well-diversified investable proxies for each factor (the scientific beta smart factor indices), it is possible to implement high-performance allocation between these indices while respecting factor risk parity constraints.

All these methodologies will be implemented without any active views (expected return forecasts) on constituents or factors; they generate portfolios that can be regarded as attractive starting points, with very substantial risk-adjusted outperformance benefits with respect to cap-weighted indices, to which additional benefits could be added by asset managers possessing skills for actively timing factor exposures.

The developed dataset extends over the 10-year period from 31 December 2003 to 31 December 2013 and uses five subregions of the global developed universe: US, UK, developed Europe (e.g. UK) and Asia Pacific (e.g. Japan). Using four smart multi-strategy indices as proxies for the value, size, momentum and volatility-rewarded tilts in each subregion, we obtain a total of $5 \times 4 = 20$ constituents.

Following an equally weighted allocation is equivalent to holding an equal dollar allocation, which does not necessarily lead to an equal risk allocation. Formally, the risk contribution of a stock to the total risk of a portfolio is given by the weight of the stock in the portfolio times the marginal contribution of the stock to total portfolio volatility. Qian [QIA 06] shows that decomposing total portfolio volatility in terms of its constituents' risk contributions is also related to the expected contributions to the portfolio losses, particularly when considering extreme losses. In what follows, we consider two approaches to managing portfolio risk: one approach based on minimizing portfolio volatility (global minimum variance or GMV approach) and another approach based on imposing equal contribution of all constituents to portfolio volatility (heuristic equal risk contribution or ERC approach).

17.2.1. Absolute risk management without factor risk exposure constraints

In our attempt to design an efficient allocation to smart factor indices, we first impose that all constituents in the portfolio have the same contribution to portfolio risk (ERC). If we make the explicit assumption that all pairwise correlation coefficients across constituents are identical, then the equal risk contribution weights can be obtained analytically and are proportional to the inverse volatility of the smart factor indices. In the general case, i.e. without the assumption of identical pairwise correlations across stocks, the risk parity methodology does not yield a closed-form solution. However, Maillard *et al.* [MAI 10] propose numerical algorithms to compute risk parity portfolios.

Overall, ERC and EW are two competing ways of implementing agnostic diversification. When looking at the empirical analysis performed in the global developed universe shown in Figure 17.2, we find that the allocation between the equally weighted and the ERC schemes can exhibit strong differences. For example, the largest average weight over the period under study in the ERC scheme is given

to the Japan low volatility smart factor index (7.45%), whereas the lowest weight is given to the Developed Europe, e.g. UK value smart factor index (3.78%), while the EW scheme maintains a 5% allocation to all indices. We also find that the ERC can lead to regional allocations that strongly deviate from the corresponding allocation within a cap-weighted index, where the larger markets (e.g. the US) strongly dominate smaller markets, such as Japan.

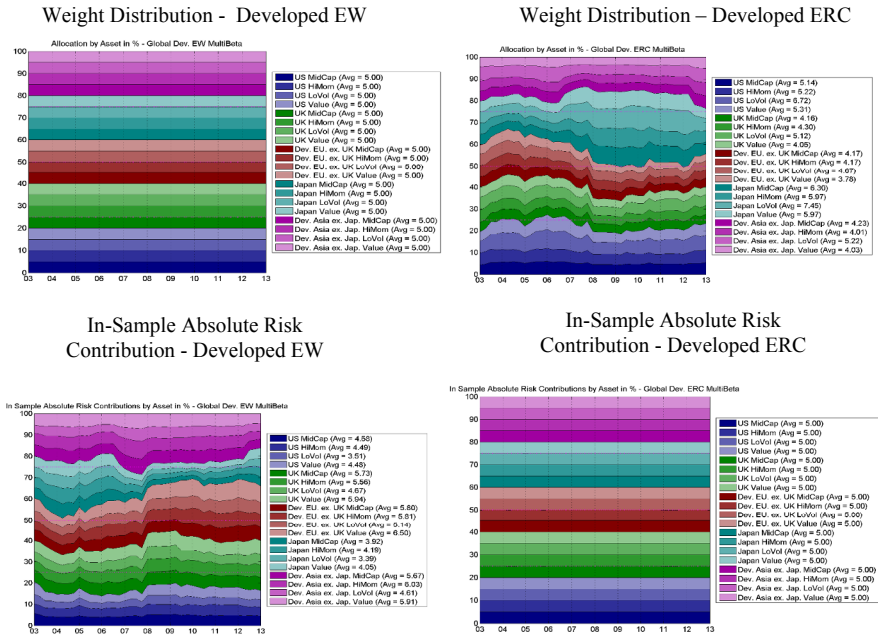


Figure 17.2. EW and ERC allocations to smart factor diversified multi-strategy indices (developed universe). The graph compares the allocation and risk contributions of diversified multi-strategy indices: the equal combination of the 20 diversified multi-strategy indices converted into US Dollars with stock selection based on mid cap, momentum, low volatility and value in the five subregions US, UK, developed Europe (e.g. UK) Japan and Asia Pacific (e.g. Japan), and the ERC combination of the same 20 constituents. The period is from 31 December 2003 to 31 December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

We have also implemented an allocation between smart factor indices based on minimizing the risk of the allocation, expressed by its volatility (GMV). In this case, the GMV portfolio of the 20 index constituents, which is the efficient portfolio that requires only covariance matrix input, the sample covariance matrix is estimated using the past 18 months of weekly data as an input. Long-only constraints are

applied to the standard minimum volatility problem, i.e. minimize portfolio volatility as given by this expression:

$$\text{Min}_{(w_i)_{i=1,\dots,N}} v(w) \equiv w' C w$$

To avoid introducing excessively strong biases with respect to the CW index, and even though the focus is not on relative risk management in this illustration, we also introduce a set of constraints dedicated to ensuring that each subregion is not too strongly under- or overrepresented with respect to its market capitalization in the CW global developed index, i.e. we define the weight to lie between half the region's market cap weight and twice its market cap weight.

More formally, we impose the following constraints (with $\delta = 2$) in each region:

$$\frac{mcap^{\text{Reg}_j}}{\delta} \leq w_{\text{tilt-MidCap}}^{\text{Reg}_j} + w_{\text{tilt-Value}}^{\text{Reg}_j} + w_{\text{tilt-HiMom}}^{\text{Reg}_j} + w_{\text{tilt-LoVol}}^{\text{Reg}_j} \leq \delta mcap^{\text{Reg}_j}$$

where *mcap* represents the market capitalizations of the different subregions, and w_{tilt} are the weights in each smart factor index of the same corresponding subregions.

Figure 17.3 shows that the GMV allocation with geographical constraints leads to a portfolio that is almost exclusively invested in the lowest volatility smart index for each subregion: on average, 52.47% low-volatility smart factor US index, 8.60% low-volatility smart factor UK index, 16.42% low-volatility smart factor developed Europe (e.g. UK) index, 12.68% low-volatility smart factor Japan index and 6.74% low-volatility smart factor Asia Pacific (e.g. Japan) index. In the end, this process leads to a dynamically managed portfolio of the 20 constituents that should achieve low volatility but that is highly concentrated.

Figure 17.3 also shows that the portfolio variance is almost exclusively driven by the low-volatility factor³, an observation that stresses the need for the introduction of risk factor budgeting constraints in order to better balance the factor contributions to the risk of the portfolio⁴.

3 See section 17.2.2 for details on measuring the contribution of the factors to portfolio risk.

4 The contribution of the low volatility factor is sometimes even greater than 100%, while other factors have a negative contribution to portfolio variance due to the presence of non-zero correlations between the smart factor indices and also between the long-short factors. For example, increasing the exposure to a factor that is negatively correlated with other factors may contribute to decreasing the portfolio variance.

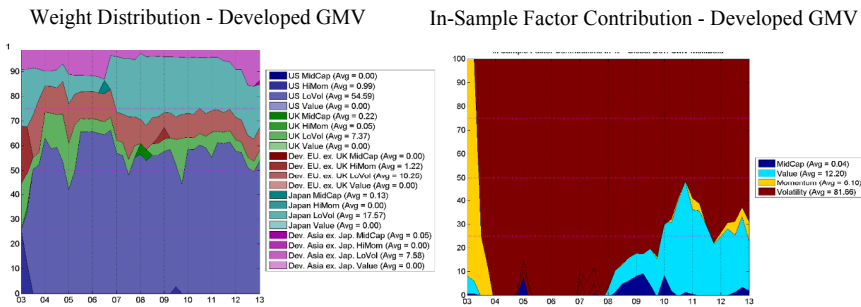


Figure 17.3. GMV allocations to smart factor diversified multi-strategy indices under geographical constraints (developed universe). The graph shows the allocation and risk contributions of the GMV allocation invested in the 20 diversified multi-strategy indices converted into US Dollars with stock selection based on mid cap, momentum, low volatility and value in the five US, UK, developed Europe (e.g. UK), Japan and Asia Pacific (e.g. Japan) subregions. Both risk parity and geographical constraints are imposed on the resulting portfolios. The period is from 31 December 2003 to 31 December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

17.2.2. Introducing risk-budgeting constraints

Having an equal contribution from the constituents to the overall portfolio risk is not identical to having an equal contribution from the factors. It is only if both the factors and the factor indices are perfectly “pure”, that is uncorrelated, that these two approaches coincide, which is not the case with smart factor indices. However, often it is the objective of investors to have an equal contribution to the underlying risk factors because risk contributions are perceived as indicators of the factor’s expected contribution to future losses (see [QIA 06]). In this way, integration of factor risk constraints in the allocation process takes into account the imperfections of existing single (smart) factor indices.

In the following, we use the factor exposure of the smart factor indices to analyze the question. We will compute exposure with respect to the equally weighted version of the factors, since they are the most neutral reference portfolios. As a neutral target, we may seek to impose an equal contribution of the factors to the variance coming from the factors. This extension of the ERC approach from the constituents to the factors leads to linear constraints in the design of the portfolio. This method of ERC of factors (along with EW of factors) is a reasonable approach for investors who are agnostic about the future performance of any single factor and therefore do not want to take a bet on one factor over another. In practice, in the absence of any active views on factors, these approaches are quite robust allocation techniques.

We introduce factor risk-budgeting constraints to the portfolio allocation process so as to avoid the domination of any one particular factor (such as the domination of the low-volatility factor). When the number of constituents N is greater than the number of factor constraints K , and long-short solutions are allowed, an infinite number of portfolios satisfy a given set of factor risk budgets (e.g. factor risk parity exposure). In a long-only context, we may have zero or multiple solutions. When no solution exists, we can start with the long-short version and rescale the weights to avoid short positions.

To be more specific, in order to measure the contribution of the factors to the portfolio variance, we employ the decomposition of the portfolio return as the sum of $K+1$ factors leading to:

$$r_p = \alpha_p + \beta_{w|mkt} r_{mkt} + \sum_{k=1}^K \beta_{w|k} r_{F_k} + \varepsilon, \text{ where } \beta_{w|k} \equiv \sum_{i=1}^N w_i \beta_{i|k} = (\beta_{1|k} \quad \dots \quad \beta_{N|k}) \underline{w}$$

Then, focusing only the contribution of the K long-short factors to the portfolio variance leads to the following expression for the contribution of factor i to the variance coming from the K factors:

$$c_i^{\text{var}}(w) = \beta_{w|i} \sum_{j=1}^K \beta_{w|j} \sigma_{F_i F_j}$$

As a neutral target, we may seek to impose an equal contribution of the factors to the variance coming from these K factors. This extension of the ERC approach from the constituents to the factors leads to the following K linear constraints in the design of the portfolio:

$$c_i^{\text{var}}(w) = c_j^{\text{var}}(w), \text{ for all } 1 \leq i, j \leq K$$

When the number of constituents N is greater than the number of factors constraints K , and long-short solutions are allowed, an infinite number of portfolios satisfy a given set of factor risk budgets (e.g. factor risk parity exposure). In a long-only context, we may have zero or multiple solutions. When no solution exists, then we can start with the long-short version and rescale the weights to avoid short positions.

However, when multiple solutions exist, we can address the diversification of specific risks, e.g. from a scientific perspective, by minimizing portfolio variance subject to factor risk parity constraints:

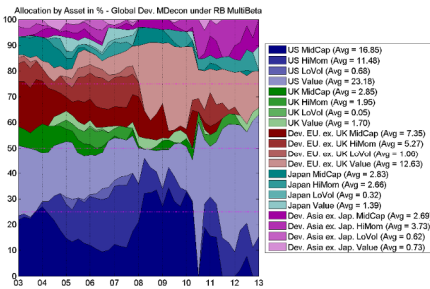
$$\text{Min}_{(w_i)_{i=1, \dots, N}} v(w) \equiv w' C w \text{ such that } c_i^{\text{var}}(w) = c_j^{\text{var}}(w), \text{ for all } 1 \leq i, j \leq K$$

We may also maximize portfolio deconcentration, measured by the effective number of constituents, again subject to factor risk parity constraints⁵:

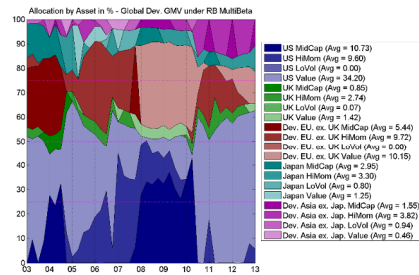
$$\text{Max}_{(w_i)_{i=1,\dots,N}} ENC \equiv \frac{1}{w'w} \text{ such that } c_i^{\text{var}}(w) = c_j^{\text{var}}(w), \text{ for all } 1 \leq i, j \leq K$$

Figure 17.4 shows max-deconcentration and GMV allocations under risk parity as well as geographical constraints.

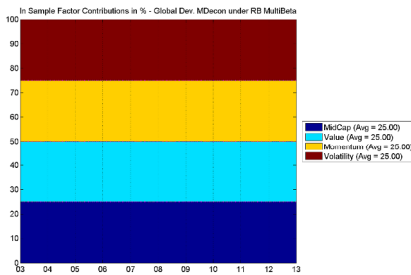
Weight Distribution – Max Deconcentration with factor risk constraints



Weight Distribution – GMV with factor risk constraints



In-Sample Factor Contribution - Max Deconcentration with factor risk constraints



In-Sample Factor Contribution - GMV with factor risk constraints

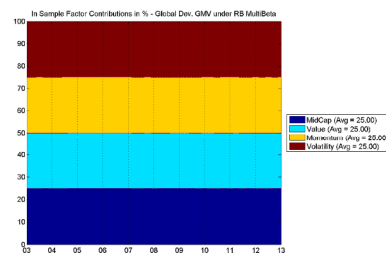


Figure 17.4. Max deconcentration and GMV allocations under risk factor and geographical constraints (developed universe). The graph shows the allocations and factor contributions of the max-deconcentration and GMV diversified multi-strategy indices invested in the 20 diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value in the five US, UK, developed Europe (e.g. UK), Japan and Asia Pacific (e.g. Japan) subregions. Both risk parity and geographical constraints are imposed onto the resulting portfolios. The period is from 31 December 2003 to 31 December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

⁵ Of course, in the absence of constraints, maximizing deconcentration simply leads to giving a weight of 1/N to each constituent in the universe.

First, we note that factor risk parity is satisfied, and that the portfolio is no longer simply invested in the low volatility constituents. Similarly to the allocation we obtained in the previous case, we also note that the aggregated weights in the different subregions appear to represent the subregion market capitalizations more fairly due to the presence of regional constraints. We also note that the max-deconcentration approach shows a more stable allocation over time compared to the GMV, which is still sensitive to changes in input parameters. Also, we see that the addition of factor risk parity constraints forces the allocations to spread the country weight more evenly among the different tilts.

Table 17.2 reports the risk and returns characteristics of various multi-smart-beta allocation portfolios, and compares the results. We note that the GMV allocation process leads to the lowest volatility. Also, we note that the EW and ERC allocations have higher returns and higher volatilities than the GMV, as is often the case. We note further that the introduction of factor risk parity constraints has led to a substantial improvement in information ratios with an information ratio above 1 for the max-deconcentration allocation under geographical and risk parity constraints. This shows that the introduction of factor risk parity constraints leads to a stabilization of the portfolio that has resulted in strong outperformance (3.37%) over the CW index, with a tracking error barely greater than 5%. The introduction of factor risk parity constraints leads to 100% outperformance probabilities over a 3-year horizon.

Developed (2004–2013)	CW (all stocks)	Diversified Multi-Strategy				
		Multi-beta EW allocation	Multi-beta ERC allocation	Multi-beta GMV allocation	Multi-beta max deconc. fact allocation	Multi-beta GMV-fact allocation
Ann returns	7.80%	11.37%	11.07%	10.57%	11.17%	10.88%
Ann volatility	17.09%	15.32%	14.33%	12.84%	17.23%	17.21%
Sharpe ratio	0.36	0.64	0.66	0.70	0.56	0.54
Max drawdown	57.13%	54.40%	51.82%	45.07%	55.22%	55.32%
Excess returns	-	3.56%	3.27%	2.76%	3.37%	3.07%
Tracking error	-	6.75%	7.51%	6.36%	3.08%	3.34%
95% Tracking error	-	13.84%	14.89%	11.85%	5.19%	5.55%
Information ratio	-	0.53	0.44	0.43	1.09	0.92
Outperf. prob. (3 years)	-	98.36%	89.34%	89.07%	100.00%	100.00%
Max rel. Drawdown	-	6.35%	9.54%	13.10%	4.03%	5.47%

Table 17.2. Multi-beta allocations across smart factor indices (developed universe). The table compares the performance of the EW, ERC and GMV and both the max-deconcentration and GMV diversified multi-strategy indices under geographical and risk parity constraints, invested in the 20 diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value in the five US, UK, developed Europe (e.g. UK), Japan and Asia Pacific (e.g. Japan) subregions. The period is from 31 December 2003 to 31 December 2013 (10 years). Outperformance probability is the probability of obtaining positive excess returns over CW if we invest in the strategy at any point in time for a period of 3 years. It is computed as the frequency of positive values in the series of excess returns assessed over a rolling window of 3 years and step size of 1 week covering the entire investment horizon

In Table 17.3, we analyze the performances in bull versus bear market regimes (defined as positive versus negative returns for the CW index). We observe that the addition of risk parity constraints to the GMV allocation tends to stabilize the returns across market conditions. For example, in the absence of factor risk parity constraints, the GMV allocation leads to a massive outperformance of 11.94% with respect to the CW index in bear markets, which is due to the almost exclusive domination of the low-volatility factor, with a defensive bias that proves extremely useful in such market conditions. However, the relative return in bull market is negative at -3.90% due to the performance drag associated with exclusively holding defensive equity exposure in bull market conditions. In this context, one key advantage of the introduction of factor risk parity constraints is that it leads to a much more balanced return profile across market conditions with positive outperformance in both bear and bull markets (at 2.66 and 3.18%, respectively).

We have shown that simple allocations that do not balance their exposures to the factors may be too exposed to the low volatility factor, which may lead to lower relative returns with respect to the cap-weighted index, particularly in bull market regimes.

Developed (2004–2013)	Diversified multi-strategy				
	Multi-beta EW allocation	Multi-beta ERC allocation	Multi-beta GMV allocation	Multi-beta max deconc. fact allocation	Multi-beta GMV-fact allocation
Ann. ret. bull	31.58%	29.55%	25.18%	32.89%	32.26%
Ann. vol. bull	11.71%	11.09%	9.42%	12.85%	12.85%
Ann. rel. ret. bull	2.50%	0.48%	-3.90%	3.81%	3.18%
Tracking error bull	5.03%	5.94%	5.06%	2.53%	2.74%
Ann. ret. bear	-24.51%	-22.42%	-17.23%	-26.67%	-26.50%
Ann. vol. bear	21.33%	19.79%	18.40%	24.42%	24.38%
Ann. rel. ret. bear	4.65%	6.74%	11.94%	2.50%	2.66%
Tracking error bear	9.64%	10.27%	8.62%	4.08%	4.43%

Table 17.3. Multi-beta allocations across smart factor indices in bull/bear regimes (developed universe). The table compares the conditional performance of the EW, ERC and GMV and both the max-deconcentration and GMV diversified multi-strategy indices under geographical and risk parity constraints, invested in the 20 diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value in the five US, UK, developed Europe (e.g. UK), Japan and Pacific Asia (e.g. Japan) subregions. The period is from 31 December 2003 to 31 December 2013 (10 years). The quarters with positive market returns are considered bullish and the quarters with negative returns are considered bearish

17.2.3. Long-term evidence in the USA universe

The limited availability of data in the global stock universe caused us to restrict the analysis to a 10-year period. In order to test the robustness of the allocation schemes, we replicate the allocations in the US stock universe for which data are available for 40 years. This period consists of varying degrees of market environments and therefore allows us to look at the performance of different allocations over time through a conditional analysis tool.

The first observation from Table 17.4 is that the results for the USA are similar in nature to those for developed. All allocations outperform the CW benchmark by a large margin (>3.8%). As expected, the information ratio of factor-risk-parity-constrained max deconcentration is 0.81, as compared to 0.76 for EW allocation, showing that the constraints fulfill their long-term objective. Table 17.5 shows that all allocations are quite stable across different market conditions. They are able to outperform the CW benchmark in both bull and bear market conditions.

USA long term (1973–2012)	CW (all stocks)	Diversified multi-strategy		
		Multi-beta EW allocation	Multi-beta ERC allocation	Multi-beta max deconc. fact allocation
Ann returns	9.74%	13.72%	13.63%	14.01%
Ann volatility	17.47%	15.75%	15.67%	16.41%
Sharpe ratio	0.24	0.52	0.52	0.52
Max drawdown	54.53%	53.86%	53.62%	56.56%
Excess returns	-	3.98%	3.89%	4.27%
Tracking error	-	5.23%	5.25%	5.27%
95% Tracking error	-	8.95%	9.10%	8.69%
Information ratio	-	0.76	0.74	0.81
Outperf. prob. (3 years)	-	80.38%	80.43%	78.83%
Max rel. drawdown	-	33.65%	43.46%	33.87%

Table 17.4. Multi-beta allocations across smart factor indices (US universe). The table compares the performance of the EW, ERC and max-deconcentration indices with risk parity constraints, invested in the four diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value in the US. The period is from 31 December 1972 to 31 December 2012 (40 years). Outperformance probability is the probability of obtaining positive excess returns over CW if we invest in the strategy at any point in time for a period of 3 years. It is computed as the frequency of positive values in the series of excess returns assessed over a rolling window of 3 years and step size of 1 week covering the entire investment horizon

USA long term (1973–2012)	Diversified multi-strategy		
	Multi-beta EW allocation	Multi-beta ERC allocation	Multi-beta max deconc. fact allocation
Ann. ret. bull	34.83%	34.57%	36.14%
Ann. vol. bull	12.94%	12.85%	13.42%
Ann. rel. ret. bull	3.03%	2.76%	4.34%
Tracking error bull	4.45%	4.46%	4.54%
Ann. ret. bear	–20.17%	–20.04%	–21.13%
Ann. vol. bear	20.23%	20.14%	21.14%
Ann. rel. ret. bear	4.83%	4.96%	3.87%
Tracking error bear	6.57%	6.58%	6.53%

Table 17.5. Multi-beta allocations across smart factor indices in bull/bear regimes (developed and USA universe). The table compares the performance of the EW, ERC and max-deconcentration indices with risk parity constraints, invested in the four diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value in the US. The period is from 31 December 1972 to 31 December 2012 (40 years). The quarters with positive market returns are considered bullish and the quarters with negative returns are considered bearish

17.3. Relative risk perspective

Many investors are seeking to improve the performance of their equity portfolios by capturing exposure to rewarded factors. Investors may thus explore a variety of portfolio strategies which can be regarded as robust attempts at generating an efficient strategic factor allocation process in the equity space for different sets of objectives and constraints. Allocation can be done in the most simple manner, such as equal dollar contribution/equal weighting (EW), or ERC, or in a more sophisticated manner of diversification, such as volatility minimization (GMV). For the objectives involving risk parameters, allocation methods can broadly be categorized into two groups depending on the risk dimension they deal with – *absolute risk* and *relative risk*. In this chapter, we focus on allocation across smart factor indices from relative risk perspective.

It is often the case that investors maintain the cap-weighted index as a benchmark, which has the merit of macroconsistency and is well understood by all stakeholders. In this context, a multi-smart-beta solution can be regarded as a reliable cost-efficient substitute for expensive active managers, and the most relevant perspective is not an absolute return perspective but a relative perspective with respect to the cap-weighted index. In what follows, we focus on two approaches:

– *naive diversification*: a relative equal risk allocation (R-ERC) portfolio, which focuses on equalizing the contribution of the smart factor-tilted indices to the portfolio tracking error;

– *scientific diversification*: a relative global minimum variance portfolio (R-GMV), also known as minimum tracking error portfolio, which focuses on minimizing the variance of the portfolio relative returns with respect to the cap-weighted index.

It should be noted that controlling for factor exposure biases from an absolute risk-budgeting perspective is useful, but this is no longer a key required ingredient since the CW index already provides a proper anchor point that is an implicit, as opposed to explicit, reference set of factor exposures. In the same manner, we find that regional constraints are no longer needed, since a portfolio seeking to equalize the contributions of the 20 constituents to the portfolio tracking error, or seeks to minimize the tracking error, will not lead to a severe overweighting of smaller regions with respect to larger regions, in contrast to what has been found from an absolute risk perspective.

17.3.1. Methodology

Relative ERC is implemented in a way similar to ERC allocation, the only difference being that tracking error contributions are equalized instead of volatility contributions. If we define the contribution of component i to portfolio tracking error as:

$$C_i^{trk}(w) = \frac{\partial \sigma_{p-cw}^2}{\partial w_i} w_i \quad \text{with} \quad \sum_{i=1}^N C_i^{trk}(w) = \sigma_{p-cw}^2$$

– the relative ERC portfolio is defined as the allocation w that satisfies the following identity:

$$\frac{C_i^{trk}(w)}{\sigma_{p-cw}^2} = \frac{1}{N} \quad \text{for all } i$$

– the relative GMV approach follows a mean variance optimization to minimize total portfolio tracking error under long-only constraints. Mathematically, it can be written as (Σ is the covariance of excess returns over the CW benchmark):

$$\text{Min}(w^T \cdot \Sigma \cdot w) \quad \text{subject to} \quad 1^T \cdot w = 1 \quad \text{and} \quad w_i \geq 0 \quad \text{for all } i$$

We discuss the composition and performance statistics of developed and US portfolios. The developed dataset extends over the 10-year period from 31 December 2003 to 31 December 2013 and uses five subregions of the global developed universe: US, UK, developed Europe (e.g. UK), Japan and Asia Pacific (e.g. Japan). Using four smart multi-strategy indices as proxies for the value, size, momentum and volatility-rewarded tilts in each subregion; we obtain a total of $5 \times 4 = 20$ constituents.

17.3.2. Risk contributions and performance

In Figure 17.5, we show the allocations of the relative GMV and relative ERC portfolios. First, we find again that the relative ERC allocation is more stable over time, which is due to the higher sensitivity of the relative GMV allocation to the parameter estimates, confirming a higher degree of robustness with the ERC approach. Even though both allocation strategies rely on risk parameter estimates, scientific diversification tends to overuse input information compared to the more agnostic risk-budgeting diversification, which makes a more parsimonious use of input estimates (see [RON 13] for more details and interpretations for the higher robustness of ERC portfolios with respect to errors in risk parameter estimates).

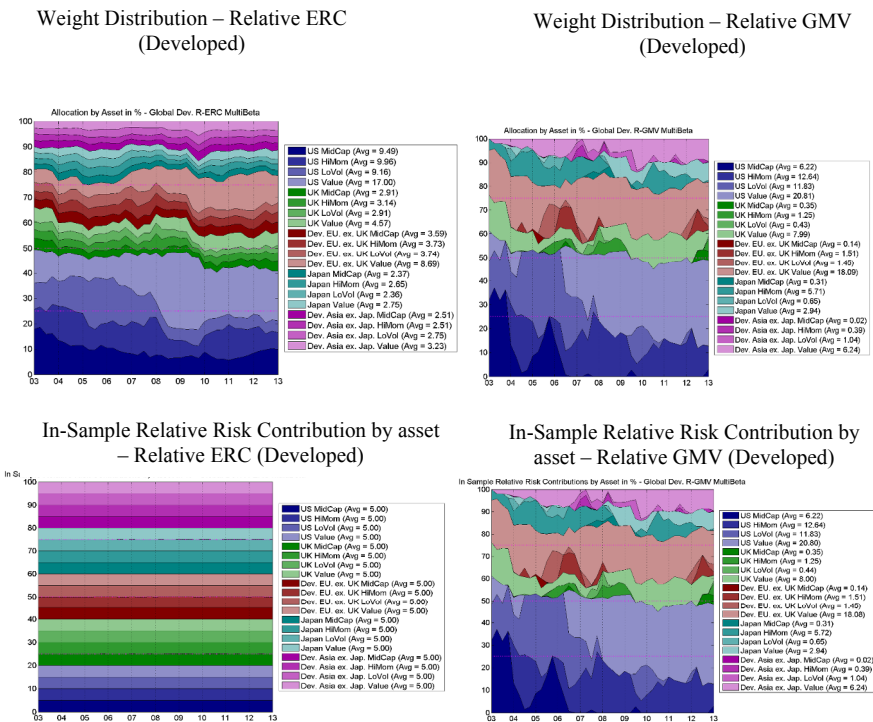


Figure 17.5. Relative GMV and relative ERC allocations to smart factor indices and risk contributions (developed universe). The graph compares the allocation and risk contributions of diversified multi-strategy indices: the relative GMV and relative ERC allocations invested in the four diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value. The period is from 31 December 2003 to 31 December 2013. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Second, by construction, we observe that the relative ERC leads to identical constituent contributions to the tracking error. However, the relative GMV portfolio involves non-equal time-varying contributions from various constituents to the tracking error of the portfolio. This observation is in line with the relative GMV objective, i.e. the components that have large tracking error are underweighted relative to the ones that have lower tracking error.

Developed (2004–2013)	CW (all stocks)	Diversified multi-strategy		Developed (2004–2013)	Diversified multi-strategy	
		Multi-beta relative ERC allocation	Multi-beta relative GMV allocation		Multi-beta relative ERC allocation	Multi-beta relative GMV allocation
Annual returns	7.80%	10.92%	9.96%	Ann. ret. bull	31.38%	31.02%
Annual volatility	17.09%	16.10%	16.64%	Ann. vol. bull	11.95%	12.25%
Sharpe ratio	0.36	0.58	0.50	Ann. rel. ret. bull	2.30%	1.95%
Max drawdown	57.13%	54.14%	55.50%	Tracking error bull	2.09%	2.01%
Excess returns	-	3.12%	2.15%	Ann. ret. bear	-25.25%	-26.93%
Tracking error	-	2.56%	2.43%	Ann. vol. bear	22.88%	23.75%
95% Tracking error	-	4.70%	4.27%	Ann. rel. ret. bear	3.92%	2.23%
Information ratio	-	1.22	0.88	Tracking error bear	3.41%	3.21%
Outperf. prob. (3 years)	-	100.00%	89.34%			
Max rel. Drawdown	-	5.10%	4.95%			

Table 17.6. *Relative ERC and relative GMV allocation (relative to the CW index) across smart factor indices (developed universe). The table compares the performance and risk of scientific beta diversified multi-strategy indices converted into US dollars. We look at relative ERC and relative GMV allocations invested in the 20 diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value in the five US, UK, developed Europe (e.g. UK), Japan and Asia Pacific (e.g. Japan) subregions. Outperformance probability is the probability of obtaining positive excess returns over CW if we invest in the strategy at any point in time for a period of 3 years. It is computed as the frequency of positive values in the series of excess returns assessed over a rolling window of 3 years and step size of 1 week covering the entire investment horizon. The quarters with positive market returns are considered bullish and the quarters with negative returns are considered bearish. The period is from 31 December 2003 to 31 December 2013*

Table 17.6 displays the risk and return characteristics of the relative ERC and GMV allocation strategies. We note that the focus on relative return leads to low tracking error levels. For example, the *ex post* tracking error is around 2.50% for these portfolios. Relative GMV, as per its objective, results in lower tracking error (2.43%) compared to relative ERC (2.56%). However, relative ERC exhibits greater outperformance (+3.12%) compared to relative GMV (+2.15%). Such low tracking error levels, associated with substantial outperformance, eventually lead to exceedingly high information ratios. In particular, the relative ERC has an information ratio of 1.22, which is the highest level among all portfolio strategies

tested so far, with an outperformance probability of 100% over any given 3-year investment horizon during the same period. We also find that the focus on relative risk leads to lower tracking errors in bull and bear market regimes compared to their absolute risk counterparts.

The benefit of exposure to multiple factors can be seen from conditional performance analysis. Both allocations are able to outperform the CW benchmark in both bull and bear market conditions. For example, relative allocation beats the CW benchmark by 2.30% in bull markets and 3.92% in bear markets.

17.3.3. Relative risk allocation using long-term USA factor indices

Since developed track records are limited to a 10-year time period, we use the US stock universe (of the 500 largest market-cap stocks) to redo the relative risk allocation exercise to ensure the robustness of our results. The US universe not only gives us the advantage of a much longer history (40 years) but also limits us to using four smart factor indices (instead of 20 indices in the international domain). For the following illustrations, long-term (40-year) data from 31 December 1973 to 31 December 2013 are used for the four smart multi-strategy indices – mid cap, momentum, low volatility and value. All other construction principles remain the same as before.

Figure 17.6 shows that over long periods, the weight distribution in the ERC allocation remains quite stable. The GMV allocation is relative time varying, overweighting the factor that is responsible for the lowest tracking error each time.

When analyzing the risk and performance indicators in Table 17.7, we observe that the relative GMV, which is supposed to minimize the tracking error, achieves a tracking error of 4.79% compared to relative ERC, with a tracking error of 4.91%. As observed in the case of the developed universe, both allocations result in high outperformance, with relative ERC slightly better (at +3.79%) than relative GMV (+3.71%). The conditional performance over the long term constitutes many market cycles, including the technology bubble and the financial crisis. The fact that both allocations outperform the benchmark in varying market conditions reconfirms the robustness of these strategies.

We find that value can be added through relative ERC and relative GMV at the allocation stage, for investors with a tracking error budget. As a result, extremely substantial levels of risk-adjusted outperformance (information ratios) can be achieved even in the absence of views on factor returns. The portfolio strategies we have presented in this chapter can be regarded as robust attempts at generating an efficient *strategic* factor allocation process in the equity space in the context of

benchmarked investment management. While possibilities for adding value through smart beta allocation are manifold, the robust performance improvements obtained through relative ERC and relative GMV allocations to the four main consensual factors displayed above in this chapter provide evidence that the benefits of multi-factor allocations exist in a context of strong relative risk constraints and are sizable.

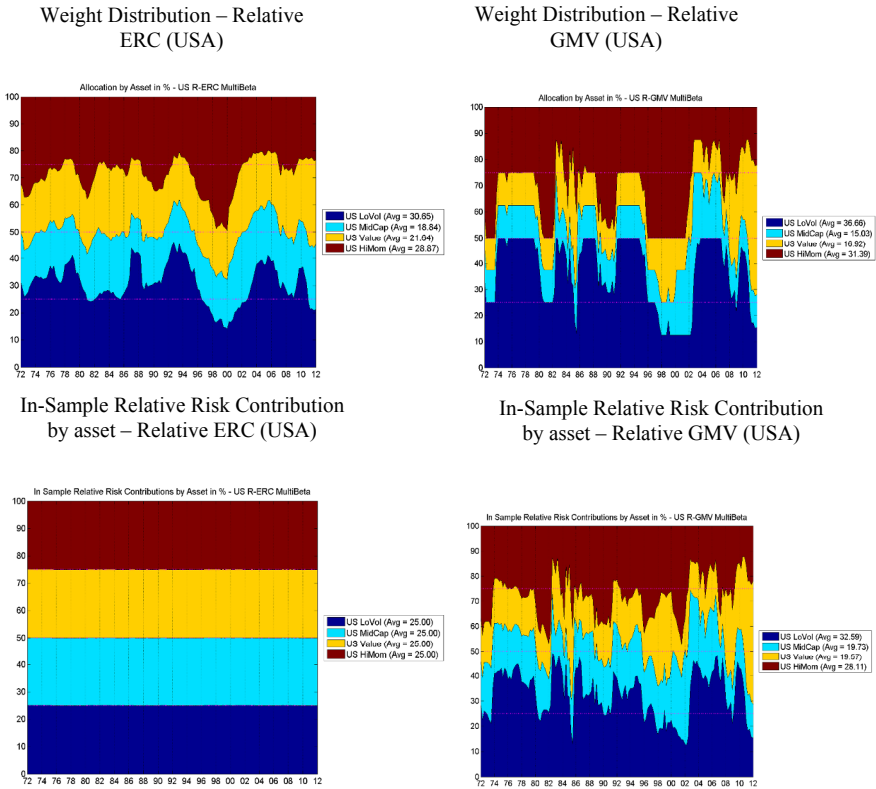


Figure 17.6. Relative GMV and relative ERC allocations to smart factor indices and risk contributions (US universe). The graph compares the allocation and risk contributions of diversified multi-strategy indices: the relative GMV and ERC allocations invested in the four diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value, and the ERC combination of the same four constituents. The relative GMV strategy has been derived with the following additional weight constraints: $1/\delta * N < w < \delta / N$, where $N=4$ constituents and $\delta = 2$. The period is from 31 December 1972 to 31 December 2012. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

US long term (1973–2012)	CW (all stocks)	Diversified multi-strategy		US long term (1973–2012)	Diversified multi-strategy	
		Multi-beta relative ERC allocation	Multi-beta relative GMV allocation		Multi-beta relative ERC allocation	Multi-beta relative GMV allocation
Annual returns	9.74%	13.53%	13.45%	Ann. ret. bull	34.72%	34.39%
Annual volatility	17.47%	15.69%	15.60%	Ann. vol. bull	12.89%	12.97%
Sharpe ratio	0.24	0.51	0.51	Ann. rel. ret. bull	2.92%	2.59%
Max drawdown	54.53%	53.30%	52.64%	Tracking error bull	4.20%	3.78%
Excess returns	-	3.79%	3.71%	Ann. ret. bear	-20.44%	-21.02%
Tracking error	-	4.91%	4.79%	Ann. vol. bear	20.14%	20.15%
95% Tracking error	-	8.11%	7.99%	Ann. rel. ret. bear	4.56%	3.97%
Information ratio	-	0.77	0.77	Tracking error bear	6.12%	5.49%
Outperf. prob. (3 years)	-	80.90%	81.31%			
Max rel. Drawdown	-	28.74%	27.00%			

Table 17.7. *Relative ERC and relative GMV allocation to the CW index across smart factor indices (US universe). The table compares the performance and risk of the scientific beta diversified multi-strategy indices. We look at relative ERC and relative GMV allocations in the four diversified multi-strategy indices with stock selection based on mid cap, momentum, low volatility and value, respectively. All statistics are annualized and daily total returns from 31 December 1972 to 31 December 2012 are used for the analysis. The S&P 500 index is used as the cap-weighted benchmark. Yield on Secondary US Treasury Bills (3M) is used as a proxy for the risk-free rate. Outperformance probability is the probability of obtaining positive excess returns over CW if we invest in the strategy at any point in time for a period of 3 years. It is computed as the frequency of positive values in the series of excess returns assessed over a rolling window of 3 years and step size of 1 week covering the entire investment horizon. The quarters with positive market returns are considered bullish and the quarters with negative returns are considered bearish*

17.4. Conclusion: index design and allocation decisions for multi-factor equity portfolios

We find that well-rewarded factor-tilted indices constitute attractive building blocks for the design of an improved equity portfolio. First-generation smart beta investment approaches only provide a partial answer to the main shortcomings of cap-weighted indices. Multi-strategy factor indices, which diversify away unrewarded risks and seek exposure to rewarded risk factors, address the two main problems of cap-weighted indices (their undesirable factor exposures and heavy concentration) simultaneously.

The results suggest that such multi-strategy factor indices lead to considerable improvements in risk-adjusted performance. For long-term US data, smart factor

indices for a range of different factor tilts roughly double the Sharpe ratio of the broad cap-weighted index. Moreover, outperformance of such indices persists at levels ranging from 2.92 to 4.46%, even when assuming unrealistically high transaction costs. The outperformance of multi-strategy factor indices over cap-weighted factor indices is observed for other developed stock markets as well. By providing explicit tilts to consensual factors, such indices improve upon many current smart beta offerings where, more often than not, factor tilts result as unintended consequences of *ad hoc* methodologies.

Moreover, additional value can be added at the allocation stage, where the investor can control for the dollar and risk contributions of various constituents or factors to the absolute (volatility) or relative risk (tracking error) of the portfolio. As a result, extremely substantial levels of risk-adjusted outperformance (information ratios) can be achieved even in the absence of views on factor returns. The portfolio strategies that we have presented in this chapter can be regarded as robust attempts at generating an efficient *strategic* factor allocation benchmark in the equity space. Obviously, active portfolio managers may generate additional value on top of this efficient benchmark by incorporating forecasts of factor returns at various points of the business cycle in the context of *tactical* factor allocation decisions.

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Factor Investing and Portfolio Construction Techniques

18.1. Introduction

Factor investing has been growing in popularity within institutional investment circles over the last few years. At the same time, risk-based portfolio construction techniques have become more mainstream, particularly following the global financial crisis of 2008. We cannot be blamed for thinking that these two concepts are very closely related: factor investing is about understanding the sources of risk that underlie a particular portfolio and taking investment decisions directly at the factor level, while risk-based portfolio construction techniques can be used to put together risk-factor portfolios. However, we will argue that factor investing is ultimately an “asset allocation” concept (even though it may not be directly used as such), whereas risk-based portfolio construction is a methodology that can be pursued in building risk-factor portfolios; in fact, under certain assumptions, it may actually be the optimal technique. Risk-based portfolio construction methods are readily used outside the risk-factor area, and, at the same time, risk-factor portfolios can be constructed through a number of different approaches such as mean-variance optimization, etc.

In sections 18.2 to 18.5 of this chapter we introduce the concept of risk-factor investing and discuss the practical implementation forms it has taken so far within the institutional asset owner space. In sections 18.6 to 18.9, we introduce two innovative risk-based portfolio construction techniques and compare them with traditional risk-based algorithms.

18.2. Risk factor investing: the new paradigm

An increasing number of asset owners are revisiting their asset allocation practices, focusing on the factors that drive portfolio returns and consequently expressing their asset allocation in terms of risk-factor exposures, as opposed to asset classes. Fundamental to this shift has been the belief that long-horizon investors are compensated for risks they take, and consequently, the decision regarding the types of risk premia they set out to harness should be at the heart of their investment process, rather than a by-product. For example, seemingly well-diversified asset-class portfolios may hide a heavily concentrated set of risk exposures. The classic 60/40 benchmark (60% invested in equities and 40% in bonds) is a case in point, as it is heavily reliant on equity risk. Moreover, asset-class correlations tend to rise during the periods of elevated market volatility and persistent liquidity draughts; in the aftermath of the most recent financial crisis, asset-class correlations remained elevated for quite some time as markets switched between binary risk-on and risk-off modes. On the contrary, correlations between risk factors tend to be lower than asset-class correlations, and in addition, they tend to be more robust to regime shifts.

Risk-factor investing builds on insights that lie at the core of asset pricing theory. In the equity space, for instance, Fama and French [FRE 92] and Carhart [CAR 97] identified a set of stock characteristics (i.e. risk factors) that tend to explain a significant portion of the return variation in the cross-section of large stock portfolios. In addition, the aforementioned studies – and numerous others that followed – presented evidence over different time periods, and across markets, which suggests that individual securities earn risk premiums over time through their exposure to rewarded factors (such as value, momentum, and size).

The notion, however, of employing risk factors at the asset allocation level gained traction when the Norway fund commissioned a study into the disappointing results of active management in 2008 and early 2009, which had wiped out 10 years' worth of cumulative outperformance. The report by professors Ang, Goetzmann and Schaefer [ANG 09] concluded that a significant component of the active risk and performance of the fund could be explained by systematic factors that happened to fare poorly during the financial crisis. In essence, in large portfolios, it is hard to find "alpha"; many mispricing opportunities tend to be small in scale and firm- (or strategy-) specific exposures tend to become swamped at the portfolio level, as many correlated individual exposures become large exposures on factors.

The broader implication of the Ang *et al.* [ANG 09] study is that exposure to such systematic sources of risk and return would actually be appropriate for a

long-horizon investor, as long as the factors earn risk premiums over the long run¹. The authors proposed that risk-factor investing must be a part of the strategic asset allocation toolkit of institutional investors, with factor exposures forming part of the portfolio and benchmark construction, helping trustees in their decision-making process.

A fundamental advantage of risk-factor investing is the opportunity it offers to capture non-traditional sources of premia that underlie rewarded factors, such as behavioral effects, supply/demand imbalances, etc. As portfolio construction aims to harvest the risk premia expected from holding exposure to rewarded factors, while minimizing exposure to unrewarded sources of risk, it seems only natural and theoretically pure to express the asset allocation decision in terms of risk factors rather than asset classes per se. We have introduced risk-factor allocation as the new investment paradigm in our 2012 research note titled “A New Asset Allocation Paradigm”², arguing in favor of allocating to uncorrelated, rewarded risk factors. Of course, to exploit the benefits of the approach fully, shorting is required to take out the market (as long-only factors would usually deliver returns close to the market return), as well as leverage to render the returns from well-diversified risk-factor portfolios meaningful.

18.3. Theory meets practice

Essentially, risk-factor investing involves looking through the asset-class “labels” to the systematic factors lying beneath the assets. The main purpose of asset allocation through the factor “lens” is to allow for a clearer demarcation of risk exposures, a better attribution of returns according to the risks taken in the portfolio and potentially a significant improvement in risk-adjusted returns through diversification by the underlying return drivers (risk premia “clusters”).

In practice, risk-factor investing has taken two (non-mutually exclusive) forms in the institutional asset owner space. We define these as the “*fundamental risk class*” and the “*risk premia*” approaches to asset allocation, respectively. Both terms have been used in practitioner journals or have appeared in practitioner magazines, and although sometimes used interchangeably, more often than not they are underpinned

¹ A risk factor does not necessarily need to be compensated by the market, whereas a risk premium does. Although a risk factor can help explain the cross-sectional variation of returns within or across asset classes, investors looking to maximize their wealth are interested in holding compensated risk premia. Risk factors are positively rewarded if, and only if, they perform poorly during “bad” times (like they did during the financial crisis for the Norway fund), but more than compensate during good times so as to generate a positive excess return, on average. The incidence of “bad” times is, of course, premium-specific.

² See [MES 12].

by a subtly different investment process. The “*fundamental risk class*” approach is closer to traditional asset allocation than the “*risk premia*” approach; the main difference to traditional asset allocation being that portfolio allocations are decided at the *risk class* level and *subsequently* flow down to the constituent asset of each risk class. In addition, risk management and portfolio attribution are performed at the risk class level rather than the asset-class level.

The *fundamental risk class* approach focuses on sources of portfolio risk that can be fundamental in nature, such as growth, inflation and liquidity. However, these are very hard to measure. Even so, some asset owners over the last few years have started thinking in terms of fundamental “risk classes³”, typically using assets as proxies, like equities for growth and commodities for inflation. For example, the strategic asset allocation of the Alaska Permanent Fund categorizes asset classes motivated by how they respond to different macro-factor risks. Effectively, Alaska’s macro-factor investing looks through asset-class labels to the underlying fundamental risks the fund is exposed to. The fund allocates to four *risk classes*. These are (see [INK 09]):

a) *company exposure* – consists of investments that tend to perform well in periods of economic growth (stocks, corporate investment grade and high yield bonds, bank loans and private equity);

b) *interest rates and cash* – intended to address to address deflation and market crises. U.S. government bonds, non-U.S. government bonds, and liquid investments with durations of less than 12 months comprise this risk class;

c) *real assets* – purpose is to hedge inflation risk and protect the real value of the Fund by investing in real estate, infrastructure, and TIPS;

d) *special opportunities* – this allocation allows the Permanent Fund to take advantage of market dislocations by, for instance, investing in absolute return strategies, special deals for distressed assets, or in illiquid investment opportunities which Alaska can exploit because of its long investment horizon.

Another good case in point is the Danish pension fund Arbejdsmarkedets TillægsPension (ATP). In order to ensure that the return stream of its investment portfolio is as stable and as independent of economic trends as possible, ATP’s portfolio has been invested in five risk classes with different risk profiles⁴. These risk classes are the interest rate risk class (proxied by interest-rate sensitive bonds), the credit risk class (the ability of issuers to repay debt obligations), the equity risk class, the inflation risk class (where part of the risk budget is allocated to liquid

³ See [ANG 14].

⁴ See [JES 11].

investments such as index-linked bonds and inflation swaps, and the rest to illiquid investments such as real estate, infrastructure, alternative energy and forestry⁵) and the commodity risk class (proxied by oil futures). Essentially, this approach has entailed investing in (long-only) asset classes in a rather stable and non-dynamic way, which has been deemed appropriate to deliver the risk premium targeted by each risk cluster – a real-life example of asset allocation through the (fundamental) risk-factor “lens”. The five risk-class portfolios provided a cushion for ATP during the global financial crisis of 2008, as portfolio diversification worked well, mainly due to the government bond portfolio rising in value at the same time as the price of equities fell.

However, more recently, and as reported by the Investment & Pensions Europe Magazine⁶, ATP has been seeking to further refine its approach in an investment environment of low prospective asset-class returns and potentially poor portfolio diversification: interest rates are expected to rise in the future from their current low levels, and a situation where risky asset classes such as equities and credit fall at the same time as interest rates are rising is not unfathomable. The five risk classes could perform poorly at the same time. To address these concerns, and following the integration of the unit which used to manage hedge fund strategies (ATP Alpha) with the broader investment platform at the end of 2012, the pension fund appears to be embracing the alternative approach of risk-factor investing. This is viewed as a way of gaining access to truly uncorrelated (to asset-class beta) sources of return, such as value, momentum and carry. It entails adopting a long-short approach to isolate the premium in question, and consequently a more dynamic investment style. We label this long-short, dynamic approach to risk-factor investing as the “*risk premia*” allocation approach.

A pioneer of the “*risk premia*” approach has been PKA, a Danish administrator for five pension funds. PKA began to implement a strategy designed to get exposure to specific risk premia in equities back in 2012. Instead of simply allocating in a long-only manner to traditional global, regional or sector equity portfolios, PKA’s strategy combines developed, emerging, frontier and small-cap mandates (traditional long-only beta) with around 15 alternative sources of risk and return implemented using dynamic long-short strategies and derivatives⁷. PKA has expanded aspects of the “*risk premia*” approach applied to equities into rates, currencies and commodities markets. Another adopter (albeit to a lesser extent and scope compared to PKA) of the “*risk premia*” approach is Swedish pension fund SPK which has

5 Apart from stable inflation-adjusted cash flows, the aim of illiquid investments is also to harness an illiquidity premium.

6 See [LII 14].

7 See [STE 14].

allocated 8% of its total assets to dynamic factors in 2014, considering this allocation as an integral part of its total portfolio⁸.

As can be gathered from the above discussion, “risk-premia” investing is not incompatible with the “fundamental risk class” approach to asset allocation and portfolio construction; rather, it can be regarded as a refinement. Although more challenging in implementation and “cutting across” asset classes, it makes use of the same principle of allocating to unique sources of risk and return, rather than assets per se, to achieve better risk-adjusted performance and superior portfolio diversification.

18.4. Taxonomy of risk premia strategies

Academics have been studying markets for years trying to identify persistent, systematic sources of return, their efforts often culminating in strategies that lacked robustness. Nevertheless, a number of investment styles have been identified (and widely supported by academics and practitioners alike) that seem able to deliver consistent long-term performance across many unrelated asset classes, during different market cycles, and in out-of-sample tests⁹. Value, momentum and carry are traditional risk factors that fit the above description. Most importantly, they are backed by sound economic reasoning: the rationale for the existence of the persistent returns of each style across asset classes is economic, behavioral or institutional/structural in nature¹⁰.

Although asset-class premia are usually measured in terms of excess returns over “riskless” Treasury bills (T-bills) and require limited portfolio rebalancing (e.g. a value-weighted market index requires some asset turnover due to, for example, new issues and new entries to an index), style premia are more dynamic in nature, typically reflecting the excess returns of one risky-asset portfolio over another risky-asset portfolio. In principle, we would like risk premia strategies to be uncorrelated with the underlying asset-class return (i.e. to be market-neutral), which is attempted by constructing the strategies in a long-short format.

For example, value seeks compensation for selecting relatively undervalued assets and shorting relatively overvalued assets in a long-term mean reversion trade. There exists a premium in this strategy for accepting the uncertainty regarding the time horizon over which the convergence trade will play out, giving up potential “upside” in the interim. Carry strategies go long of high-yielding assets and short of

8 Others have treated a bit more cautiously and created a separate bucket for dynamic long-short risk premia strategies, like some of the Swedish AP funds (see [LII 15]).

9 See, for instance, [ASN 13].

10 For a more detailed exposition of the theories that have been brought forward to rationalize the value, momentum and carry risk premia see [MES 12].

low-yielding assets from the same asset class, seeking a relative compensation for taking on the higher risk associated with higher yielding assets. The momentum premium is associated with accepting the risk of reversals in strategies that go long of past winners and short of past losers from the same asset pool, in the belief that market participants will extrapolate past trends into the future.

The style-based risk premia universe can be broadened further to include another two categories that permeate asset-class boundaries too. The first category relates to the volatility risk premium, underpinned by the growth in derivative product markets and related microstructure issues, as well as supply/demand imbalances. To be more precise, systematically selling volatility to generate carry could also be grouped together with the rest of the carry premia, however, we believe that the common properties of volatility – across asset classes – of persistence and mean-reversion¹¹, the more elaborate premium extraction strategies often involved for the implied versus realized volatility spread, as well as the insurance-like features of the volatility premium, justify the designation of volatility into a class of its own¹². On the downside, data availability for the (short) volatility risk premium are limited compared to the more traditional risk factors (academic evidence in favor of which stretches back many decades in some cases).

The second category comprises systematic anomalies that are rather incompatible with the notion of offering compensation for some type of economic risk, and seem to be purely related to behavioral or market structure issues. Two of these “anomalies” in the equity space are quality and low beta, which we classify under the heading of low risk. Such strategies capitalize on the fact that lower beta stocks or stocks with lower fundamental risk (higher quality) offer a better risk-adjusted return than higher risk stocks. There is increasing evidence that the low risk anomaly exists in asset classes other than equities, most predominantly in fixed income¹³.

11 Volatility tends to stay low for long periods, while transitions to a high volatility regime tend to be sudden and difficult to predict.

12 Investors tend to be divided between those with a preference for long volatility exposure as a form of portfolio insurance, and those who prefer to take the opposite side of this trade and capture an embedded risk premium over time. The payout profile of a short volatility strategy involves frequent smaller profits punctuated by occasional larger losses. The most obvious parallel to this is the insurance market where premiums are gathered on a continual basis and used to cover occasional large losses; profits are made if the insurance premiums are set at a level that exceeds losses over time.

13 See [LEO 14]. The authors present some compelling empirical evidence in favor of a low-risk anomaly in government bonds, quasi-government bonds, securitized and collateralized bonds, corporate investment grade and high yield bonds, emerging market bonds, and aggregates of some of these universes. The results proved invariant to currency and appeared robust to using different measures of risk.

Overall, we believe there are three dimensions to risk premia strategies: the *style* of investing dimension, the *asset class* dimension (where style strategies select securities from within the same asset pool) and a *hybrid* dimension (where style strategies go long/short assets themselves, like in trend investing)¹⁴.

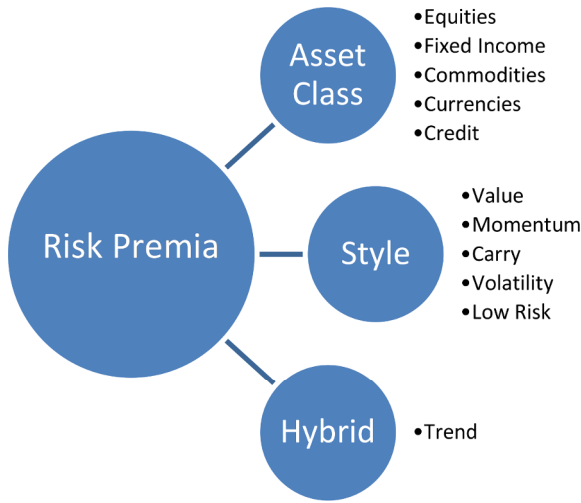


Figure 18.1. *Taxonomy of risk premia strategies (source: Deutsche Bank Quantitative Strategy)*

Portfolios of risk premia strategies can be built either independently, i.e. by allowing allocations to flow freely between the individual strategies, or by first restricting the allocations along any of the aforementioned three dimensions (followed by subsequent combination of the different “buckets”). Building risk premia portfolios within an asset class or style “bucket” first has the advantage of improving estimation efficiency by reducing the dimensionality of the problem at hand. To this end, it may prove even more powerful to opt for an empirical form of “bucketing” of the risk premia strategies. We could think of “bucketing” as a half-way house between building a risk-factor model and employing a factor variance-covariance matrix for estimation as opposed to using the individual asset-by-asset covariance matrix¹⁵:

¹⁴ See [NAT 13].

¹⁵ With a large set of strategies we may consider clustering techniques to assign strategies into homogeneous groups, taking into account the full distributional characteristics of each factor.

– *bucketing by asset class*: this is simple, but may result in foregone diversification benefits by not allowing factors to diversify across asset classes; for example, the weight on equity value would only depend on how it interacts with the other equity factors, rather than its relationship with factors across asset classes;

– *bucketing by style*: while conceptually appealing for combining volatility or momentum strategies, other style “groups” may not have common return drivers across asset classes. Moreover, some strategies (such as quality and size) may not fall into predefined style definitions;

– *bucketing by degree of cyclicity*: another way of thinking about the risk premia space is grouping strategies by the degree in which they exhibit cyclical behavior with the equity market over the long run. This classification may result in an empirically more homogeneous grouping of the strategies. Historically, and irrespective of the exact factor implementation, strategies that try to extract the (implied) volatility premium using derivatives instruments, as well as certain other factors such as FX Carry, have more procyclical tendencies than, say, momentum/trend strategies and factors such as equity quality.

Empirical support for “bucketing” strategies according to their degree of cyclicity with the equity market is provided by Principal Component Analysis (PCA) on the historical correlations of our risk premia universe provides empirical support for grouping strategies according to their degree of cyclicity. This is particularly true since 2000 when the vast majority of factors enter our database (we require at least 5 years of history before adding a factor returns series to our analysis). The PCA analysis is carried out on monthly return series for 17 underlying long-short risk premia strategies across asset classes¹⁶.

Of course, as the underlying investments are long-short strategies across asset classes capturing different sources of risk and return, the first PCA (PC1) explains only about 20% of the variation in the returns of the risk-factor strategies; nevertheless, the returns of PC1 are remarkably highly correlated with equity returns since 2000 (about 80%, see Figure 18.2), suggesting that whatever common variation there exists in the performance of our risk premia strategies, it can be largely explained by movements in the equity market. Last, but not least, the factors that we have identified before as procyclical have a positive loading to PC1, while the counter-cyclical factors load negatively on PC1, respectively.

¹⁶ Details regarding factor construction and the underlying data of our risk premia universe are available from the authors upon request.

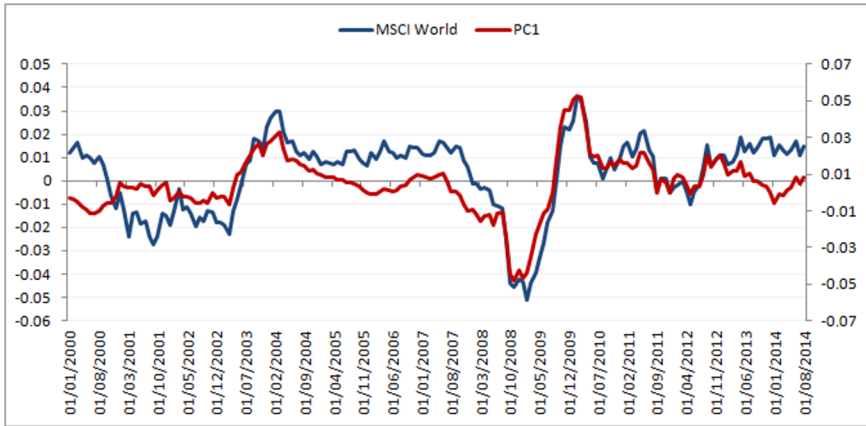


Figure 18.2. Monthly returns of the MSCI World Index and the first principal component (PC1) since January 2000 (source: Bloomberg Finance LLP, MSCI, Factset, Thomson Reuters, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

18.5. Is risk premia allocation inherently superior to asset-class allocation?

Yes for some, no for others. Some papers (such as [BEN 10] and [PAG 11]) have argued that risk-factor-based asset allocation is inherently superior to allocation based on asset classes. Idzorek and Kowara [IDZ 13] criticize the aforementioned authors for not offering real proof of their argument and carrying out an apples-to-oranges comparison between a relatively simple asset class set and a risk-factor set that includes many more potential exposures. Furthermore, many of the presumed gains result from the fact that comparisons involve different inherent constraints, as for risk factors the long-only constraint is relaxed. On the contrary, in a simplified world in which the number of factors equals the number of assets, the asset class returns are completely determined by the risk factors, and the risk factors are completely determined by the asset-class returns (i.e. there is a one-to-one mapping between asset class and risk-factor returns). Idzorek and Kowara present a simple mathematical proof that there is no gain in efficiency from performing unconstrained optimizations in risk-factor space compared to the asset class space. The empirical evidence they produce seems to point in the same direction [IDZ 13].

Nevertheless, Ilmanen and Kizer [ILM 12] show that although imposing a long-only constraint to risk-factor portfolios reduces the effectiveness of risk-factor-based

asset allocation, it does not eliminate it. Their findings concur with those of Blitz [BLI 11] that the benefits of introducing explicit factor allocations to asset class portfolios are meaningful even for long-only investors. Therefore, relaxing the long-only constraint when moving from an asset-class based to a risk-premia-based allocation framework cannot account for the total incremental benefit earned.

Our own research and analysis suggests that introducing risk premia portfolios alongside traditional asset-class beta appears to confer significant benefits in terms of risk-adjusted return and left tail behavior, irrespective of the portfolio construction technique adopted. Factor diversification is more effective at reducing portfolio volatility and market directionality than asset class diversification, and is probably the best answer for many investors whose portfolio risk is dominated by (stock) market directionality¹⁷.

On the flipside, full adoption of a risk-factor-based asset allocation model by large institutional asset owners – particularly where risk factors are defined and captured through dynamic strategy styles – presents itself with practical challenges. The first is that most institutional investors are still uncomfortable with positions involving shorting and leverage that are implied by risk factors, and which are needed in order to fully harness the benefits of this approach. Second, it is not possible for the whole world to completely embrace risk premia allocation, as most factors require offsetting long and short positions; it is not possible for the whole world to be, for instance, simultaneously long the size premium (which goes long small-cap and shorts large-cap stocks), as this would require everyone to short large-caps. Not all investors may hold the same portfolio should they wish to do so, and capacity considerations inevitably surface.

18.5.1. Efficient frontier analysis

We proceed to create two efficient frontiers – one comprising only asset classes that tend to appear in the asset allocation portfolio of an endowment fund (equities, fixed income, commodities, private equity, real estate and hedge funds), and the other containing our risk premia strategies. The Endowment proxy portfolio has been built using an annual average of the allocations (assumes annual rebalancing) of the Yale Endowment and Harvard Endowment policy portfolios respectively, between 2005 and 2013. The allocations are obtained from the annual report

¹⁷ See, for instance, [MES 12], where they have shown that even the endowment model which emphasizes allocation to alternative assets such as real estate and private equity, as well as hedge funds, failed to protect portfolios during (and in the aftermath) the global financial crisis in 2008 as the embedded – in alternatives – equity and liquidity risk consumed any diversification potential among the different asset classes.

documents of the aforementioned endowments¹⁸. The 60/40 portfolio was built assuming 60% notional invested in Global Equities, proxied by the MSCI World Price Return Index, and 40% invested in Bonds (represented by the JP Morgan Global Government Bond Index Hedged to USD), with monthly rebalancing.

Because we are dealing with investments with asymmetric and fat-tailed risk, for the most part, we create efficient frontiers based on both standard deviation and conditional value at risk (CVaR) (see Figure 18.3). It is important to emphasize that the mean-variance and mean-CVaR efficient frontiers have been created in-sample, that is with full knowledge of the realized asset class/risk-factor returns time series.

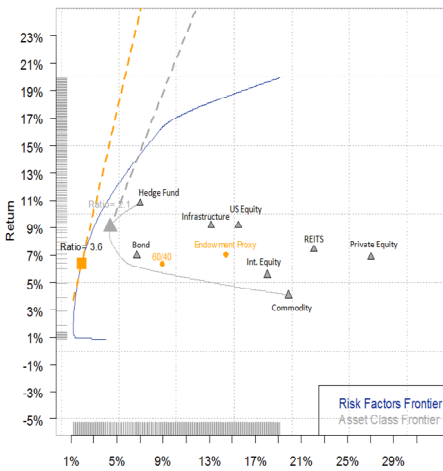
Before discussing the efficient frontier analysis, it is interesting to observe that the 60/40 portfolio (which refers to the “classic” asset allocation benchmark of 60% equities/40% bonds) lies closer to the asset-class efficient frontier compared to the Endowment proxy portfolio (see Figure 18.3: Panel A and Panel B). The 60/40 portfolio has had a lower return and lower risk compared to the Endowment proxy. This is partly a function of the fact that some of the time series we use to proxy the asset classes with, such as private equity, have relatively short time series histories, and are thus heavily influenced by the global financial crisis in 2008/2009.

It is obvious that the set of mean-variance and mean-CVaR optimal cross-asset risk-factor portfolios (*risk factors frontiers* in Figure 18.3) would have significantly outperformed the corresponding optimal mean-variance and mean-CVaR portfolios that employ asset classes only (*asset class frontiers* in Figure 18.3). The tangency risk premia portfolio generates a Sharpe ratio of 3.6 versus a Sharpe ratio of 2.1 for a portfolio comprising of asset classes (both standard and alternative). The mean-variance tangency risk premium portfolio has a significantly lower volatility and somewhat lower return compared to the tangency asset-class portfolio. Equivalently, the return to CVaR ratio of the risk-factor tangency portfolio stands at 2.1 versus 1.1 for the corresponding asset class tangency

18 For example, the 2013 documents can be found at www.hmc.harvard.edu/docs/Final_Annual_Report_2013.pdf and investments.yale.edu/images/documents/Yale_Endowment_13.pdf, respectively. The 2013 average allocations of the Harvard and Yale Endowments were as follows: 16.5% allocation to the absolute return/hedge fund asset class, proxied with the HFRI Fund of Funds fund-weighted composite index (Bloomberg Ticker HFRIFWI), 8.5% allocation to domestic equity (for which we use the MSCI US Total Return Index, Bloomberg Ticker NDDUUS), 16% to foreign equity (represented by the MSCI All Country World-ex-US Total Return Index, Bloomberg Ticker NDUEACWZ), 8% to fixed income (we use as proxy the JP Morgan Global Government Bond Index Hedged to USD, Bloomberg Ticker JHDCG10R), 11.5% to natural resources (proxies are the S&P Goldman Sachs Commodity Index, Bloomberg Ticker SPGCC1), 24.2% to private equity (proxied by the S&P Listed Private Equity Index, Bloomberg Ticker SPLPEQTR) and 15.3% to real estate (FTSE NARET Global REITS Index, Bloomberg Ticker TENHGU).

portfolio. It is interesting to note that, on the risk side, the CVaR reduction in moving from the tangency mean-CVaR asset-class portfolio to the corresponding mean-CVaR risk-factor portfolio is higher compared to the volatility reduction achieved when the tangency mean-variance portfolios are likewise compared. This suggests that a significant degree of tail diversification can be achieved in portfolios of risk factors.

Panel A: historical MV frontier



Panel B: Historical MCVaR frontier

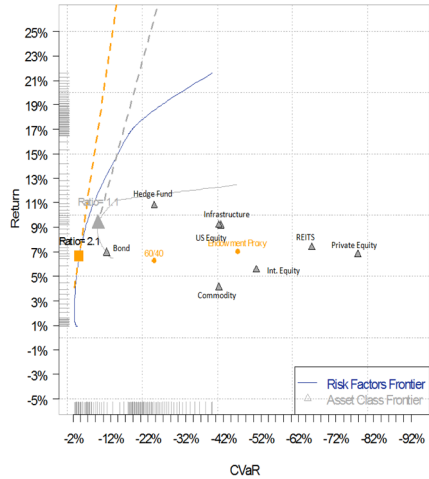


Figure 18.3. Efficient frontiers (source: Deutsche Bank, Bloomberg, Factset, MSCI). We use proxies for the asset classes as follows. Bonds: JP Morgan Global Government Bond Index Hedged to USD (Bloomberg Ticker JHDCG10, start date February 1998). Hedge Fund: HFRI fund-weighted single manager non-FOF Index (Bloomberg Ticker HFRIFWI, start date January 1990). US Equity: MSCI USA Total Return Index (Bloomberg Ticker NDDUUS, January 1970). Intl. Equity: MSCI All-Country World Ex-USA Total Return Index (Bloomberg Ticker NDUEACWZ, start date January 1999). Commodity: S&P GSCI Total Return Index (Bloomberg Ticker SPGCC1, start date February 1970). Infrastructure: UBS World Infrastructure & Utilities Total Return Index (Bloomberg Ticker UIAUGLTR, start date January 1995). Real Estate: FTSE NARET Global REITS Index (Bloomberg Ticker TENHGU, start date March 2005). Private Equity: S&P Listed Private Equity Total Return Index (Bloomberg Ticker SPLPEQT, start date December 2003). Average annualized returns over the whole history of each index were used to create the frontier, with covariances between two time series calculated over their common overlapping time periods. The 60/40 portfolio was built assuming 60% notional invested in Global Equities proxied by the MSCI World Price Return Index (Bloomberg Ticker MXWO, start date January 1970) and 40% invested in our Bond proxy (see above), with monthly rebalancing. See footnote 18 for the Endowment proxy portfolio). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Of course, these tangency portfolio optimal figures reflect the in-sample opportunity set, given complete knowledge of realized return distributions. In order not to rely solely on the historical realizations of risk-factor and asset-class returns, we perform a number of independent simulations for each time series assuming a t-distribution for returns and randomly perturbing the expected return of each series using breakpoints from the uniform distribution¹⁹. We simulate 4 million mean-variance risk factor and 4 million asset-class portfolios, with which we plot 200 efficient frontiers, respectively (20,000 portfolios per frontier), shown in Figure 18.4: Panel A (blue radial lines represent the generated risk-factor efficient frontiers). Although there is some overlap between the risk factor and asset-class frontiers, it appears that the risk-factor mean-variance tangency portfolios lie to the left of the asset-class tangency portfolios, and a bit further down – i.e. typically risk-factor tangency portfolios are lower risk – lower return.

Comparing the simulated portfolios in Sharpe ratio terms (Figure 18.4 Panel B), the degree of overlap of the tangency portfolios' distributions is almost 29%; in other words, the likelihood of a cross-asset risk-factor mean-variance optimal portfolio (at least one simulated using our cross-asset risk premia universe) having a strictly higher Sharpe ratio to a mean-variance optimal asset portfolio is 72%²⁰.

18.6. Portfolio construction techniques

In what follows, we will introduce two innovative risk-based portfolio construction techniques that can be used, among other applications, to build risk-factor portfolios, and compare them with traditional risk-based algorithms. Risk-based allocation has become one of the most popular fields in both academic and investment circles in recent years. On the one hand, it is generally perceived that risk is easier to predict than returns; therefore, risk-based portfolio construction techniques are potentially more robust than traditional active portfolios that rely more on return forecasts. On the other hand, given the prolonged global financial crisis in 2008 and the subsequent risk-on/risk-off environment, not only is managing risk becoming more paramount than outperforming a benchmark, but also risk-based

19 Let μ represent the (annualized) historical mean of a returns time series and σ the annualized standard deviation. A random value ρ between 0 and 1 is generated from the uniform distribution and is used to generate the quantile value of the t-distribution and subsequently the density of the distribution. This is then divided by ρ to obtain the multiple κ by which the historical standard deviation is scaled to arrive at the new (simulated) mean μ' : $\mu' = \mu \pm \kappa\sigma$. This is repeated 200 times for each asset/risk factor.

20 This number is much higher for a mean-CVaR optimal portfolio; mean-CVaR tangency risk factor portfolios tend to have superior mean-CVaR figures compared to corresponding tangency portfolios formed solely from asset classes. Results available upon request from the authors.

allocation techniques actually do indeed outperform many active strategies that require return prediction. The two techniques can be further extended to account for return forecasts (see [LUO 13c] for details).

Panel A: resampled efficient frontiers

Panel B: distribution of the Sharpe ratios of the resampled MSR portfolios

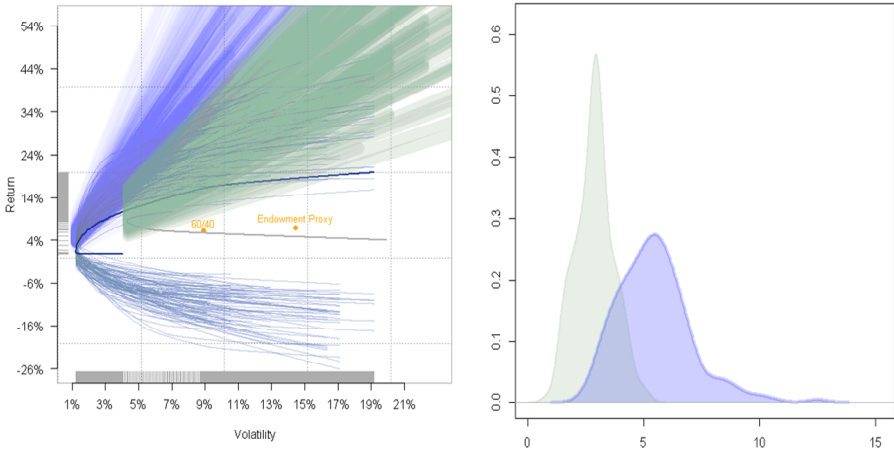


Figure 18.4. Simulated efficient frontiers and MSR Sharpe ratios (source: Deutsche Bank, Bloomberg, Factset, MSCI). Two hundred efficient frontiers plotted with risk factors and asset classes, respectively, using 4 million mean-variance portfolios in each case. The blue lines in Panel A are the generated risk factor efficient frontiers. For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

Portfolio construction itself does not introduce new assets (i.e. breadth) or new insights in return prediction (i.e. skill). The heart of portfolio construction is about diversification and risk reduction. In the next section, we introduce two innovative ways to define diversification and risk reduction.

We all know that classic finance theory, such as Markowitz's mean-variance optimization, heavily depends on the assumption that asset returns are jointly normally distributed. We also know that empirical evidence almost universally rejects the normality assumption. The traditional statistical tools (e.g. Pearson's correlation coefficient, portfolio volatility, etc.) are mostly based on this problematic assumption.

In this chapter, we empirically backtest seven risk-based allocation techniques, compared to traditional capitalization-weighted benchmarks in different contexts from asset allocation, multi-asset (bonds, commodities and alternative betas), country/sector portfolios, to equity portfolios:

- naive diversification: equally weighted, inverse volatility/volatility parity, risk parity/equal risk contribution;
- sophisticated diversification techniques: maximum diversification²¹ and minimum tail dependence;
- sophisticated risk reduction techniques: global minimum variance and minimum CVaR.

Figure 18.5 shows the scatterplot of the returns for the respective equity indices of six countries. The correlations among all six countries appear to be very strong. In addition, we see some clear nonlinear relationships and some heavy tails, especially among the peripheral European countries. The traditional assumption that asset returns follow multivariate normal distribution is clearly rejected by almost any statistical tests.

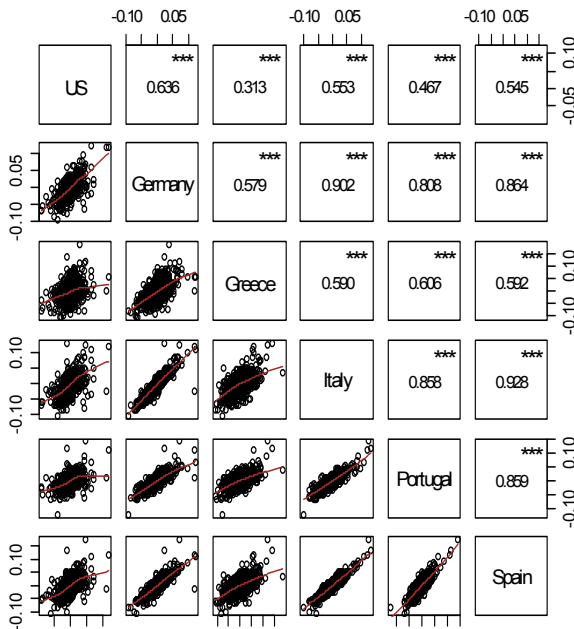


Figure 18.5. Scatterplot of six countries (source: Bloomberg Finance LLP, MSCI, Deutsche Bank Quantitative Strategy) Note: the x- and y-axes and numbers are correlation coefficients between two assets. *** indicates statistical significance (p value less than 0.01)

21 This is TOBAM's maximum diversification technique or most diversified portfolio. Both are trademarks of TOBAM.

For all portfolio backtestings in this chapter, we follow the following procedures. All backtestings are completely out-of-sample. All data used in this research are point-in-time. For example, we will use the point-in-time index constituents as our investment universe. All risk models are also point-in-time, either using commercial risk models²² or our own calculated covariance matrices.

All portfolios are rebalanced monthly. However, all risk models are computed daily, using typically a rolling 1 year (or 5 years) of daily returns. Portfolio performance is also measured daily, which is essential for downside risk metrics (e.g. tail dependence, CVaR, etc.) in particular. All portfolios in this section are long only, fully invested.

18.7. An alternative approach for defining diversification

In this section, we first define an alternative approach to measure co-movement in asset returns. Rather than relying on Pearson's correlation, we measure tail dependence between two assets using a copula model. In [CAH 13], we demonstrated an interesting way to measure the crowdedness of systematic equity strategies – median tail dependence. In this research, we extend the tail dependence concept by designing a strategy that proactively avoids crowded trades in what we call the minimum tail-dependent portfolio (MinTailDependence).

The purpose of TOBAM's MaxDiversification (see [CHO 08] and [CHO 13]) is to build a portfolio that is as diversified as possible, where diversification is measured by Pearson's correlation. The Pearson's correlation coefficient only measures the dependence between two random variables correctly if they are jointly normally distributed. Empirically, asset returns are almost never jointly multivariate normally distributed. We may argue that the dependence in the left tail (e.g. the chance of both assets suffering extreme losses at the same time) is more relevant in risk management and portfolio construction.

Pfaff [PFA 12] first introduced the minimum tail dependence portfolio (MinTailDependence) concept. Similar to MaxDiversification, with MinTailDependence we try to build a portfolio that is as “diversified” as possible, where “diversification” is measured by tail dependence.

²² We use Axioma's medium horizon fundamental risk models for all of our equity, country and industry portfolios.

18.7.1. Introducing the copula model

The copula model was first introduced by Sklar [SKL 59]. A more recent textbook explanation can be found in [MCN 05]. A copula models the dependence between assets in a multivariate distribution. Copula models allow for the combination of multivariate dependence with univariate marginals. In a non-technical sense²³,

$$\text{Joint Distribution} = \text{Copula} + \text{Marginal Distribution}$$

Therefore, a copula model gives us the flexibility to model joint asset return distributions. For example, we could fit an exponential GARCH model for each asset's marginal distribution, while at the same time, model the joint distribution using a t-copula model.

In a simple example, tail dependence coefficient can be estimated as:

$$\tau_{i,j} = \frac{\sum_{t=1}^T 1_A}{\|\sqrt{N}\|}$$

where:

– $\tau_{i,j}$ is the empirical tail dependence coefficient between asset i and j ;

– $1_A := \begin{cases} 1, & \text{if } (\text{rank}(r_i) \leq \|\sqrt{N}\| \ \& \ \text{rank}(r_j) \leq \|\sqrt{N}\|) \\ 0, & \text{otherwise} \end{cases}$;

– $\|\sqrt{N}\|$ is the squared root of the number of observations rounded to the nearest integer.

Figure 18.6 shows the difference between Pearson's correlation and copula-based tail dependence. Tail dependence coefficients among our six countries are clearly higher than Pearson's correlation coefficients – as expected, assets are more likely to fall at the same time than the average. Some differences are strikingly large. For example, the Pearson's correlation between Portugal and Spain is only 56%, which is quite normal compared to other pairs of countries (e.g. US and Germany). However, the tail-dependent coefficient is 86%, which is clearly on the high end.

²³ See [MEU 11].

	US	Germany	Greece	Italy	Portugal	Spain
US	100%	64%	31%	55%	47%	54%
Germany	53%	100%	58%	90%	81%	86%
Greece	25%	33%	100%	59%	61%	59%
Italy	44%	72%	36%	100%	86%	93%
Portugal	28%	53%	36%	58%	100%	86%
Spain	44%	64%	36%	81%	56%	100%

Figure 18.6. Correlation versus tail dependence (lower triangle = correlation/upper triangle = copula tail dependence) (source: Bloomberg Finance LLP, Compustat, MSCI, Russell, Thomson Reuters, Deutsche Bank Quantitative Strategy)

To visually examine how Pearson’s correlation coefficient underestimates the true dependence, let us compare the theoretical bivariate normal distribution between Portugal and Spain (see Figure 18.7: Panel A), with the empirical distribution (see Figure 18.7: Panel B). The empirical distribution between Portugal and Spain is clearly bimodal with two distinct peaks (or modes), i.e. the probabilities of these two countries both move higher or fall lower are much higher than any other combinations.

Panel A: Bivariate Gaussian distribution

Panel B: Empirical distribution

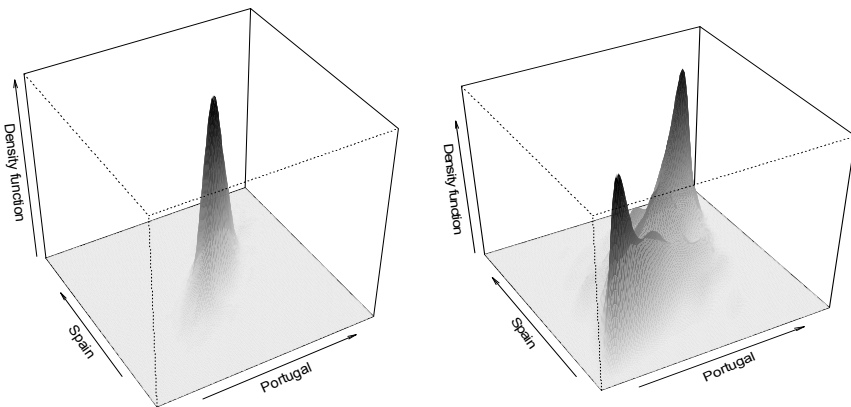


Figure 18.7 Theoretical Gaussian and empirical bivariate distributions (source: Bloomberg Finance LLP, MSCI, Deutsche Bank Quantitative Strategy)

18.7.2. Minimum tail dependence portfolio optimization algorithm

Numerically²⁴, the MinTailDependence portfolio can be solved easily by minimizing $\psi' \mathbf{T} \psi$, where \mathbf{T} is the tail dependence matrix. Therefore, the MinTailDependence optimization is almost exactly the same as MaxDiversification, by replacing the correlation matrix with tail dependence matrix. The final weights are then retrieved by rescaling the intermediate weight vector (optimized using the tail dependence matrix) with the standard deviations of the assets' returns.

Step 1

$$\arg \min_w \frac{1}{2} \psi_t' \mathbf{T}_t \psi_t$$

subject to:

$$\begin{aligned} \psi_t' \mathbf{1} &= 1 \\ \psi_t &\geq 0 \end{aligned}$$

where:

- w_t is the first intermediate vector of asset weights at time t ;
- \mathbf{T}_t is the asset-by-asset tail dependence matrix at time t .

Step 2

Then, we need to rescale the first intermediate vector of asset weights w_t by each asset's volatility $\sigma_{i,t}$:

$$\xi_t = D_t^{-1/2} \psi_t \text{ or } \xi_{i,t} = \psi_{i,t} / \sigma_{i,t}$$

where:

- ξ_t is the second intermediate vector of asset weights at time t ;
- D_t is the diagonal matrix of asset variance at time t with $\sigma_{i,t}^2$ at its i, i element and zero on all off-diagonal elements.

²⁴ Please note that the calculation follows closely the maximum diversification, as shown by Chouefaty and Coignard [CHO 08].

Step 3

Finally, we rescale the second intermediate asset weight vector of the total weight, so the sum of the final weights equal to 100%, i.e. no leverage.

$$\omega_{i,t} = \frac{\xi_{i,t}}{\sum_{j=1}^N \xi_{j,t}}$$

18.7.3. Alternative beta portfolios

Now, let us demonstrate the benefit of our MinTailDependence portfolio in the context of alternative beta portfolios. Investing along the alternative beta portfolios (also called risk premia or risk factors) has become fairly popular in recent years, especially among asset owners.

Here, we define five simple alternative beta factor portfolios in global equities:

- value, based on trailing earnings yield;
- momentum, based on 12-month total returns excluding the most recent month;
- quality, based on return on equity;
- size: MSCI World SmallCap total return index – MSCI World LargeCap total return index;
- low volatility/low risk, based on trailing 1-year daily realized volatility.

All portfolios (other than size) are constructed for the MSCI World universe, in a regional and sector neutral way, by forming a long/short quintile portfolio, equally weighted within the long and short portfolios. We divide the MSCI World into the following regions: US, Canada, Europe (e.g. UK), UK, Asia (e.g. Japan) and Australia/New Zealand. We select equal number of stocks from each region in each of the 10 GICS sectors. For those region/sector buckets where we have less than five stocks, we do not invest in those buckets.

It is interesting to see that all portfolios constructed on alternative betas significantly outperform the benchmark MSCI World Index, with much higher Sharpe ratios (see Figure 18.8) and much lower downside risks (see Figure 18.9). Our MinTailDependence strategy displays the highest Sharpe ratio and the lowest downside risk.

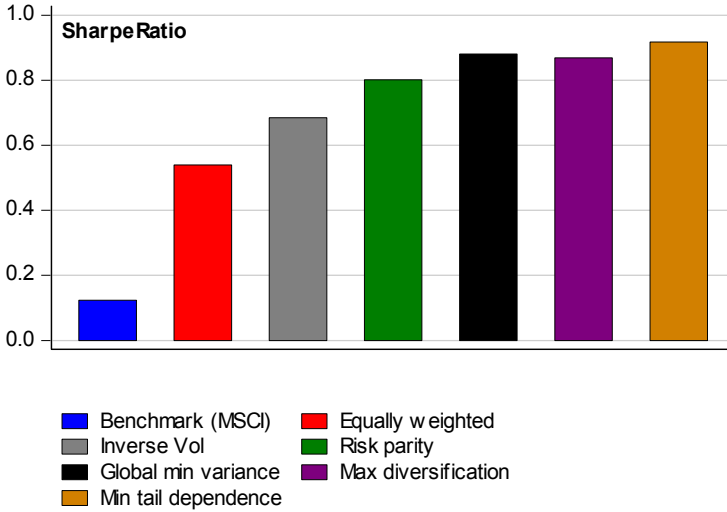


Figure 18.8. Sharpe ratio (source: Bloomberg Finance LLP, Compustat, MSCI, Russell, Thomson Reuters, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

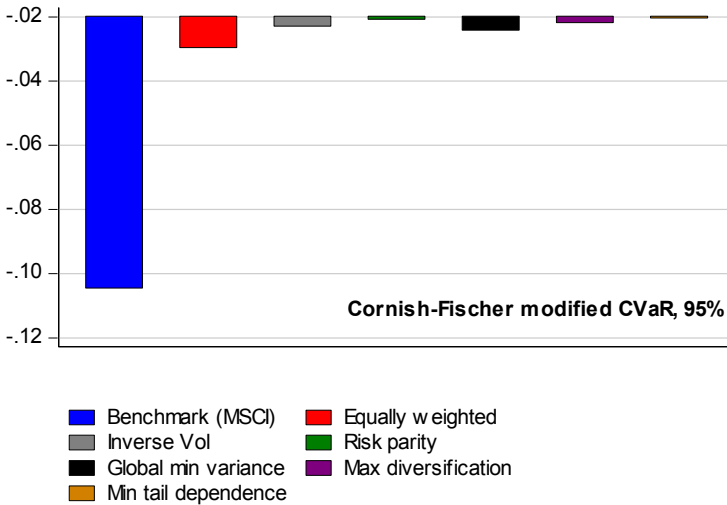


Figure 18.9. CVaR/expected shortfall²⁵ (source: Bloomberg Finance LLP, Compustat, MSCI, Russell, Thomson Reuters, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

27 We compute the realized CVaR using the Cornish–Fisher expansion explained in [FAV 02]

18.8. An alternative definition of risk

Second, we give a practical introduction to conditional value at risk (also called expected shortfall) and how to construct a portfolio that minimizes expected CVaR, i.e. the MinCVaR portfolio. We further develop a new algorithm by combining robust optimization and CVaR optimization into what we call robust CVaR optimization, which shows great promise by outperforming other portfolio construction techniques in terms of both Sharpe ratio and downside risk.

The MinTailDependence portfolio (along with maximum diversification) helps us better capture diversification benefits, while the MinCVaR strategy (along with global minimum variance portfolio) attempts to manage risk better.

18.8.1. Minimum CVaR portfolio

CVaR or conditional value at risk is a statistical measure of tail risk, measured by assessing the likelihood (at a specific confidence level) that a specific loss will exceed the VaR. Mathematically speaking, CVaR is derived by taking a weighted average between the VaR and losses exceeding the VaR.

Rockafellar and Uryasev [ROC 00, ROC 01] first introduced portfolio optimization with CVaR or conditional value at risk. In the risk management literature, VaR or value at risk is criticized as being an incoherent risk measure. However, CVaR, which is also referred as expected shortfall, is a coherent risk measure. In portfolio optimization, CVaR is a convex function, while VaR is not necessarily convex²⁶. More importantly, as shown in [ROC 01], CVaR optimization can be transformed into linear optimization, which tends to be easier to solve.

Despite the theoretical soundness of the CVaR methodology, in practice, we have to estimate CVaR empirically and face all the usual problems with estimation errors. It is the same trade-off between model error versus estimation error, i.e. we could have a better model, but may have larger estimation error.

Let us use country equity portfolio as an example. Our sample includes 45 countries comprising the MSCI All Country World Index (ACWI). We use daily total returns in USD to perform all portfolio backtests below. All strategies are monthly rebalanced and constrained to be long only.

²⁶ The problem with non-convex optimization is that we may get local optima instead of global optima. Therefore, a global optimizer is typically required, while global optimizations tend to be very slow.

18.8.2. CVaR optimization theory

Here, we will only give a very brief description of the CVaR optimization problem and corresponding algorithm. A more detailed exposure can be found in [ROC 00, ROC 01].

In the CVaR optimization setup, let us define ω as the vector of asset weights (i.e. our decision variable). The asset return distribution is defined by vector r . The loss function is further defined as $f(\omega, r)$. γ is the value-at-risk and α is our confidence level. The minimum CVaR optimization can be transformed as:

$$\arg \min_{\omega} = \gamma + \frac{1}{(1-\alpha)S} \sum_{s=1}^S (f(\omega, r_s) - \gamma)^+$$

Subject to:

$$\omega'_t = 1$$

$$\omega_i \geq 0$$

Therefore, the minimum CVaR optimization problem can be solved by linear programming algorithms.

18.8.3. Mean-CVaR efficient frontier and minimum CVaR portfolio

Similar to the mean-variance efficient frontier, we can define the mean-CVaR efficient frontier as a hyperbola containing portfolios with the following characteristics: for given level of risks (defined as CVaR), they have the highest expected returns. The portfolio that separates the efficient frontier from the lower border of the feasible set is the global minimum CVaR portfolio (i.e. the MinCVaR portfolio). Figure 18.10 shows the mean-CVaR efficient frontier, using our simple six-country equity index example.

18.8.4. Robust minimum CVaR optimization

The traditional CVaR optimization conducts the linear optimization using historical returns. To ensure the optimized weights are robust to a specific set of observed returns, we propose a new optimization technique that we will call “robust minimum CVaR optimization” or RobMinCVaR. We borrow the ideas from robust optimization (see [MIC 98] and traditional CVaR optimization above). In summary, we fit historical returns to a prespecified multivariate distribution. Next, we simulate multiple sets of historical returns.

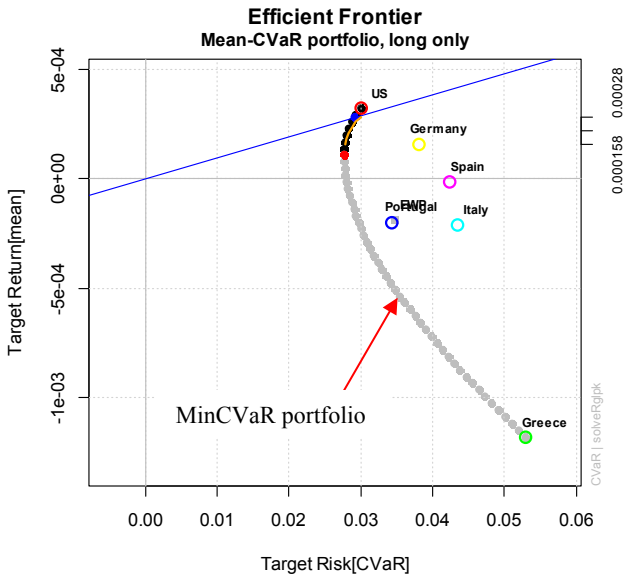


Figure 18.10. Mean-CVaR efficient frontier (source: Bloomberg Finance LLP, MSCI, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

18.8.4.1. Fitting a multivariate skew-t distribution

To fully account for the nature of non-multivariate normal distribution in asset return data, we fit our 45-country return data at each month end, using 5 years of rolling daily returns to a multivariate skew-t distribution. The family of multivariate skew-t distributions is an extension of the multivariate Student's t family, via the introduction of a shape parameter which regulates skewness. The fits are done using maximum likelihood estimation (see [AZZ 99] and [AZZ 03]).

18.8.4.2. Robust minimum CVaR optimization

We then simulate 50 time series of the same 5 years of daily country returns with the above fitted multivariate skew-t distribution, for these 45 countries at each month end. Then, we can construct 50 MinCVaR portfolios – one for each simulated data. We further construct our final portfolio using three approaches:

- *average (RobMinCVaR-Avg)*: we simply average the weights of the 50 asset weight vectors;

– *the most conservative portfolio (RobMinCVaR-Conservative)*: for each of the 50 MinCVaR portfolios, we calculate the expected CVaR, then we take the portfolio with the lowest CVaR, i.e. the worst-case scenario portfolio, as our final portfolio;

– *the most optimistic portfolio (RobMinCVaR-Optimistic)*: for each of the 50 MinCVaR portfolios, we calculate the expected CVaR, then we take the portfolio with the highest CVaR, i.e. the most optimistic case scenario portfolio, as our final portfolio.

In the following simulation, we fix α at the 10% level. All three RobMinCVaR portfolios outperform the traditional MinCVaR strategy, with higher Sharpe ratios (see Figure 18.11) and slightly higher downside risks (see Figure 18.12). The RobMinCVaR-conservative portfolio, in particular, shows a decent Sharpe ratio. Indeed, even compared to all other risk-based allocation techniques, the RobMinCVaR portfolio shows the highest Sharpe ratio (see Figure 18.13) and the second lowest downside risk (see Figure 18.14).

The biggest challenge in our RobMinCVaR optimization is computational speed. It can be slow even when the number of assets is modest. For example, with 45 countries, it takes about 30 s per period. If we run 50 simulations over the past 14 years, it can take around 3 days for a complete backtesting.

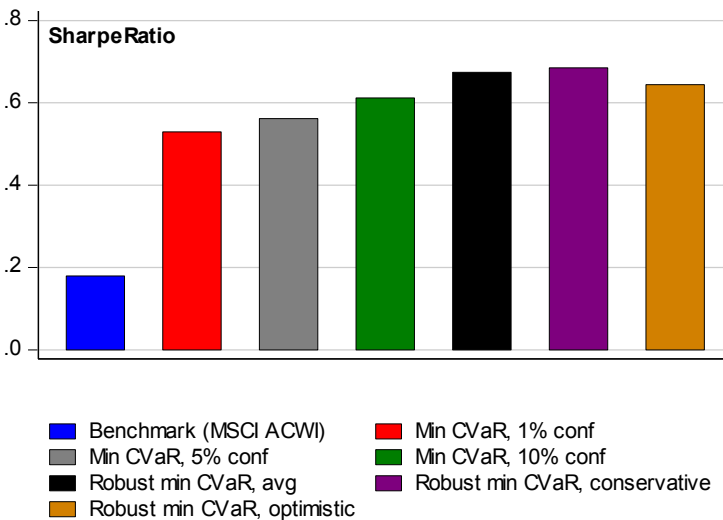


Figure 18.11. Sharpe ratio (source: Bloomberg Finance LLP, MSCI, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

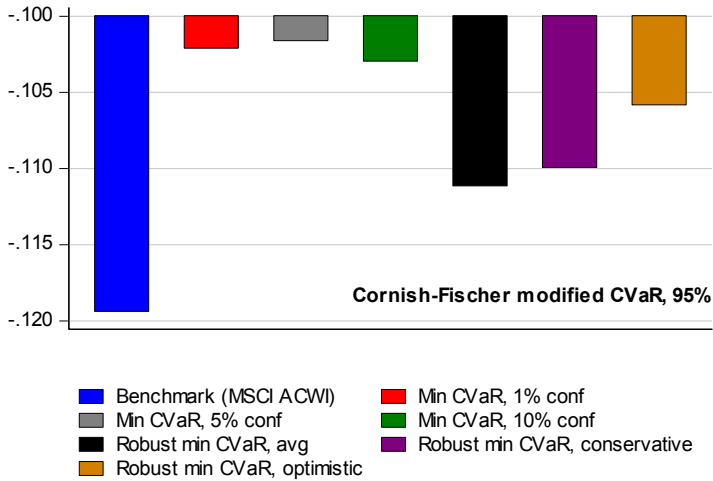


Figure 18.12. CVaR/expected shortfall (source: Bloomberg Finance LLP, MSCI, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

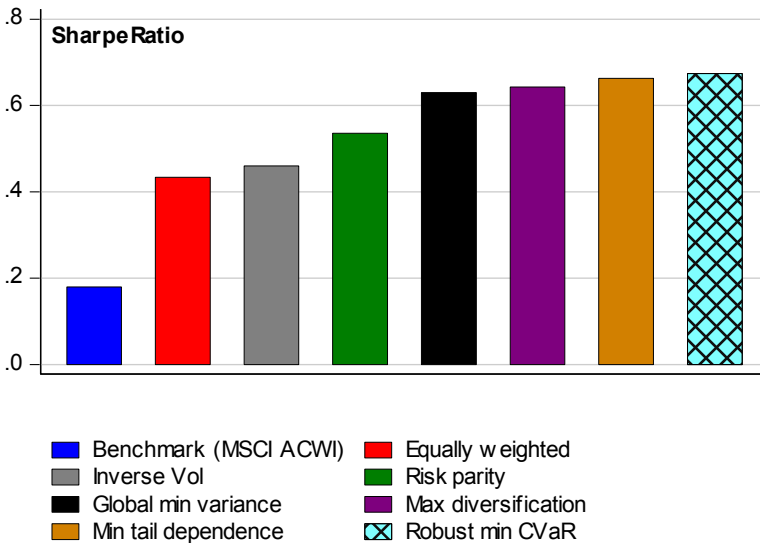


Figure 18.13. Sharpe ratio – risk-based allocations (source: Axioma, Bloomberg Finance LLP, Compustat, MSCI, Russell, Thomson Reuters, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

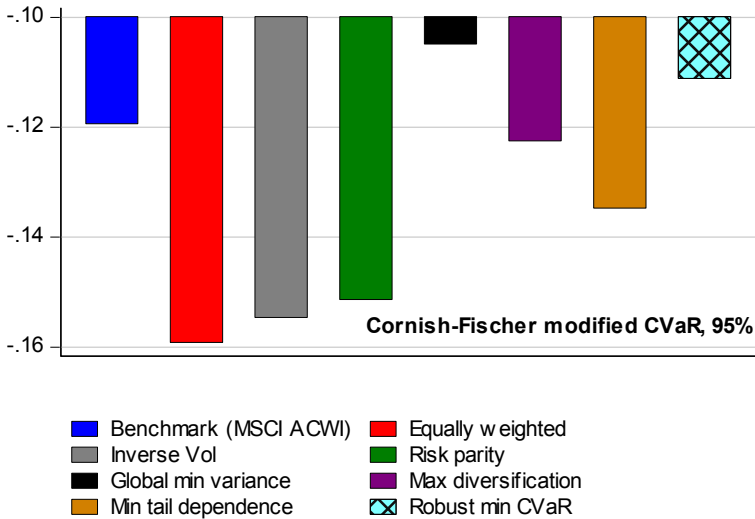


Figure 18.14. CVaR/expected shortfall (source: Axioma, Bloomberg Finance LLP, Compustat, MSCI, Russell, Thomson Reuters, Deutsche Bank Quantitative Strategy). For a color version of the figure, see www.iste.co.uk/jurczenko/risk.zip

18.8.5. Choice of alpha parameter

In MinCVaR optimization, one of the key input parameters is the confidence level α . The choice of α is more art than science. Therefore, let us first experiment with three different levels of α and study the impact of parameter sensitivity. We use our country portfolio as an example by investing in the 45 countries comprising the MSCI ACWI. We further set the α at 1, 5 and 10%. As shown in Figure 18.12 and 18.13, the MinCVaR portfolio is not very sensitive to the choice of α . More importantly, all three portfolios significantly outperform the capitalization-weighted benchmark.

18.9. Comparison of different risk-based portfolio construction techniques

18.9.1. The philosophy of portfolio construction

In summary, we survey seven risk-based allocations and compare their performance with the more traditional capitalization-weighted benchmark. Using our country equity portfolio example explained in the CVaR optimization section 18.8, we perform a cluster analysis on the monthly returns (from 1999 to 2013) of seven risk-based portfolios, along with the benchmark (MSCI ACWI). Among the

seven portfolio construction techniques, we can see that they form four distinct clusters²⁷ (see Figure 18.15):

- benchmark;
- naive diversification: equally weighted, inverse volatility/volatility parity, risk parity/equal risk contribution;
- sophisticated diversification techniques: maximum diversification²⁸ and minimum tail dependence;
- sophisticated risk reduction techniques: global minimum variance and minimum CVaR.

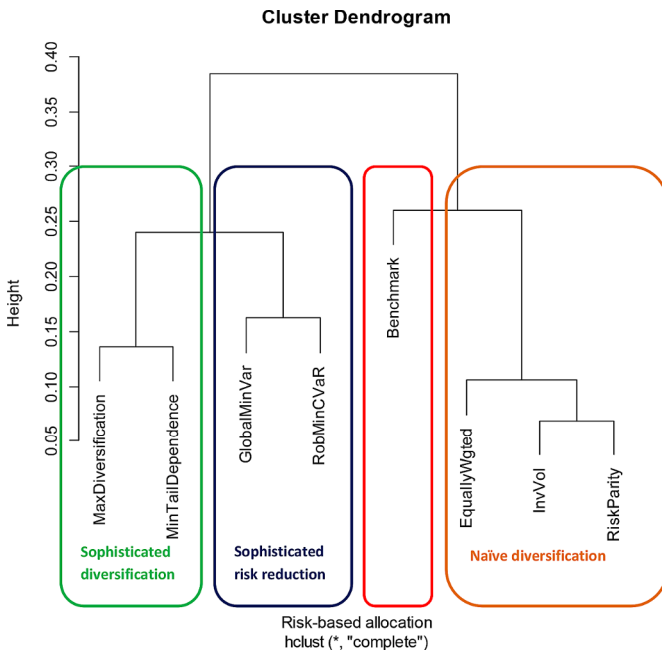


Figure 18.15. Cluster analysis (source: Bloomberg Finance LLP, MSCI, Deutsche Bank Quantitative Strategy)

27 A dendrogram shows the similarity among a group of entities. The arrangement of the branches (called “leaves” in cluster analysis) tells us how similar they are. The more closely they are connected to each other, the higher the correlation. The height of the branch also indicates similarity. The greater the height, the greater the difference.

28 This is TOBAM’s maximum diversification technique or the most diversified portfolio. Both are trademarks of TOBAM.

18.9.2. A horse race of risk-based portfolio construction techniques

To obtain a fair comparison of the seven risk-based portfolio construction techniques, we backtest their performance in three different contexts, over 17 portfolios. All portfolios are constructed out-of-sample, long-only, monthly rebalanced, from 1999 to 2013²⁹:

– *asset allocation*: multi-asset allocation, global sovereign bonds, commodities and alternative betas;

– *country, sector and industry allocation*: global countries, economic risk-hedged global countries³⁰, global sectors, US sectors, European sectors, global industries and region x sector combinations;

– *equities*: US equities, European equities, Asia, (e.g. Japan equities), Japanese equities, emerging markets equities and global equities.

In terms of Sharpe ratio (see Figure 18.16), GlobalMinVar achieves the highest Sharpe ratio overall, followed by MinTailDependence and MaxDiversification. Capitalization-weighted benchmark delivers the lowest Sharpe ratio, followed by naïve EquallyWgtd and InvVol strategies. The comparison along other risk dimensions can be found in [LUO 13c].

18.10. Conclusion

In this chapter, we have discussed how investors are starting to implement risk premia investing as a top-down element of their overall investment strategy, and highlighted the benefits this is likely to confer versus traditional asset-class allocation practices. Despite practical challenges to full adoption of risk premia investing as the new asset allocation paradigm, particularly by large institutional investors, we believe factor investing is here to stay. We have also introduced two innovative risk-based portfolio construction techniques that can be used, among other applications, to build risk-factor portfolios, and compared them with traditional risk-based algorithms. In particular, we have developed a new algorithm that combines robust optimization with CVaR optimization into what we call robust CVaR optimization, which shows great promise by outperforming other portfolio construction techniques in terms of both Sharpe ratio and downside risk.

29 The exact specification of each portfolio can be found in [LUO 13b]. We do not include robust minimum CVaR in this simulation, due to computational speed concerns.

30 Defined in [LUO 13a].

Sharpe ratio	Benchmark	Equally Wgted	Inv Vol	Risk Parity	Global Min Var	Max Diversification	Min Tail Dependence
<i>Asset allocation, avg ranking</i>	6	7	5	4	2	1	3
Multi-assets	7	6	4	2	5	1	3
Sovereign bonds	4	7	6	5	1	2	3
Commodities	6	6	4	5	1	2	3
Alternative betas	7	6	5	4	2	3	1
<i>Country/sector allocation, avg ranking</i>	7	6	5	3	1	4	2
Countries, MSCI ACWI	7	6	5	4	3	2	1
Countries, MEAM	7	5	6	4	2	3	1
Sectors, MSCI	7	5	2	3	1	6	4
Sectors, US	7	5	2	3	1	6	4
Sectors, Europe	7	6	3	2	1	4	5
Industries, MSCI	7	6	5	3	1	4	2
Regions x sectors, MSCI	7	6	5	3	1	2	4
<i>Equities, avg ranking</i>	7	6	5	4	3	3	3
US	7	6	5	4	2	1	3
Europe	7	2	1	3	6	5	4
Asia, e.g. Japan	7	6	5	4	2	3	1
Japan	7	5	4	3	1	2	6
Emerging markets	7	4	2	3	5	6	1
Global	7	6	5	4	2	1	3
<i>Overall ranking</i>	7	6	5	4	1	3	2

Figure 18.16. Sharpe ratio ranking (source: Axioma, Bloomberg Finance LLP, Compustat, MSCI, Russell, Thomson Reuters, Deutsche Bank Quantitative Strategy)

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Multi-Factor Portfolio Construction for Passively Managed Factor Portfolios

Transparent rule-based index-tracking portfolios that employ alternative weighting schemes have grown rapidly in the last decade, especially within equities. These passively managed factor portfolios can be constructed in many ways, ranging from relatively simple rule-based approaches that specify weights as a function of factor characteristics to more complex optimization-based ways. Both single factor and multiple factor portfolios can be constructed. In the latter case, an often asked question is whether it is better to combine individual factor portfolios or build a multi-factor portfolio from the security level. Here, we show that a bottom-up approach to multi-factor portfolio construction can produce superior results than a combination of individual single factor portfolios, at least for well-known factors such as value, quality, low volatility and momentum. Because the bottom-up approach assigns weights to securities on multiple factor dimensions simultaneously, it accounts for cross-sectional interaction effects in a way that combining single-factor portfolios does not.

19.1. A short history of passively managed factor portfolios

Transparent rule-based index-tracking portfolios that employ alternative weighting schemes have grown rapidly in the last decade, especially within equities. Today, these types of non-market cap-weighted portfolios go by the term “advanced beta”, “smart beta”, “systematic strategies”, “factor-based investing” and more. These strategies are passively implemented in the same way as traditional passive portfolios. Because of this, they retain the benefits of passive management including full transparency and low costs, with the potential to earn higher returns and/or

deliver lower volatility than market cap-weighted portfolios. Particularly, over long periods, many investors are increasingly viewing them as a more cost-effective way to enhance returns relative to traditional active management. A wealth of papers have been written on these non-market cap-weighted strategies, which we refer to as passively managed factor (PMF) strategies for the remainder of this chapter¹.

The earliest PMF strategies applied an alternative weighting scheme to market capitalization weighting. Examples include Gross Domestic Product (GDP)-weighted portfolios in the 1980s, equal-weighted portfolios in the 1990s and more recently, fundamental-weighted portfolios in the 2000s. GDP weighting applies GDP weights as country weights in a global equity portfolio, while equal weighting and fundamental weighting assign equal weights to securities or weights based on company fundamentals such as book value, respectively. Proponents of these strategies were typically critical of cap weighting and argued that these alternative weighting schemes were superior either because they were more representative of investment value or more diversified (less concentrated).

Since 2008, an alternate way of viewing PMF portfolios has emerged, one that focuses on the underlying factors. This approach focuses on what “pure” factors (value, size, quality, momentum, etc.) the portfolios are exposed to, and derive their returns, from. The pure factors, beginning with the multi-factor models of Ross [ROS 76] are those that have been widely researched in the academic literature, have strong theoretical foundations and have exhibited persistence over multiple decades. Viewing PMF as a way to capture pure factors means it is consistent with the way academics have viewed factors, most widely popularized by Fama and French’s seminal three-factor model, and extended over the years by countless others. It also grounds PMF investing in the same broad investing principles that underlie many active management approaches.

These factors represent systematic sources of return and risk, “risk premia” or arise because of mispricing of securities by investors which fail to be arbitrated away or because of market frictions. Increasing familiarity with traditionally academic factor models and newer commercially available factor models has driven greater adoption of this factor-based approach.

19.2. Single-factor portfolio construction

There are a range of techniques that can be used to build single-factor portfolios. Equal weighting, GDP weighting, fundamental indexation and its close companion

¹ See [ANG 09, ANG 13, URW 11, BEN 13a] and [BEN 13b].

wealth weighting were compelling because they were intuitive and did not employ a black box algorithm such as optimization, which meant security weights could be directly tied to the securities' observable characteristics. Subsequent factor-based approaches could also be constructed in a similar manner; these can be viewed as "heuristic" or rule-based methods which use a set of rules to specify security weights as a function of the factor characteristics. Fama–French factor portfolios, for instance, while typically not viewed as PMF, are in fact rules-based factor portfolios. However, these were never meant to be investable portfolios, their long-short construct being difficult to scale.

Heuristic methods fall under one of the two categories: benchmark-independent or benchmark-relative. A benchmark-independent approach specifies a function for determining the weights that does not recognize the role of a benchmark (a market cap-weighted portfolio). A benchmark-dependent approach, however, does. Fundamental indexation, equal weighting and risk weighting are all examples of benchmark-independent approaches as shown in equations [19.1]–[19.3].

$$\text{Equal weights: } w_i = \frac{1}{N} \quad [19.1]$$

$$\text{Fundamental indexation weights: } w_i = \frac{F_i}{\sum_i F_i} \quad [19.2]$$

$$\text{Risk weights: } w_i = \frac{\frac{1}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2}} \quad [19.3]$$

where w_i is the weight of stock i in the portfolio, N is the number of stocks in the universe, F_i is the fundamental value of stock i (e.g. book value, earnings, etc.) and σ_i^2 is the variance of stock i .

Benchmark-relative approaches on the other hand incorporate market cap-weighting explicitly. For instance, one popular way is to apply multipliers to market cap weights:

$$\text{Tilted factor portfolio weights: } w_i = w_{i,mkcap} \gamma_i \quad [19.4]$$

where γ_i is a scalar applied to the market cap weight of each stock. The scalar γ_i can be specified in many ways. It can be the result of a mapping function based on

the security's factor characteristics. It can be nonlinear or linear cross-sectionally, and it can be unique for each security or unique for groups of securities.

In addition to the weighting scheme, stock screening decisions also drive the performance and characteristics of the portfolio. These two decisions together determine the main characteristics of the portfolio (risk, return, excess return, relative risk, liquidity, concentration, etc.). In tilted portfolios, for instance, the greater the amount of stocks screened or the more aggressively the weighting scheme departs from cap weighting, the higher the relative risk (or tracking error), the higher the turnover, the lower the liquidity and so forth.

Benchmark-relative approaches have appeared to become the more preferred route in recent years primarily because of several reasons. First, there has been a broad adoption of market cap-weighted indices as performance and policy benchmarks by institutional investors. In this context, factor exposures are viewed as active exposures relative to the market cap-weighted index. Second, benchmark-relative approaches are more consistent with academic models such as Fama–French. In this model, for instance, the market factor is the first factor such that size Small Minus Big (SMB) and value High Minus Low (HML) are meant to capture effects excess or net of the market. When benchmark-relative portfolios are regressed on Fama–French factors, the signs of the exposures are consistent with the targeted factors. In the same vein, “anti-tilted” portfolios which tilt away from a particular factor do in fact underperform the benchmark and exhibit the opposite signs on Fama–French exposures. Further discussion appears in [BEN 15].

Besides heuristic or rules-based approaches, more complex portfolio construction methods can be used. PMF does not preclude per se the use of quadratic optimization or linear or nonlinear algorithms. Optimization has widely been accepted for building minimum volatility portfolios, for instance not least because it is by far the most efficient way to do so. Standard mean-variance quadratic optimization could be used for PMF portfolios. Recall the unconstrained optimal solution from [GRI 00] which maximizes the function:

$$\max \alpha'w - \frac{\lambda}{2} w' \Sigma w \quad [19.5]$$

where w is the vector of active weights, Σ is the covariance matrix, α is the vector of alphas and λ is the risk aversion parameter. The optimal portfolio is given by:

$$w^* = \frac{1}{\lambda} \Sigma^{-1} \alpha \quad [19.6]$$

Equation [19.6] is not dissimilar from equations [19.1]–[19.4]. Security weights are a function of alpha (which could just be some normalized factor characteristic or exposure), risk and risk aversion. In practice, however, constraints are typically required to arrive at realistic portfolios, since quadratic optimization tends to select extreme outcomes if no constraints are set and to potentially become error-maximizers, such that estimation noise in the inputs is magnified in the optimal weights; see [MIC 98]. Once constraints are introduced, the closed form solution in equation [19.6] no longer holds and the link between optimization inputs and portfolio weights quickly becomes less clear.

While the use of more complex portfolio construction techniques is not barred in PMF portfolios, because they tend to run contrary to transparency, their usage is likely to be limited. This issue arises because the active decision to own factors is made by the investors, and not by the asset managers. Passive managers hired to track a factor index cannot be held accountable if the factor underperforms since their objective is to track the index. Because the investors own the factor investing decision, they must be comfortable in understanding the methodology behind the indices. More generally, if the goal is broad exposure to one or more factors, which we believe it should be in PMF, we believe that both approaches will achieve the desired result.

19.3. Why combine multiple factors?

So far, we have discussed single-factor portfolio construction in broad terms without referencing the actual factors, arguably the most important point in PMF investing. Factor research (also known as the asset pricing anomaly research) comprises a vast body of academic literature. The most widely discussed factors include the original Fama–French–Carhart factors – value, size and momentum – and a handful of additional factors which have received moderate treatment – (low) volatility, quality, liquidity and yield. Numerous other stock characteristics have also been studied, spanning across income statement and balance sheet measures such as earnings revisions and accruals, technical indicators such as volatility and relative strength (momentum) and even non-financial factors such as media coverage, Internet hits and environmental, social and governance (ESG) themes.

There are several main camps in the debate over what drives factor returns. In the first camp are those who argue that factors earn excess returns because there is systematic risk attached to them. Markets are generally efficient and these factors reflect “systematic” sources of risk. In the second camp, factors are thought to earn excess returns because of investors’ systematic errors which lead to persistent mispricing. These systematic behaviors are a result of investors collectively exhibiting behavioral biases and barriers which prevent these from being arbitrated away. A third camp focuses on market frictions giving rise to these anomalies, for

instance the fact that many investors cannot use leverage. All three rationales have been proposed for value and size, while momentum, quality and low volatility tend to rest on investor mispricing mistakes or market frictions.

Since factors are unobservable, there is a limit to how certain investors can be around their existence and persistence. Following the old adage that “it is better not to put all of one’s eggs in one basket”, employing multiple factors has been one manner in which investors have diversified this information uncertainty. This is the first leg of the diversification argument for multi-factor investing.

The second leg of the diversification argument is that empirically, factors also have exhibited variation in performance over time, such that they diversify each other. In Table 19.1, we summarize the performance of factor portfolios over the past two decades. These portfolios are developed market securities formed from the MSCI World Index universe, where securities ranked higher on the relevant factor are overweight relative to the benchmark and securities ranked lower are underweight. (The details behind these portfolios, including the actual metrics used for these factors, are discussed in section 19.6). All five portfolios have historically outperformed the market cap-weighted benchmark, the MSCI World Index, and have exhibited higher return-to-risk ratios and moderate-to-robust information ratios (Table 19.1).

	Value	Volatility	Size	Momentum	Quality	
	Valuation tilt	(Low) Volatility tilt	(Low) Size tilt	Momentum tilt	Quality tilt	MSCI World
Annualized returns	9.15%	8.39%	8.86%	8.69%	9.04%	7.82%
Annualized standard deviation	15.74%	13.01%	15.26%	14.66%	13.85%	14.98%
Return to risk ratio	0.58	0.64	0.58	0.59	0.65	0.52
Annualized excess returns	1.34%	0.57%	1.04%	0.87%	1.22%	–
Tracking error	3.39%	3.44%	2.87%	4.40%	2.36%	–
Information ratio	0.39	0.17	0.36	0.20	0.52	–

Table 19.1. *Tilted factor portfolios, performance (Gross USD Monthly Returns, March 1993–December 2014, Global, Universe = MSCI World Index). The valuation tilted strategy tilts toward stocks with higher than average book-to-price, sales-to-price, earnings-to-price, cash flow to price and dividend yield. The low volatility tilted strategy tilts toward stocks with lower than average historical return volatility, while the low size tilted strategy tilts toward smaller cap stocks in the MSCI World Index (i.e. mid caps). The momentum tilted strategy tilts toward stocks with higher than average trailing 12-month returns, while the quality tilted strategy tilts toward stocks with higher than average return-on-assets and lower than average earnings-per-share variability and long-term debt to equity. Underlying factors are shown in the first row with the relevant tilted strategy underneath. Excess returns are the returns to the tilted strategies minus the benchmark (MSCI World). Tracking error is the standard deviation of excess returns annualized. The information ratio is excess returns divided by tracking error*

Next, Table 19.2 displays the correlations of the factor portfolios over the March 1993–December 2014 period. Correlations are generally low, sometimes negative. The highest correlations are between value and size, and low volatility and quality. The lowest correlations are between value and momentum, value and quality, and size and quality.

	Value	Volatility	Size	Momentum	Quality
	Valuation tilt	(Low) Volatility tilt	(Low) Size tilt	Momentum tilt	Quality tilt
Valuation tilt	1.00				
(Low) Volatility tilt	0.13	1.00			
(Low) Size tilt	0.60	-0.02	1.00		
Momentum tilt	-0.42	0.23	-0.22	1.00	
Quality tilt	-0.38	0.53	-0.36	0.31	1.00

Table 19.2. *Correlation of excess returns (Gross USD Monthly Returns, March 1993–December 2014). Underlying factors are shown in the first row with the relevant tilted strategy underneath. Excess returns are the returns to the tilted strategies minus the benchmark (MSCI World) (source: SSgA, Factset)*

The correlations above are just one view of diversification, measuring month-to-month co-movement between two PMF strategies. Another view of diversification is through their co-movement over longer periods. Rolling excess returns averaged over the preceding 3 years are shown in Figure 19.1. For example, between 2004 and 2007, quality and momentum significantly underperformed the market, but value and size significantly outperformed the market. There have been only a handful of periods, all short-lived, where all the factors performed poorly, for instance in 2003.

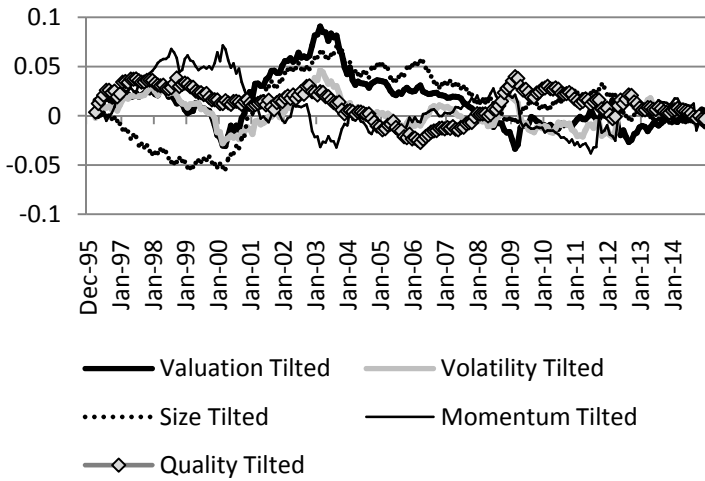


Figure 19.1. Alleviating timing risk as seen through returns over time (Gross USD Monthly Returns, March 1993–December 2014). The relevant tilted strategies are shown for each underlying factor. Excess returns are the returns to the tilted strategies minus the benchmark (MSCI World) (source: SSgA, Factset)

Having said that, as evidenced in Figure 19.1, all five factors have historically undergone prolonged periods of underperformance. Combining multiple factors alleviates the problem but does not completely eliminate it. Even if multiple factors are employed, PMF investing requires patience to harvest premiums over the long run.

19.4. Multi-factor portfolio construction

Multi-factor portfolios can be constructed into two main ways. The simplest way is to combine single-factor portfolios, such as the ones shown in Tables 19.1 and 19.2 and Figure 19.1, into one portfolio. Another way is to build the portfolio from the security-level up (“bottom-up”), incorporating all the factor characteristics simultaneously. Intuitively, the latter approach is more compelling since this approach evaluates securities on the multiple dimensions simultaneously. Asness [ASN 97] highlighted, for instance, interaction effects between value and momentum. Mixing portfolios independently constructed may miss these interaction effects. However, analogous to working with “building blocks,” combining single-factor portfolios does have benefits for performance attribution and reallocation across factors.

We focus on comparing the “bottom-up” versus the “combination” methods within the context of the tilted portfolio approach where market cap weights are

scaled by a multiplier. First, it may be helpful to point out that there is only one condition under which the two approaches will be identical:

– the starting weight (which the multiplier is applied to) is equal weight;

– multipliers do not capture information about the cross-sectional distribution of securities. For instance, ranks are used to identify the relative attractiveness of the 10 securities, not scores, and no two securities have the exact same rank (among any of the factors).

To illustrate this, we show portfolios of 10 securities in Table 19.3 (left panel) that blend three simple “factors” – dividend yield, book-to-price and return-on-assets as of 31 December 2014. For the combination portfolio, we rank securities for each of the three factors. We assign multipliers to equal weights (10%), and cap weights, which are identical to the stocks rank (e.g. a stock ranked 5 has a multiplier of 5). The scaled weights are then rescaled to sum to 100%. Then, we blend the three resulting portfolios into one using equal weights. For the bottom-up Portfolio, after we rank securities, we compute an average (equally weighted) rank across the three factors and multiply this combined rank by security market cap weights (or equal weights). We rescale the weights to sum to 100%.

	Rank-based approach (left panel)				Score-based approach (right panel)			
	Applied to equal weights		Applied to market cap weights		Applied to equal weights		Applied to market cap weights	
	Comb. portfolio	Bottom-up	Comb. portfolio	Bottom-up	Comb. portfolio	Bottom-up	Comb. portfolio	Bottom-up
633987	10.30%	10.30%	3.61%	2.95%	10.42%	12.73%	3.61%	3.15%
B1YW44	6.67%	6.67%	0.28%	0.25%	6.71%	1.82%	0.28%	0.06%
88579Y10	10.91%	10.91%	9.30%	10.04%	10.92%	14.55%	9.30%	11.58%
425304	8.48%	8.48%	10.49%	9.97%	8.54%	7.27%	10.49%	7.39%
425305	7.88%	7.88%	8.91%	8.94%	7.33%	3.64%	8.91%	3.57%
710889	8.48%	8.48%	4.33%	3.59%	8.56%	10.91%	4.33%	3.99%
00282410	14.55%	14.55%	9.29%	8.51%	14.65%	18.18%	9.29%	9.21%
00287Y10	10.30%	10.30%	9.21%	9.27%	10.33%	9.09%	9.21%	7.08%
629210	12.12%	12.12%	43.97%	45.98%	12.15%	16.36%	43.97%	53.73%
000312	10.30%	10.30%	0.61%	0.51%	10.39%	5.45%	0.61%	0.23%
Correlation	1.00		1.00		0.87		0.99	
Absolute sum of weight differences	0.00%		5.70%		32.05%		24.09%	

Table 19.3. *Bottom-up versus combination method, 10-stock example: rank versus score-based (31 December 2014)*

If we apply the multipliers to equal weights, we see in Table 19.3 that the two resulting portfolios are identical; that is the bottom-up multi-factor portfolio is identical to the combination portfolio in the left panel. This is the only scenario in which the two portfolios can be the same; when no cross-sectional distributional information is captured in the multiplier (e.g. ranks are used, each security receives one unique rank and no two ranks are the same), and the multipliers are applied to equal weights. Note that in the right panel, while the correlation between the weights in the two portfolios is 1.0, because the “excess weight” (the difference between the sum of the weights and the 100% target weight) is distributed unevenly across securities so that higher ranked larger stocks receive more weight, the weights for the two portfolios are different.

Similarly, if we use scores instead of ranks, the two outcomes are different, whether or not we apply the multipliers to equal weights or cap weights. For scores, we normalize the raw metrics for each security by subtracting the mean and dividing by the standard deviation across securities, for each factor. Scores preserve the distributional characteristics of each factor in a way that ranks do not. If a security has an extremely high price-to-book relative to the other securities, that “extremeness” will be captured by the score. Since multipliers cannot be negative for long-only portfolios, we use the ranks of the scores, and not the scores themselves, as the multipliers. Note that average rank is not the same as a rank based on average score. The latter is what we use in the bottom-up portfolio and it does indeed capture distributional characteristics. Table 19.3 (right panel) summarizes the difference between the rank-based approach and the score-based approach.

The score-based approach results in portfolios that are meaningfully more different from each other than the rank-based approach. In the case where multipliers are applied to equal weights, the correlation falls to 0.87 and the absolute sum of the weight differences is 32%, considerably higher than both rank-based examples. In the case where multipliers are applied to market cap weights, the correlation remains high at 0.99 but the absolute sum of the weight differences remains quite high at 24%.

We have seen with a highly stylized hypothetical 10-security example that scoring has a far greater impact than ranking when comparing combination versus bottom-up approaches to multi-factor portfolio construction. But, how much of an impact does it have in more realistic portfolios that employ a much larger number of securities?

We conduct the following simulations to further our understanding:

– combination portfolio: for the combination portfolio, we create the following single-factor portfolios: value, low volatility, quality and momentum. The definitions for factors are the same used in the tilted portfolios (section 19.6). Our

universe is the MSCI World Index. First, we rank the securities by each metric. Second, we group the securities into 20 fractiles, which we also refer to as subportfolios. We apply a fixed set of multipliers (linearly interpolated between 0.25 and 2.0 in increments of 0.25) to the market cap weight of the security depending on which fractile (subportfolio) it falls in. Finally, we rescale the weights such that they sum to 100%. The combination portfolio is an equally weighted average of the four individual factor portfolios. All portfolios are rebalanced monthly;

– bottom-up portfolio: first, we assign scores to securities for each factor. Second, we average the scores (equally weighting the factors). Third, we group the securities into 20 fractiles/subportfolios based on their average scores. We then apply the same fixed set of multipliers as in the combination portfolio depending on the fractile the security falls in. Finally, the weights are rescaled such that they sum to 100%. The factor definitions, universe, rebalancing frequency are the same as above.

The results are summarized in Table 19.4. The bottom-up returns are higher than any of the underlying component factor returns, and higher than the combinations. The difference is not insignificant, a spread of 86 basis points. Moreover, the volatility of the bottom-up approach is significantly lower and risk-adjusted return increases from 0.73 to 0.84 between the two approaches. This suggests that there are interaction effects important to capture. While these relationships are not likely to be the same across all factor combinations, we suspect that for the most well-known factors – value, size, volatility, momentum, etc. – these interaction effects will exist.

	Value portfolio	Low volatility portfolio	Quality portfolio	Momentum portfolio	Combination portfolio	Bottom-up
Annualized return	11.63%	10.69%	10.40%	10.91%	10.94%	11.80%
Annualized volatility	17.05%	13.77%	15.05%	15.07%	15.06%	14.12%
Risk-adjusted return	0.68	0.78	0.69	0.72	0.73	0.84
Excess return	3.49%	2.55%	2.26%	2.77%	2.80%	3.66%
Tracking error	7.12%	5.19%	4.43%	4.52%	4.78%	5.10%
Information ratio	0.49	0.49	0.51	0.61	0.59	0.72

Table 19.4. *Combination versus bottom-up approach, four-factor portfolios (January 1993–March 2015, Gross USD Returns)*

19.5. Conclusion

PMF portfolios have emerged in recent years as an alternative to investors dissatisfied with market-cap weighting or as an explicit way to achieve exposure to well-known factors that have been shown to drive stock returns. These portfolios employ indices and portfolios are managed to these indices just as in traditional

passive investing. Thus, PMF investing has the same benefits as traditional passive investing – transparency, implementation efficiency and low costs. These innovations are changing the investment landscape, which until recently, was composed of traditional passive investing and active management.

Portfolios can be constructed in many ways, ranging from relatively simple rules-based approaches that specify weights as a function of factor characteristics to more quantitatively-oriented ways that utilize more complex functions. Within multi-factor portfolio construction, we show that a bottom-up approach can produce superior results than a combination of individual single-factor portfolios, at least for well-known factors such as value, quality, low volatility and momentum.

19.6. Appendix A: description of tilted factor portfolios

The tilted factor portfolios shown in this chapter employ the following methodology. Each factor uses either a single metric or several metrics (in the latter case, they are equally weighted) as follows:

- value (valuation tilted strategy): price to fundamental (five fundamentals used: earnings, cashflow, sales, dividend and book value);
- low volatility (volatility tilted strategy): trailing 60-month variance of total returns;
- quality (quality tilted strategy): return-on-assets, variability in earnings per share² and leverage³;
- size (size tilted strategy): free float-adjusted market capitalization;
- momentum (momentum tilted strategy): trailing 12-month return.

For each tilted portfolio, we rank all stocks in the benchmark universe by the variable shown. (In the case of value and quality, a normalized score is first calculated and averaged across the individual metrics before ranking.) The stocks are next assigned to 20 ranked subportfolios such that each subportfolio holds 5% of the market capitalization of the universe⁴. The subportfolio with the highest ranking

2 Earnings variability is measured by the standard deviation of earnings per share (EPS) divided by the median earnings for the past 5 years. Dividing the median earnings normalizes the volatility and makes it more comparable across different companies.

3 Leverage is measured by total liabilities divided by shareholders equity. It indicates what percentage of equity and debt companies use to finance their assets. The lower the indicator, the more sound a company's financial strength is, and the higher quality it is, holding all other factors constant.

4 In the simulated strategies shown in this chapter, shares of a stock can straddle two subportfolios.

is subportfolio 20, while the subportfolio with the lowest ranking is subportfolio 1. Stocks within each subportfolio are cap-weighted. Next, a multiplier⁵ is assigned to each subportfolio with subportfolios in which the lower ranked subportfolios receive a multiplier less than 1 and the higher ranked subportfolios receive a multiplier greater than 1. This multiplier is then applied to each stock in the subportfolio's market cap weight. All weights are then rescaled to sum to 100%. All simulated strategies are rebalanced annually in March, while the momentum strategy is rebalanced quarterly.

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⁵ The multiplier used for the valuation tilted strategy is based on the ratio of the universe's weighted valuation ratio relative to each subportfolio's weighted valuation ratio. The same logic is applied to the size tilted strategy and volatility tilted strategies. In these last two, a maximum multiplier of 3 is allowed. For the quality tilted and momentum tilted strategies, the multiplier is 1.95 for subportfolio 20 and 0.05 for subportfolio 1, with a linear interpolation for the subportfolios in between.

Statistical Overfitting and Backtest Performance

In the field of mathematical finance, a “backtest” is the usage of historical market data to assess the performance of a proposed trading strategy. It is a relatively simple matter for a present-day computer system to explore thousands, millions or even billions of variations of a proposed strategy, and pick the best performing variant as the “optimal” strategy “in sample” (i.e. on the input dataset). Unfortunately, such an “optimal” strategy often performs very poorly “out of sample” (i.e. on another dataset), because the parameters of the investment strategy have been overfit to the in-sample data, a situation known as “backtest overfitting” [BAI 12, BAI 14a, BAI 14b, BAI 15].

While the mathematics of backtest overfitting has been examined in several recent theoretical studies, here we pursue a more tangible analysis of this problem, in the form of an online simulator tool. Given an input random walk time series, the tool develops an “optimal” variant of a simple strategy by exhaustively exploring all integer parameter values among a handful of parameters. This “optimal” strategy is overfit, since by definition a random walk is unpredictable. Then, the tool tests the resulting “optimal” strategy on a second random walk time series. In most runs using our online tool, the “optimal” strategy derived from the first time series performs poorly on the second time series, demonstrating how hard it is *not* to overfit a backtest. We offer this online tool to facilitate further research in this area.

20.1. Introduction

Modern high-performance computing technology, accelerated by the relentless advance of Moore’s Law, has enabled researchers in many fields to perform

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computations that would have been unthinkable in earlier eras. For example, in the July 2014 edition of the Top 500 list of the world's most powerful supercomputers (see Figure 20.1), the best system performs at over 30 Pflop/s (i.e. 30 “petaflops” or 30 quadrillion floating-point operations per second), a level that exceeds the sum of the top 500 performance figures approximately 10 years earlier [SIM 15]. Note also that a 2014-era Apple MacPro workstation, which features approximately 7 Tflop/s (i.e. 7 “teraflops” or 7 trillion floating-point operations per second) peak performance, is roughly on a par with the #1 system of the Top 500 list from 15 years earlier (assuming that the MacPro’s Linpack performance is at least 15% of its peak performance).

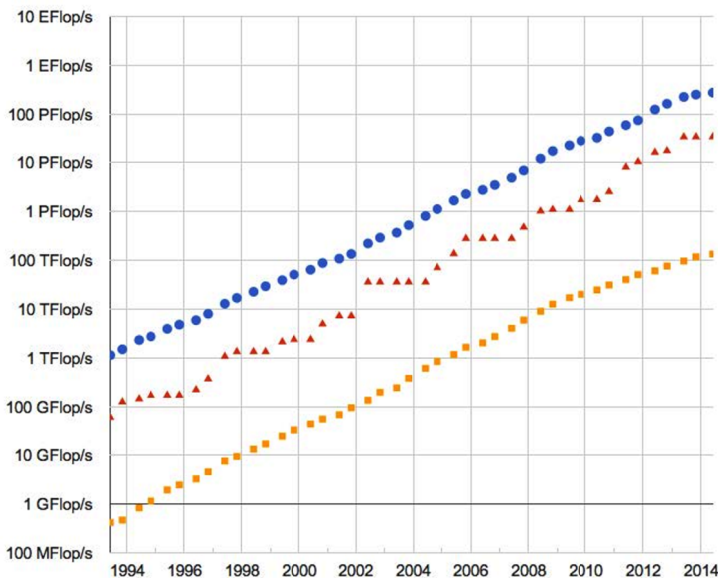


Figure 20.1. Performance of the Top 500 computers: Triangle = #1 system; Squares = #500 system; Circles = sum of #1 through #500

These powerful computer systems make it possible to analyze very large datasets and also to simulate very complex natural phenomena. However, they also permit researchers to explore thousands, millions or even billions of variations of a proposed model on a given dataset, and thus greatly magnify the potential for statistical overfitting errors.

In this context, statistical overfitting means either proposing a model for an input dataset that inherently possesses a higher level of complexity than that of the input dataset being used to generate or test it, or else trying many variations of a model on an input dataset and then only presenting results from the one model variation that appears to best fit the data. In many such cases, the model fits the data well only by fluke, since it is really fitting only the idiosyncrasies of the specific dataset in question, and has little or no descriptive or predictive power beyond the particular dataset used in the analysis.

Statistical overfitting can be thought of as an instance of “selection bias”, wherein a researcher presents the results of only those tests that support a predefined hypothesis. These types of errors are discussed in David Hand’s very readable 2014 book *The Improbability Principle* [HAN 14].

Statistical overfitting and “selection bias” are thought to be at the root of some of the reproducibility problems that have plagued several fields of scientific research in recent years. For example, in the biomedical field, there have been numerous instances of pharmaceutical products that look promising based on initial clinical tests and trials, but later disappoint in real-world implementation. The success rates for new drug development projects in Phase II trials have recently dropped from 28 to 18% [PRI 11]. The principal reason for these disappointments is now thought to be the fact that pharmaceutical firms, intentionally or not, typically only publish the results of successful trials, thus introducing a fundamental bias into the results.

Recently, the U.S. Securities and Exchange Commission announced that its examination of hedge funds uncovered a number of issues that are examples of “selection bias”. According to a *Wall Street Journal* report, the SEC “uncovered marketing and advertising issues, with some firms potentially misleading clients on past performance by ‘cherry picking’ their results from fund to fund” [ACK 14].

20.2. Backtest overfitting in finance and investments

In the field of mathematical finance, statistical overfitting most often arises when using “backtests” to develop and/or refine an investment strategy, a phenomenon known as “backtest overfitting”. The term “backtest” means using historical market data (e.g. the past 10 years of daily S&P500 closing averages) to evaluate how a proposed strategy would have performed had it been fielded over the past time frame in question. Since even a modestly powerful desktop or workstation can explore thousands, millions or even billions of variations of a strategy, it is not only possible but in fact quite likely that some variation of the strategy will perform well on this backtest dataset, yet in reality not have any useful predictive power, since the proposed strategy is only fitting idiosyncrasies in the “noise” of the dataset.

Overfitting can also occur when a statistical test is carried out multiple times on the same dataset, without controlling for the steady increase in the false positive rate (this is known as the “multiple-testing problem”).

As an example, if someone rolls a set of 10 six-sided dices, the probability of seeing all sixes (or any other particular prespecified combination) is approximately 1.65×10^{-8} , or in other words, roughly one chance in 60 million. But, as she/he rolls the 10 dices together over and over again, her/his chances of getting all sixes increase, until, after tens of millions of trials, she/he is almost guaranteed to see a roll with 10 sixes. If we had seen only the one all-sixes result of this experiment, we might have been justified in concluding that the dices are “loaded,” and that future rolls are likely to produce disproportionate numbers of sixes, but this is not the case.

Rolling 10 dices 60,000,000 times is perhaps not a practical real-world scenario. But using a computer to explore 60,000,000 variations of an investment strategy is a relatively minor task, something that could be done in a few minutes on a present-day system. Hence, such computer “experiments” are vastly more likely to result in overfitting errors.

As another illustrative example, suppose that a financial advisor sends out 10,240 ($= 10 \times 2^{10}$) letters to prospective clients, with half predicting that some stock or other security would go up in market value, and half predicting that it would go down. One month later, the advisor sends out a set of 5,120 letters, only to those who were earlier sent the correct prediction, again with half predicting some security will go up and half predicting it will go down. After 10 repetitions of this process, the final 10 recipients, were they not aware of the many letters to other clients, doubtless would be impressed at the advisor’s remarkable prescience. The set of often correct predictions sent to each of these final 10 recipients can be thought of as the equivalent of the string of 10 consecutive sixes in the first example above.

As a third example, suppose that an investor believes that there are daily, weekly or monthly patterns in historical stock market data, and he or she seeks a strategy that can exploit these patterns for financial gain. One very basic strategy would be to buy a set of stocks each Monday and then sell them on Wednesday. Another would be to buy stocks on the sixth day of each month and sell them on the 19th. A computer program can easily explore many thousands of such variations. The strategy could then be refined further by selling the portfolio at any time that it drops in value more than 10% from its initial price, or to purchase shares only when they increase in value by 10% over their value at the start of the trading period, as part of a strategy to capture “momentum” in market prices. There are enormous numbers of such combinations – millions just for this simple example – which are the equivalent to rolling the dice many times in the first example above. And, for the same reason,

it is highly likely that one of these parameter combinations will perform well on the historical dataset, but this is merely a “selection bias” statistical fluke.

Harvey and Liu [HAR 14] and Harvey *et al.* [HAR 14] report hundreds of examples where multiple testing and selection bias have taken place in the factor investing literature. The list is by no means exhaustive. In fact, it is very difficult to find publications where multiple testing has been controlled for when discovering new factors. This leads these authors to conclude that “*most claimed research findings in financial economics are likely false*”. Bailey and Lopez de Prado [BAI 14a], Bailey *et al.* [BAI 15] and Harvey and Liu [HAR 15a, HAR 15b] have proposed practical procedures to correct for the increased false positive probability that results from multiple testing.

Backtested models are usually based on a hypothetical phenomenon governing financial markets. However, as some have noted, “poorly performing strategies are discarded or optimized to create the final product” [BEA 13], thus unwittingly introducing a bias into the analysis. Indeed, it now appears that backtest overfitting errors are much more pervasive in the field than commonly recognized, and are likely to be the reason why many systematic funds, which strategies rely on backtests, often disappoint [BAI 14a]. Such errors are evidently an unfortunate byproduct of fast, computer-based tools used by analysts to explore, develop and refine potential models and strategies.

20.3. Quantifying backtest overfitting effects

How can we quantify the effects of backtest overfitting? A common statistic utilized to measure performance is the Sharpe ratio, as it can be “used to quantify the backtested strategy’s return on risk” [LOP 13]. The Sharpe ratio, informally speaking, is defined as the performance of an investment over a given period, normalized by the standard deviation of the investment’s changing value over that period. For a more precise definition and other technical details, see [LOP 13] or [BAI 14b].

While the Sharpe ratio on the backtest dataset is important, we must also consider the Sharpe ratio for the algorithm on new data. In an attempt to avoid backtest overfitting, researchers and analysts often use the “hold-out” method to a strategy, which consists of splitting the data into two subsets. The model or strategy is trained in one subset (called the in sample, or IS dataset) and tested on another subset (called the out of sample, or OOS dataset). While this form of cross-validation is useful for some purposes, unfortunately it is not a guarantee against overfitting, since the hold-out method does not control for the number of trials involved in a discovery. And, if we try hard enough, we can find an “optimal”

strategy that performs well on both the in sample and out of sample datasets, yet still has no substantive “skill”.

Two of the present authors, together with two other colleagues, co-authored some recent studies on backtest overfitting. In the first study [BAI 14b], a formula was derived relating the number of variations attempted in the development of a strategy to the size of the backtest dataset. For example, it was shown that if only 5 years of daily market data are available, and if 45 or more independent variations of a strategy are tried, it is more than likely that the best strategy selected in this process has a Sharpe ratio of 1.0 or better, indicating one standard deviation above the mean in performance. In the second study [BAI 15], a formula was derived for the probability of backtest overfitting. Numerous other results are presented in both papers. One particularly troubling consequence of this theory is that overfitting a backtest on time series with memory (e.g. autoregressive processes) leads to persistent losses, rather than just zero expected performance.

While a theoretically rigorous basis for backtest overfitting is important for fundamental understanding, it is clear that some additional research tools are needed. For example, as mentioned above, some researchers and analysts believe that markets may act cyclically or seasonally. While that hypothesis is likely to be true in some specific cases, a carefully designed experiment is required to reach any conclusion with a satisfactory degree of confidence. Preliminary studies have shown that in most cases, there simply are not enough data points to determine the statistical significance of these cyclical behaviors, after controlling for the number of trials involved in making those supposed discoveries. Several other hypotheses of this sort could be listed. Which of these ideas have merit and which do not?

20.4. An online demonstration of backtest overfitting

In this work, we present an online tool that allows analysts and researchers to experiment with the phenomenon of backtest overfitting. We have based the tool on a popular investment strategy, with a very limited number of possible parameter choices, in part to emphasize the fact that backtest overfitting can arise even in simple contexts, and is not only a potential problem for highly sophisticated strategies. For our test data, we simulate a series of daily prices by drawing returns from a Gaussian (normal) distribution, using a high-quality pseudorandom number generator.

The current version of the tool is available online at this URL:

<http://datagrid.lbl.gov/backtest>

20.4.1. Simple example of backtest overfitting (SEBO)

The investment strategy considered in our tool is particularly simple. We assume an investment decision is being made monthly, and only one equity with the simulated price is considered. A day of the month is chosen as the entry day for the investment. The strategy enters the market on either the buy side (long) or the sell side (short), and it always commits all available funds to this single position. The strategy exits the market either after it has held the equity for the specified number of days or once it triggers a stop loss condition. The variables controlling this simple strategy are then adjusted to produce good results, measured by the Sharpe ratio, based on the input time series dataset.

Our implementation of this Website consists of text introducing the parameters that control the process. A Python program then accepts input values from users, performs the computation and displays the output results. For convenience of discussion, we have named the program “Simple Example of Backtest Overfitting” (SEBO). The following is a detailed description of its operation:

1) SEBO first constructs a time series simulating stock market data. The daily price fluctuations are simulated by drawing returns from a standard Gaussian (normal) distribution, which are then compounded to derive a price time series. We use the pseudorandom number generator *random.gauss* from the Python programming language. The simulated prices generated this way are split into two equal parts, the first of which will be our in sample data and the second half will be the out of sample data.

2) SEBO then explores all possible variations of the trading strategy, based on the following parameters specified by the user:

i) *stop loss*: this is the maximum percent loss that can be sustained before the position is liquidated. To limit the number of choices, the user only chooses a maximum integer, so that the tool explores integer values up to the upper boundary;

ii) *holding period*: this is the maximum length of time that stock can be held before it is sold. This is given in terms of trading days per month, with a value that cannot exceed 22. The test tool examines all possible number of days between 1 and the maximum number of holding period specified by the user;

iii) *entry day*: this is the business day that the strategy enters the market in each trading month. All 22 trading days of a month are tried. The user does not control this parameter;

iv) *side*: this is the type of trading strategy, either “long” (profits are to be made when the stock is rising) or “short” (profits are to be made when the stock is falling). Our tool examines both choices of long and short for every combination of other parameters.

3) For each combination of parameters, SEBO computes the Sharpe ratio on the given input time series. When a set of parameters achieves a better Sharpe ratio than the current best set, SEBO records the parameter set and plots the value of the investment. After SEBO has examined all possible combination of parameters, the set of parameters it has on record is the “optimal” variation of the investment strategy.

4) The program then generates a second pseudorandom time series to test the strategy on a different time period (an “out of sample” dataset). The “optimal” variation of the investment strategy is applied to this second time series, and a Sharpe ratio is computed.

5) The program then outputs, on the result page, a “movie” showing the progression of the generation of the optimal strategy on in sample (backtest) data on the left-hand side of the result page, with the performance of the final, “optimal” strategy on out of sample data shown in the graph on the right-hand side of the result page.

20.4.2. How SEBO is used

The tool has an online form for the user to specify the parameters mentioned above. The values that can be assigned by the user are the maximum holding period, maximum stop loss, length of the backtest, standard deviation of the Gaussian distribution and seed of the pseudorandom generator. As explained above, the last three parameters control the simulated prices, while the first two parameters control how to exit the market. The SEBO program employs an extremely simple choice for when to enter the market: it simply chooses a fixed trading day of the month to enter the market. It tries all 22 possible choices in this case.

If a user is unsure what parameters to use, she/he may use the parameters that generated the example in Figure 20.2, or request the tool to choose a set of random parameters within the acceptable ranges. If we input a value for a parameter that is out of the acceptable range, the software uses a preset value for the parameter that falls within the acceptable range. The intent here is to permit the tool to be used by persons with a wide range of expertise in the field, from elementary to advanced.

After the execution of SEBO program, two figures are generated on the output page. In the examples from Figure 20.2, the green line is the underlying time series, and the blue line shows the performance of the strategy. “SR” denotes the Sharpe ratio. In most runs, the SR of the right-hand graph (i.e. the final strategy on out of sample data) is either negative or much lower than the SR of the final left-hand graph (i.e. the “optimal” strategy on in sample data), indicating that the strategy has been overfit on the in sample (backtest) data.

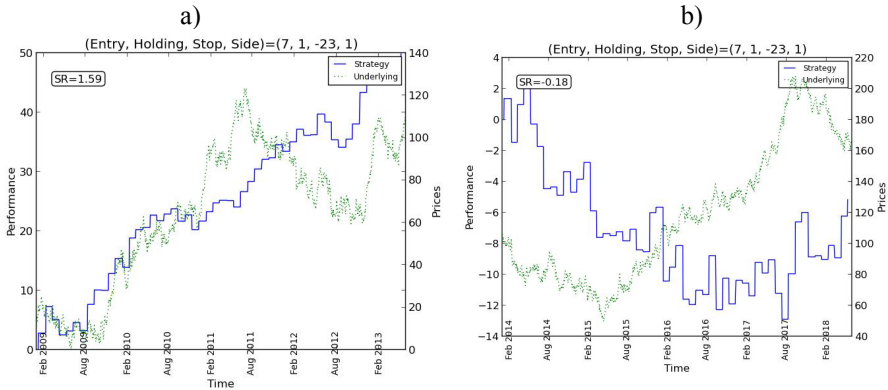


Figure 20.2. The “optimal” strategy’s performance a) on the in sample dataset, and b) on the out of sample dataset

In the specific example shown in Figure 20.2(a), note that the SR of the final optimized strategy, when applied to the input (in sample) dataset, is 1.59, indicating a fairly promising strategy (the annualized rate of return is 1.59 times the risk undertaken). However, when this same “optimized” strategy is applied to the second (out of sample) dataset, as shown in Figure 20.2(b), the resulting SR is -0.18 , indicating a completely ineffective strategy (it is actually somewhat prone to lose money). Even though in both cases, the underlying prices seem to oscillate in the similar way, on the in sample data, the investment represented by the blue line goes steadily up, while the same line on the out of sample data on the right goes steadily down. This suggests that the “optimal” strategy’s excellent performance on the in sample dataset was only a statistical fluke – the strategy was optimized to the particular characteristics of that data and had no fundamental “intelligence” to deal with any other dataset.

20.4.3. Understanding the results

Since the sample prices are generated with Gaussian distribution centered on zero, we expect the average investment strategy to have a Sharpe ratio of zero. This is indeed the case on the out of sample data in 400 test runs with different seeds for the random numbers (see Figure 20.3). In contrast, on this set of tests, the Sharpe ratios on the in sample data are centered on 0.9, which is significantly higher than zero.

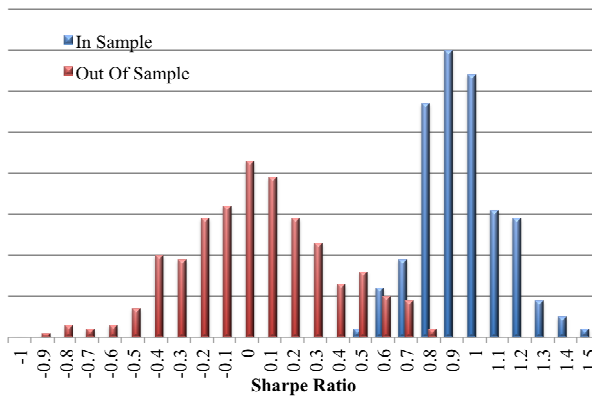


Figure 20.3. *Distribution of Sharpe ratios from 400 test runs with same parameters except the seeds to the random number generator*

Half of the test runs use the same set of parameters except the seed for the pseudorandom number generator. These parameter values are maximum holding period of 25, maximum stop loss percentage of 50, backtest length of 2,500 (simulating 10-year worth of daily data) and a standard deviation of 2. Given these parameters, SEBO will investigate 55,000 different parameter combinations on the in sample data. In other words, only a few tens of thousands of different variants of the proposed investment strategy are examined in each test run. Had more variants been examined, the Sharpe ratio of the tested strategies on in sample data would doubtless have been even higher. The other half of the test cases use random parameter values for maximum holding period, maximum stop loss percentage, backtest length and standard deviation. The seeds for random number generator in SEBO vary from 201 to 400.

Keep in mind that we are optimizing only a couple of parameters. Investment strategies often involve many more parameters. If we add another parameter, like a profit taking threshold, the overfit SR will be boosted further, to any desirable level. But, our goal is to show that even the simplest of the investment strategies can be easily overfit. An important conclusion is that there is no SR threshold or haircut that can be considered safe.

Although there is much still to be learned about the phenomenon of backtest overfitting, it is clear from running just a modest number of cases that when attempting to produce an “optimal” strategy, it is very hard to avoid backtest overfitting. Indeed, the online tool demonstrates how easily false trading strategies can be derived from purely random data. And, if we do not know how many

variations of a strategy have been attempted when developing a strategy, there is no way to know *a priori*, one way or the other, whether the resulting strategy is overfit.

Another conclusion from using the online tool is that the “hold-out” method is not very effective in preventing backtest overfitting. If the web application is run once, it is very likely that the optimized strategy will perform well on the in sample dataset but poorly on the out of sample dataset. However, if enough cases are tried using the online tool, a strategy that performs well both in sample and out of sample can be discovered. And yet, as before, the strategy cannot have any innate “intelligence,” since it is generated based on a pseudorandom dataset.

It should also be emphasized that while this tool was designed to demonstrate the effect of backtest overfitting in mathematical finance, the fundamental underlying principle of statistical overfitting applies very broadly nonetheless. Thus, this tool could be easily modified to demonstrate a much broader class of overfitting problems. Indeed, by simply renaming of the input parameters and output results (i.e. renaming the Sharpe ratio, and suitably changing the output plots), we could consider the online tool to be a test of statistical overfitting when attempting to “guess” the future course of any process that can be modeled by a random walk.

20.5. Conclusion

We have developed an online tool to demonstrate the dangers of backtest overfitting in the mathematical finance field, although, as we emphasized above, the underlying mathematics and software design could easily be considered to be a demonstration of the much broader problem of statistical overfitting of a random walk process.

By using the tool to generate even a modest number of trials, it is immediately clear that it is extremely easy, by using a computer to explore the parameter space of variations of a basic strategy, to “discover” what appears to be an “optimal” trading strategy that gives great-looking performance, based on standard financial performance statistics such as the Sharpe ratio, but yet is completely impotent when presented with any other dataset. The problem, as emphasized above, is that the resulting “optimal” strategy is statistically overfit, since far more variations of the strategy have been tried that can be justified given the size of the input dataset. For this reason, it is actually quite likely that even a modestly sophisticated search process will identify what mistakenly appears to be a promising strategy.

We hope that this research will help investors understand the dangers of backtest overfitting in particular, and selection bias in general. There is still much to be learned about this perplexing phenomenon.

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