Improving the Investment Process with a Custom Risk Model: *A Case Study with the GLER Model*

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Markowitz [1952, 1991] developed the mean–variance optimization (MVO) model to construct portfolios that optimally trade off risk and return. The three important ingredients in an MVO model are the alpha vector representing expected returns, the risk model that is used to measure the variance of the portfolio, and a set of constraints representing the portfolio managers' mandates and choices. Misalignment arises when the alpha vector is not completely spanned by the factors in the risk model. It results in the optimizer taking large exposures on factors that have systematic risk but are missing from the risk model. With constraints, misalignment appears between the implied alpha and the risk model. Misalignment in MVO results in optimal portfolios that suffer from risk underestimation, undesired exposures to factors with hidden systematic risk, a consistent failure of the portfolio manager to achieve ex-ante performance targets, and an intrinsic inability to transform superior alphas into outperforming portfolios; see Saxena and Stubbs [2013] and Ceria et al. [2012]. Saxena and Stubbs [2010] provided a theoretical framework to show that the *alpha alignment factor* (AAF) (Renshaw et al. [2006]) alleviate the misalignment problem. Moreover, they present empirical results with the AAF showing improved ex-post performance. Saxena and Stubbs [2012] highlighted the efficacy of the AAF in addressing the misalignment issues that occur with the U.S. expected returns (USER) model; see also the foreword by Markowitz [2012] for comments on this approach.

Consider an MVO model with an investment universe of n assets. Let h_i denote the weight invested in the ith asset. Let α_i denote the portfolio managers' estimate of the expected return for the ith asset. We will assume that the risk is measured by a factor model. We will differentiate between two types of factors: *alpha* factors, which have a positive long-term risk premium, and *risk* factors, which explain the cross-section of asset returns but do not have such a long-term trend. Examples of alpha factors include value, momentum, and growth, while examples of risk factors include industries and countries. Let B_A and B_R denote the asset exposures to the alpha and risk factors, respectively. Let us suppose that the alpha signal is a linear combination of the factors in B_A, i.e., $\alpha = B_A\omega$. The risk model is given by

$$Q = B\Omega B^T + \Delta^2 \tag{1}$$

where

$$B = [B_A \, B_R]$$

is the combined matrix of factor exposures,

$$\Omega = \begin{bmatrix} \Omega_A & \Omega_{AR} \\ \Omega_{AR}^T & \Omega_R \end{bmatrix}$$

is the factor covariance matrix, and Δ^2 is a diagonal matrix of specific variances. Consider the following MVO model

$$\max_{Ah \le b} \alpha^T h - \frac{\lambda}{2} h^T Q h \qquad (2)$$

where $\lambda > 0$ is an appropriate risk-aversion parameter, and $Ah \le b$ is the set of all constraints imposed by the portfolio manager. For ease of exposition, we consider only linear constraints in the MVO model in this section. The discussion can also easily be extended to include nonlinear constraints. Examples of portfolio constraints in the MVO model include asset bounds, sector exposure bounds, limit on the number of names, turnover, and so on.

Note that our risk model in (2) also contains the factors that are used in the construction of the alpha vector. Consider what would happen if the alpha signal is not spanned by all the factors in the risk model. First, consider the unconstrained case with $\Delta^2 = \sigma_s^2 I$, i.e., all assets have the same specific risk. The alpha signal can be decomposed as

$$\alpha = \alpha_{B_R} + \alpha_{B_R^\perp} \qquad (3)$$

where $\alpha_{B_R} = B_R (B_R^T B_R)^{-1} B_R^T \alpha$ is the portion of α that is spanned by the exposures in the risk model and $\alpha_{B_R^\perp} = \alpha - \alpha_{B_R}$ is the portion of α that is orthogonal to the risk exposures. The optimal solution to the unconstrained problem is given by

$$\begin{aligned} h^* &= \frac{1}{\lambda} Q^{-1} \alpha, \\ &= \frac{1}{\lambda} (B_R \Omega_R B_R^T + \sigma_s^2 I)^{-1} \alpha \\ &= \frac{1}{\lambda \sigma_s^2} \alpha_{B_R^\perp} \frac{1}{\lambda \sigma_s^2} \left(I - B_R (B_R^T B_R + \sigma_s^2 \Omega_R^{-1})^{-1} B_R^T \right) \alpha_{B_R} \end{aligned} \qquad (4)$$

Lee and Stefek [2008] noted that the optimal solution is dominated by the first term that is simply $\alpha_{B_R^\perp}$ scaled by the specific variance. So, the optimizer overweights $\alpha_{B_R^\perp}$ relative to α_{B_R} in the final portfolio. In doing so, the optimizer takes excessive exposure to factors that have systematic risk but are missing from the factor risk model. This leads to the MVO model badly underestimating the actual risk associated with the optimal portfolio. Including the alpha factor or all its components in the risk model ensures that the optimizer correctly trades off the risk and the return of the alpha signal; in this case, $\alpha_{B_R^\perp} = 0$ and $\alpha_B = \alpha$, so only the second component of the optimal solution in Equation (4) is non-zero.

Now consider the following risk model

$$\bar{Q} = [B_R \; \alpha] \begin{bmatrix} \bar{\Omega}_R & \bar{\Omega}_{R\alpha} \\ \bar{\Omega}_{R\alpha}^T & \bar{\omega}_\alpha \end{bmatrix} \begin{bmatrix} B_R^T \\ \alpha^T \end{bmatrix} + \Delta_b^2 \qquad (5)$$

that explicitly includes the alpha signal as a custom factor, in addition to the risk factors in the model. It is worth emphasizing that the difference between this model and the factor model in Equation (1) is that the former contains the alpha signal rather than its components that are in B_A. This risk model is also aligned with the alpha signal in the unconstrained case. However, the constrained case is very different. Constraints introduce additional misalignment between the alpha vector and the factors in the risk model. To see this, one can use the theory of convex optimization to replace the constrained MVO model (2) with the following unconstrained model

$$\max \, (\alpha^I)^T h - \frac{\lambda}{2} h^T Q h \qquad (6)$$

where

$$\alpha^I = \alpha - A^T \pi \qquad (7)$$

is the *implied alpha*, where π contains the optimal dual multipliers to the linear constraints $Ah \le b$. Now suppose that the alpha signal is made of two factors α^1 and α^2. Furthermore, assume that the MVO model has a constraint imposing an upper bound on the exposure that the portfolio takes to α^2. Suppose the optimal dual multiplier to this constraint exactly matches the weight of α^2 in the alpha factor. For this example, the implied alpha is given by

$$\begin{aligned} \alpha^I &= \omega_1 \alpha^1 + \omega_2 \alpha^2 - \pi \alpha^2 \\ &= \omega_1 \alpha^1 \end{aligned}$$

In other words, the implied alpha signal is a multiple of the α^1 factor. In this case, the implied alpha signal will be misaligned with the risk model (5) that contains only the alpha signal. On the other hand, the implied alpha signal is still aligned with the factor model (1) that contains both α^1 and α^2 in B_A. This remains true for any value of π.

The optimal solution to (2) can be written as

$$h^* = \frac{1}{\lambda\sigma_s^2}\alpha_{B^\perp}^1 \frac{1}{\lambda\sigma_s^2}\left(I - B(B^T B + \sigma_s^2\Omega^{-1})^{-1}B^T\right)\alpha_B^I \qquad (8)$$

where $\alpha^I = \alpha_{B^\perp}^I + \alpha_B^I$ is a decomposition of the implied alpha with $\alpha_B^I = B(B^T B)^{-1} B^T \alpha^I$. Note that α_B^I will generally explain a larger portion of α^I with the original factor model (1) than a risk model that does not explicitly contain the alpha factors. Consequently, the second component along α_B^I in (8) is better represented in the optimal solution. Therefore, a better way to persuade the optimizer to correctly identify the systematic risk associated with taking bets on the alpha vector in the presence of constraints is to introduce each of the components of the alpha vector, namely the factors in B_A, as factors in a custom risk model. We will use a custom risk model (CRM) that contains all the components of the alpha vector for a case study with the global expected returns (GLER) model in this article.

Our aim is to showcase the following desirable features of a custom model in this article:

1. Correct for risk underestimation.
2. Better represent the alpha signal in the final portfolio in an optimal risk-adjusted fashion. By doing so, improve the IR (information ratio) of the active portfolio, i.e., push the realized frontier upward.
3. Generate a more intuitive and useful ex-post performance attribution analysis of the portfolio.

The article is organized as follows: The next section describes the GLER model. The next two sections describe the construction of the custom risk model and the alpha signal for the GLER study. The following section presents the case study. We report our conclusions in the final section.

THE GLER MODEL

The GLER model uses fundamental valuation factors, which use reported earnings and other financial data and momentum (see Guerard et al. [2012b], for the role of momentum in predicting asset returns) to construct expected return estimates for assets. Guerard et al. [2013a] have a detailed description of the GLER model. Guerard, et al. [2012a, 2013b] integrated the USER and GLER models in several portfolio construction strategies to generate portfolios with attractive ex-post properties.

We will give a brief description of the important features of this model in this section as it pertains to our study. The GLER is a multi-factor model given by

$$\begin{aligned} \text{TR} = {} & w_0 + w_1\text{EP} + w_2\text{BP} + w_3\text{CP} \\ & + w_4\text{SP} + w_5\text{REP} + w_6\text{RBP} + w_7\text{RCP} \\ & + w_8\text{RSP} + w_9\text{CTEF} + w_{10}\text{PM} + e_t \end{aligned}$$

where

- TR = Asset return from period t to period $t + 1$;
- EP = Earnings–price ratio = earnings per share/price per share;
- BP = Book–price ratio = book value per share/price per share;
- CP = Cash–price ratio = cash flow per share/price per share;
- SP = Sales–price ratio = net sales per share/price per share;
- REP = Relative earnings–price ratio = earnings–price ratio/average earnings–price ratio over the past 5 years;
- RBP = Relative book–price ratio = book–price ratio/average book–price ratio over the past 5 years;
- RCP = Relative cash–price ratio = cash–price ratio/average cash–price ratio over the past 5 years;
- RSP = Relative sales–price ratio = sales–price ratio/average sales–price ratio over the past 5 years;
- CTEF = Consensus earnings-per-share I/B/E/S forecast, revisions, and breadth;
- PM = Price momentum = price at time t–1 (a month ago)/price at time t–12 (a year ago);
- e_t = randomly distributed residual term.

These estimates are altered over time as company attributes and investing fashions change. The GLER model is estimated using weighted latent root regression analysis to identify variables that are statistically significant at the 10% level; it uses the normalized coefficients as weights and averages the variable weights over the past 12 months.

The CTEF attribute is generated from forward forecast information. It is an equally weighted version of the following attributes

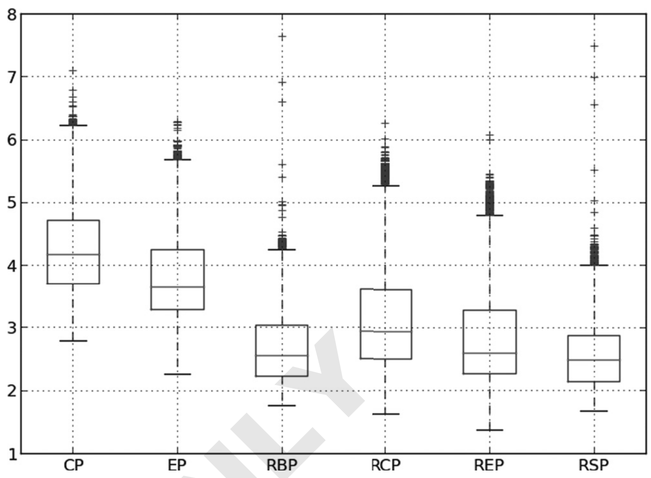
- FEP1 = One-year ahead forecast earnings per share/price per share;
- FEP2 = Two-year ahead forecast earnings per share/price per share;
- RV1 = One-year ahead forecast earnings per share monthly revision/price per share;
- RV2 = Two-year ahead forecast earnings per share monthly revision/price per share;
- BR1 = One-year ahead forecast earnings per share monthly breadth;
- BR2 = Two-year ahead forecast earnings per share monthly breadth.

The GLER attributes are available as *ranks* between 0–99, with 0 being the *least desirable* and 99 being the *most desirable*. These data are available on the last trading day of the month between January 1999 and November 2011. Note that nine of the attributes, namely, EP, BP, CP, SP, REP, RBP, RCP, RSP, and CTEF, are *Value* factors, while the PM attribute is a *Momentum* factor. We want to construct a custom risk model called *CRM* from *BaseFund*, which includes the GLER attributes, and excludes the *Value* and *Short-Term Momentum* factors for the GLER study in this article. We removed the Value factor since we have several proxies for Value among the GLER composite factors. Moreover, we remove the Short-Term Momentum factor because it is too short-term for the strategy that we will consider in our study. Note that we retain the base model's Medium-Term Momentum factor in the CRM.

Collinear factors introduce estimation errors in the regressions used to estimate the factor returns; this is the reason that a weighted latent root regression was employed to calculate the coefficients in the GLER model. So we want to identify and coalesce the collinear factors. There is a high degree of collinearity between the various Value attributes. To highlight the

EXHIBIT 1

VIFs for Collinear Factors in *CollinearCRM*



collinearity issue, we construct a custom risk model called *CollinearCRM* from the Axioma fundamental model, *WW21AxiomaMH* (hereafter referred to as *BaseFund*), which includes the 10 GLER factors and excludes the Value and the Short-Term Momentum factors from *BaseFund*. Exhibit 1 contains a box plot of the variance inflation factor (VIF) for six suspected collinear factors (EP, CP, REP, RCP, RSP, RBP) in *CollinearCRM*. The top and the bottom of the box represent the first and the third quartiles, and the band inside the box represents the median of the VIF distribution. Let us define the interquantile range (IQR) as the difference between the third and the first quartiles. The lower whisker of the box plot represents the value that is 1.5 IQR below the bottom of the box. Similarly, the upper whisker is the value that is 1.5 IQR above the top of the box. Values outside the whiskers are regarded as outliers and plotted with crosses. The high VIF values (>5) for some of the factors indicate multicollinearity in this risk model. We first conduct the following *collinearity* study of the GLER model:

1. Standardize each rank attribute along the same lines as the style factors in the fundamental model. In particular, this is done for each rank attribute b as follows:
   a. Calculate the capitalization-weighted mean-exposure $\overline{b} = b^T h_u$, where h_u contains the market-cap weights for all the assets in the *estimation universe* of the *BaseFund* model and is 0 otherwise.

b. Calculate the equal-weighted standard deviation

$$\sigma = \sqrt{\frac{1}{n-1}\left(\sum_{i=1}^{n}(b_i - \bar{b}_i)^2\right)}$$

of the exposure values in b about the market-cap-weighted mean $\bar{b}$ and n includes all the assets in the original attribute.

c. The standardized attribute $\hat{b}$ is then given by

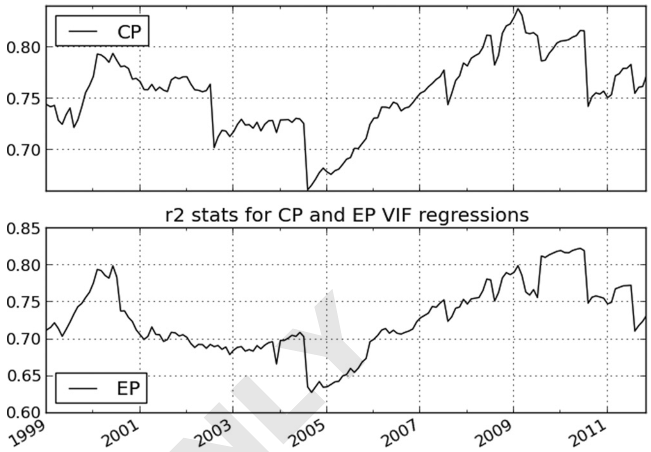
$$\hat{b}_i = \frac{b_i - \bar{b}}{\sigma}, i = 1, \ldots, n$$

2. Calculate VIF statistics where we regress each standardized attribute against the remaining attributes, and all the other style factors in the *BaseFund* model, excluding Value and Short-Term Momentum. Carry out a square root of market-cap-weighted regression with an intercept term. Examine the beta coefficients, the *t*-statistics, and the coefficient of determination from the regression.
3. Coalesce the two most collinear factors from the VIF regression. A composite GLER rank factor is obtained by equally weighting the two collinear rank factors. We then standardize this composite factor using Step (1) and carry out the VIF regression in Step (2). The process is repeated until the VIF regression reveals that there are no collinear factors.

The collinearity study revealed that EP and CP were the two most collinear factors in the first round. Let us briefly describe how we arrived at this decision. We run an end-of-month cross-sectional regression for all the standardized attributes between January 1999 and November 2011. Exhibit 2 plots the r-squared (coefficient of determination) for the EP and CP VIF regressions. Note that the r-squared values of the two regressions are high, indicating that EP and CP are collinear with other factors in the VIF regression model. Exhibits 3(a) and 3(b) contain box plots of the beta coefficients for the various factors in the EP and CP VIF regressions, respectively. Exhibits 3(c) and 3(d) contain the box plots of the *t*-stats in the EP and CP VIF regressions, respectively. These statistics indicate that EP and CP are highly collinear and need to be coalesced together. The EP-CP rank attribute was obtained by

E X H I B I T 2

Coefficient of Determination for the EP and CP VIF Regressions



equally weighting these two attributes together. The second round revealed that RSP and RBP should be equally combined to form the composite RSP-RBP attribute. Finally, the third round revealed that REP, RCP, and EP-CP were collinear. So, the four attributes, EP, CP, REP, and RCP, were equally weighted together (25% weight each) to form the composite EP-CP-REP-RCP attribute. To summarize, we now have the following six composite attributes:

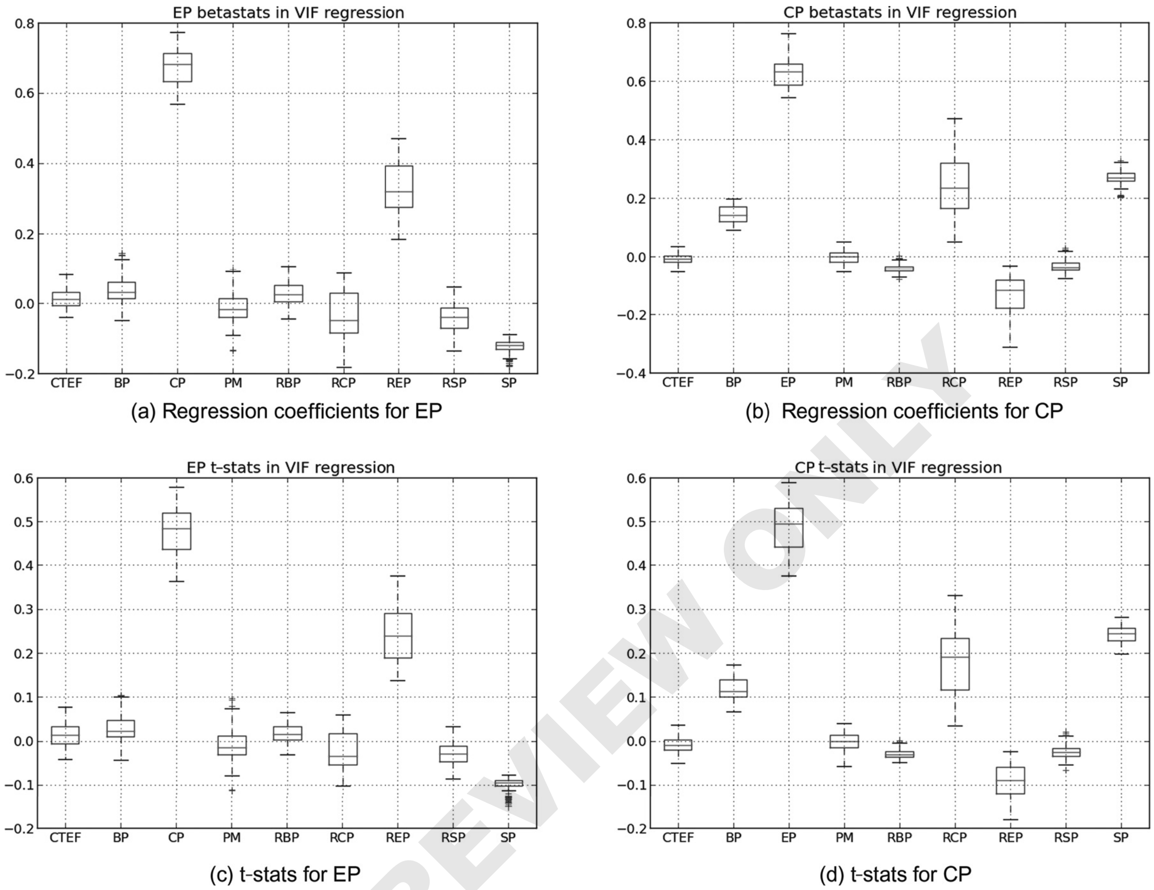
$$\begin{aligned} \text{EP-CP-REP-RCP} &= 0.25\text{EP} + 0.25\text{CP} \\ &\quad + 0.25\text{REP} + 0.25\text{RECP} \\ \text{RSP-RBP} &= 0.5\text{RSP} + 0.5\text{RBP} \\ \text{SP} &= \text{SP} \\ \text{BP} &= \text{BP} \\ \text{CTEF} &= \text{CTEF} \\ \text{PM} &= \text{PM} \end{aligned} \tag{9}$$

Note that the equal weighting employed to construct the composite attributes in Exhibit 17 is arbitrary, but sufficient for our illustrative purposes.

CUSTOM RISK MODEL

This section will briefly describe the steps used in the construction of a custom risk model for the GLER study. The primary custom risk model used in this article will henceforth be labeled as *CRM*. This custom risk model is generated from the *BaseFund* model. It includes

Exhibit 3
Comparing the Regression Coefficients and *t*-Stats for the EP and CP VIF Regressions



(a) Regression coefficients for EP (b) Regression coefficients for CP

(c) t-stats for EP (d) t-stats for CP

the composite factors EP-CP-REP-RCP, RSP-RBP, SP, BP, CTEF, and PM, and excludes the Value and Short-Term Momentum factors from *BaseFund*.

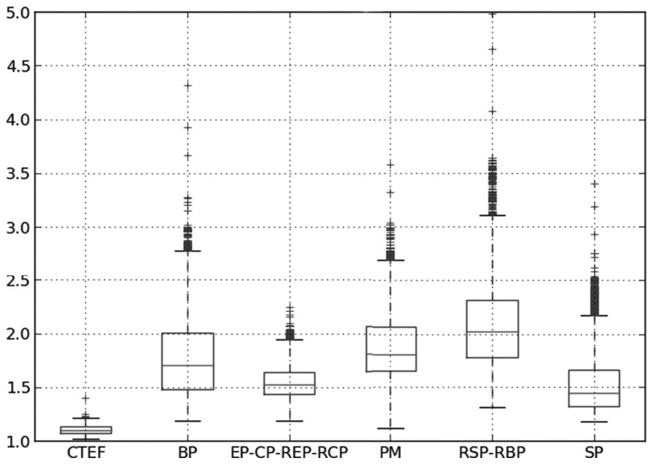
Recall that we have the GLER rank attributes for the last trading day of each month between January 1999 and November 2011. We first construct a matrix of daily factor exposures for each trading date between January 1999 and November 2011. This matrix includes the standardized GLER composite factors as well as a subset of the Axioma style, industry, country, currency, and global market factors from the fundamental model. The standardized GLER composite factors from the last trading day of a month are used as proxies for all the trading days for the succeeding month, excluding the last trading day for which data are available.

The CRM is constructed in the same way as *BaseFund* except for the difference in the style factors. We employ a robust regression scheme to construct the factor returns from the factor exposures and asset returns. The factor covariance matrix is then constructed from these factor returns. We refer the reader to Axioma [2013] for more details. To recap, some of the original GLER attributes were combined to avoid collinear factors. This is done to reduce the estimation errors in the regressions that generate the factor returns. Exhibit 4 contains a box plot of the variance inflation factor (VIF) for each of the GLER attributes in the final model. A word on how we computed the VIFs is in order. The VIF for a composite GLER factor is obtained by regressing this factor against all the styles in the custom model. We use a weighted regression, with the weights determined by the robust regression (to estimate factor returns) used in the construction of the risk model.

This regression is carried out on each trading day between the end of January 1999 and the end of November 2011. The VIFs for the composite GLER

Exhibit 4

VIFs for Composite GLER Factors in *CRM*



attributes are below 5. Therefore, coalescing the original 10 attributes into 6 composite attributes did alleviate the multicollinearity issue among these attributes.

GENERATING AN ALPHA FROM COMPOSITE GLER ATTRIBUTES

We will briefly describe how we generate our primary alpha signal, hereafter labeled *Alpha1*, from the GLER composite attributes in this section. We do so by first transforming these rank attributes into portfolios. A factor mimicking portfolio (FMP) is a long–short, dollar-neutral portfolio that represents a factor. Let B_A denote the factor matrix containing the standardized GLER composite attributes. We standardize the rank attributes by subtracting the market-cap-weighted mean over the estimation universe of the fundamental model and dividing the result by the equally weighed standard deviation about the market-cap-weighted mean as described in the section on the GLER model. Let B_R contain the standardized exposures for all the factors in the *BaseFund* minus the Value and the Short-Term Momentum factors. The FMP associated with the *j*th composite attribute is the solution h^j, to

$$\begin{aligned} \min \quad & h^T W h \\ \text{s.t.} \quad & B_R^T h = 0 \\ & B_A^T h = e_j \end{aligned} \tag{10}$$

where W is a diagonal matrix containing the square root of the asset market caps, and e_j is the vector with 1 in the j^{th} position and zeros elsewhere. Note that the FMP associated with an attribute has an unit exposure to this attribute and is neutral to all the other factors in the risk model. The FMP returns represent the *pure* attribute returns. In the Axioma fundamental model, these returns are computed using cross-sectional regressions. We use the square root of market cap weighting in the FMPs so that it is consistent with the initial weighting employed in the robust cross-sectional regression in Axioma's fundamental model. Our alpha signal for the study is given by

$$\alpha = B_A \omega \tag{11}$$

where

$$\omega_j = \frac{1}{T} \sum_t (r^t)^T (h^{jt}), \; j = 1, \ldots, m \tag{12}$$

T is the total number of time periods, and r^t and h^{jt} denote the time series of realized asset returns and FMP holdings, respectively.

We generate *Alpha1* for the GLER study from the composite GLER factors as follows: For each of the 6 composite GLER factors, we run an end-of-month backtest between January 1999 and November 2011, where each rebalancing constructs an FMP for the appropriate factor by solving (10). The FMP investment universe in each period includes all the assets in the GLER attribute that are also in the *BaseFund* model. The B_A matrix for each FMP includes the 6 composite GLER attributes, and the B_R matrix includes all the factors in the *BaseFund* model, excluding the Value and Short-Term Momentum styles. The composite alpha signal is then constructed using (11), where the weights in (12) represent the long-term average returns of the FMP portfolios.

Exhibit 5 contains a box plot of the monthly returns of the 6 FMP portfolios between the end of January 1999 and the end of November 2011. Exhibit 6 contains the annualized average FMP returns for these 6 composite attributes.

ILLUSTRATIVE EXAMPLE

In this section, we present an outline of the GLER study in the first subsection. The second subsection illus-

Exhibit 5
Monthly Returns of the GLER Composite FMPs

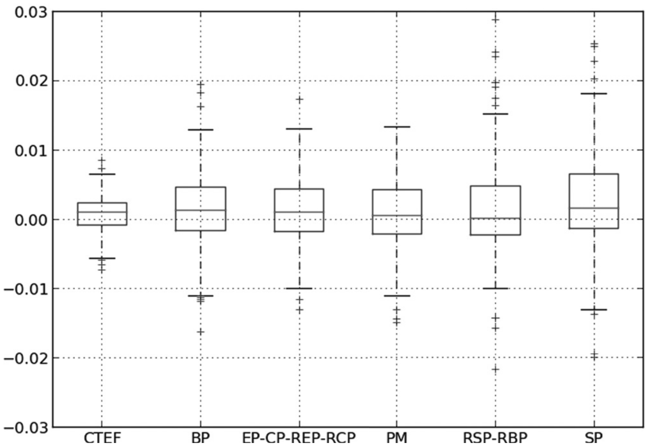


Exhibit 6
Annualized Average FMP Returns for GLER Composite Factors

Attribute	Annualized Average FMP Return
EP-CP-REP-RCP	1.62%
RSP-RBP	2.03%
SP	2.44%
BP	1.89%
CTEF	0.91%
PM	1.00%

trates the use of our custom model *CRM* in improving risk underestimation and IR in the MVO model. The third subsection highlights the intuitive and helpful ex-post PA summary given by *CRM*.

GLER Study Setup

We first describe the GLER study that is used to highlight the role of the custom risk model. We want to combine the GLER attributes EP, BP, CP, SP, REP, RBP, RCP, RSP, CTEF, and PM into a composite alpha signal that delivers a high IR in an end-of-month rebalancing between January 2000 and November 2011. These attributes are available on the last trading day of the month between January 1999 and November 2011. We can use the risk model of our choice to measure the risk taken by the portfolio. We experiment with the *BaseFund* model, the Axioma statistical model *WW21AxiomaMH-S* (hereafter referred to as *BaseStat* in this article), and our custom model *CRM*. The section on the custom model describes the construction of CRM. The investment universe during each rebalancing period includes all the master assets that are also in the risk model employed in the optimization; the size steadily increases from 8,000-odd assets in January 2000 to 10,500-odd assets in November 2011. The portfolio also must satisfy the following mandates.

1. Long-only and fully invested portfolio.
2. Target a realized active risk of 4% with respect to a global cap-weighted benchmark.[1] The strategy has a tracking error constraint with respect to this benchmark and the chosen risk model to achieve this goal.
3. Asset bounds of 4%.
4. Minimum threshold holdings of 0.35%.
5. Maximum round-trip turnover of 16%.

We construct a composite alpha signal called *Alpha1* from the 6 composite GLER factors, where the weights are based on the long-term expected return of the FMPs corresponding to these factors (see Exhibit 6). Exhibit 7 contains these weights.

Use of Custom Model in Improving Risk Underestimation and IR

We run three end-of-month frontier backtests with the *Alpha1* signal that use the *BaseStat*, *BaseFund*, and *CRM* custom risk models. The three resulting portfolios are labeled as *Alpha1-BaseStat*, *Alpha1-BaseFund*, and *Alpha1-CRM*, respectively. The turnover constraint is allowed to be relaxed during the backtest in order to achieve a feasible rebalancing in each period. Exhibit 8 presents the realized backtest summary for the *Alpha1-BaseStat* portfolio for TE varying between 1.9% to 7%. Note that the *BaseStat* model underestimates the risk associated with the portfolio. The bias

Exhibit 7
GLER Attribute Weights in *Alpha1*

Attribute	Weight
EP-CP-REP-RCP	16.4%
RSP-RBP	20.53%
SP	24.7%
BP	19.06%
CTEF	9.2%
PM	10.10%

EXHIBIT 8
Frontier Summary for the *Alpha1-BaseStat* Portfolio

	Tracking Error						
	1.9%	**2.5%**	**3%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	4.26%	6.47%	8.75%	12.25%	14.39%	14.60%	16.86%
Avg Annualized Active Risk	4.18%	5.44%	6.46%	8.54%	10.18%	11.75%	13.42%
Avg Turnover	18.06%	17.21%	16.46%	16.30%	16.19%	16.14%	16.08%
Information Ratio	1.02	1.19	1.36	1.43	1.41	1.24	1.26

statistic (ratio of the realized risk to the predicted risk) is around 2.1. Running the backtest with a TE of roughly 1.9% gives a portfolio with a realized risk of 4.0%, and the IR of this portfolio is 1.02. Exhibit 9 presents the backtest summary for the *Alpha1-BaseFund* portfolio for TE varying between 2.7% to 7.0%. The *BaseFund* model also underestimates the risk associated with the portfolio, though to a lesser extent. The bias statistic is around 1.5. Running the backtest with a TE of roughly 2.7% gives a realized risk of 4.0%, and the IR of this portfolio is 1.02. Exhibit 10 presents the realized backtest summary for the *Alpha1-CRM* portfolio for TE varying between 3.0% to 7.0%. The *CRM* has a bias statistic of 1.15; it has the least risk underestimation among the three risk models. The perceptive reader might wonder why the *CRM* model also underestimates risk when there is no misalignment between the alpha factor and the risk model. This is due to the misalignment between the implied alpha and the risk model, especially caused by the asset bound and turnover constraints that are unlikely to be spanned by the factors in the risk model.

Running the backtest with a TE of roughly 3.5% gives a portfolio with a realized risk of 4.0%, and the IR of this portfolio is 1.48. Exhibit 11 plots the realized frontiers (realized return versus realized risk) for the *Alpha1-BaseStat*, *Alpha1-BaseFund*, and *Alpha1-CRM* portfolios for varying TE. Clearly, the *CRM* is able to push the frontier upward in addition to correcting for risk underestimation.

Exhibit 12 compares the average predicted risk and the realized risk associated with these three portfolios. The average predicted risk is also subdivided into the factor and specific risk portions. Clearly, the *BaseStat* model is underestimating total risk. Moreover, this model is attributing most of this total risk to the specific risk, and the predicted factor risk is very small. The predicted specific risks for *BaseFund* and *CRM* are about the same

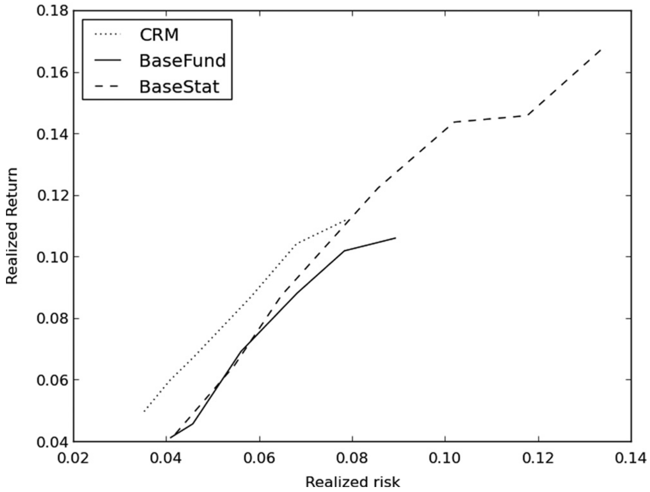
EXHIBIT 9
Frontier Summary for the *Alpha1-BaseFund* Portfolio

	Tracking Error					
	2.7%	**3%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	4.14%	4.59%	6.95%	8.84%	10.21%	10.62%
Avg Annualized Active Risk	4.07%	4.56%	5.60%	6.80%	7.81%	8.91%
Avg Turnover	16.91%	16.40%	16.16%	16.07%	16.07%	16.09%
Information Ratio	1.02	1.01	1.24	1.30	1.31	1.19

EXHIBIT 10
Frontier Summary for the *Alpha1-CRM* Portfolio

	Tracking Error					
	3%	**3.5%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	5.0%	6.0%	6.78%	8.71%	10.44%	11.24%
Avg Annualized Active Risk	3.50%	4.06%	4.58%	5.80%	6.79%	7.89%
Avg Turnover	16.37%	16.21%	16.15%	16.07%	16.07%	16.09%
Information Ratio	1.42	1.48	1.48	1.50	1.54	1.42

EXHIBIT 11
Realized Frontiers for the *Alpha1* Frontiers for Varying TE



in Exhibit 12. Since the ex-ante total risk as seen by *CRM* is larger than *BaseFund*, the *CRM* is correcting for risk underestimation by better capturing the systematic risk of the portfolio. Also, note that the risk underestimation is most severe when the same risk model is used to both generate and measure the risk of the portfolio.

Performance attribution results with the custom risk model are presented in Exhibits 13 and 14. The three portfolios in Exhibit 13 all have a realized risk of 4% over the duration of the backtest, so it is instructive to compare their realized active returns over this period. The *Alpha1-CRM* portfolio has a realized active return of 6%, compared to 4.26% and 4.15% for the *Alpha1-BaseStat* and *Alpha1-BaseFund* portfolios, respectively. Furthermore, Exhibit 13 indicates that the extra active return for the *Alpha1-CRM* portfolio comes primarily from bets on the GLER (custom) style attributes. Exhibit 14 shows the return contributions and the exposures taken by the three portfolios to the GLER and Axioma style attributes in the custom risk model. Note that the *Alpha1-CRM* portfolio consistently takes large positive exposures on the GLER attributes, and these, in turn, translate into better realized active returns. All portfolios take a negative exposure to the GLER PM attribute, and Axioma's Growth and Medium-Term Momentum attributes. Exhibit 15 gives the average exposure of the benchmark to the GLER attributes. The benchmark takes large negative exposures to the BP and SP attributes. To target a realized risk of 4%, we had to generate the *Alpha1-BaseStat* and *Alpha1-BaseFund* portfolios with TEs of 1.9% and 2.7%, respectively, as opposed to 3.5% for *Alpha1-CRM*. Since the *Alpha1-BaseStat* and *Alpha1-BaseFund* portfolios need to track the benchmark more closely on an ex-ante basis, they take smaller exposures to the GLER attributes, especially SP and BP. The *Alpha-CRM* portfolio gets most of its active return from SP and BP, as can be seen in Exhibit 14.

We mention in the introduction that simply adding the alpha factor in the risk model is not enough to address the misalignment that is caused by constraints. To illustrate this case, we construct a second custom risk model labeled *CRMCompositeAlpha* from *BaseFund,* as follows:

1. Add the *Alpha1* signal as an additional style factor.
2. Remove the Short-Term Momentum factor in *BaseFund*.
3. Retain the Value factor in *BaseFund* that was removed in the *CRM* model.

We run a frontier backtest with the *CRMCompositeAlpha* model and the *Alpha1* signal with TE varying from 3% to 7%. Exhibit 16 presents the frontier backtest summary. The left-hand panel of Exhibit 17 compares the realized frontiers of these two risk models. The original custom model *CRM* that has all the constituents of the alpha signal is better able to push the frontier upward. Note also that *CRM* has a bias statistic

EXHIBIT 12
Ex-Ante and Ex-Post Risk Associated with *Alpha1* Portfolios

Portfolio	Ex-Post Risk	Average Ex-Ante Total Risk (Factor, Specific)		
		BaseStat	BaseFund	CRM
Alpha1-BaseStat	4.18%	1.9% (0.58%, 1.82%)	3.51% (2.77%, 2.14%)	3.86% (3.21%, 2.12%)
Alpha1-BaseFund	4.08%	2.6% (1.52%, 2.1%)	2.7% (1.6%, 2.17%)	3.48% (2.7%, 2.16%)
Alpha1-CRM	4.06%	2.82% (1.64%, 2.26%)	3.19% (2.12%, 2.35%)	3.5% (2.62%, 2.34%)

Exhibit 13
Factor Return Contribution in Active Portfolios with *Alpha1* Signal

Source of Return	BaseStat	BaseFund	CRM
Active Portfolio	4.26%	4.15%	6.00%
Factor Contribution	4.96%	4.74%	5.74%
Axioma Style	–0.48%	–0.50%	–0.40%
GLER Style	4.14%	4.74%	5.46%
Country	0.92%	0.39%	0.46%
Industry	0.27%	–0.16%	0.26%
Currency	0.03%	0.00%	–0.06%
Local	0.08%	0.01%	0.02%
Market	0.00%	–0.02%	0.00%
Specific Return	–0.70%	–0.30%	0.26%

Exhibit 14
Factors with the Largest Exposures in Active Portfolios with *Alpha1* Signal

Source of Return	Return Contribution (Avg Exposure)		
Notable Factors	**BaseStat**	**BaseFund**	**CRM**
Custom Styles			
BP	2.11% (123.96%)	1.96% (127.83%)	2.23% (135.03%)
SP	1.30% (118.19%)	1.72% (129.89%)	1.96% (133.49%)
EP-CP-REP-RCP	0.42% (39.32%)	0.68% (59.35%)	0.8% (65.60%)
RSP-RBP	0.42% (39.32%)	0.45% (42.94%)	0.52% (45.55%)
CTEF	0.08% (7.5%)	0.10% (9.62%)	0.17% (13.33%)
PM	–0.18% (–20.08%)	–0.18% (–13.69%)	–0.22% (–13.54%)
Axioma Styles			
Size	0.89% (–48.71%)	0.64% (–29.39%)	0.75% (–35.97%)
Growth	–0.47% (–23.56%)	–0.42% (–20.40%)	–0.30% (–14.01%)
Liquidity	–0.17% (–11.43%)	–0.06% (–4.11%)	–0.09% (–6.0%)
Medium-Term Momentum	–0.46% (–13.43%)	–0.34% (–7.34%)	–0.36% (–5.42%)
Volatility	–0.13% (4.16%)	–0.25% (3.70%)	–0.33% (–3.97%)
Leverage	–0.11% (11.38%)	–0.04% (5.44%)	–0.05% (–3.92%)
Exchange Rate Sensitivity	–0.02% (0.47%)	–0.03% (–1.54%)	–0.03% (–2.92%)

Exhibit 15
Average Benchmark Exposures to GLER Attributes

Factor	Avg Benchmark Exposure
BP	–40.46%
SP	–29.43%
EP-CP-REP-RCP	–7.3%
RSP-RBP	0.94%
CTEF	0.08%
PM	–1.4%

of 1.15, which is lower than the 1.2 for *CRMCompositeAlpha* when the ex-ante risk is 3.5% (target realized risk of 4%).

The custom risk model *CRM* is correcting the misalignment that ensues between the implied alpha signal and the risk model. To highlight this point, we construct another alpha signal *MAlpha1* from SP, BP, CTEF, PM; and

$$\text{MEP-CP-REP-RCP} = 0.2\text{EP} + 0.3\text{CP} + 0.3\text{REP} + 0.2\text{RCP}$$

$$\text{MRSP-RBP} = 0.6\text{RSP} + 0.4\text{RBP}$$

which serve as misaligned versions of EP-CP-REP-RCP and RSP-RBP, respectively. The weights assigned to the attributes, including the misaligned ones, are the same as those in the *Alpha1* signal; see Exhibit 7 for the values. We wish to emphasize that the only difference

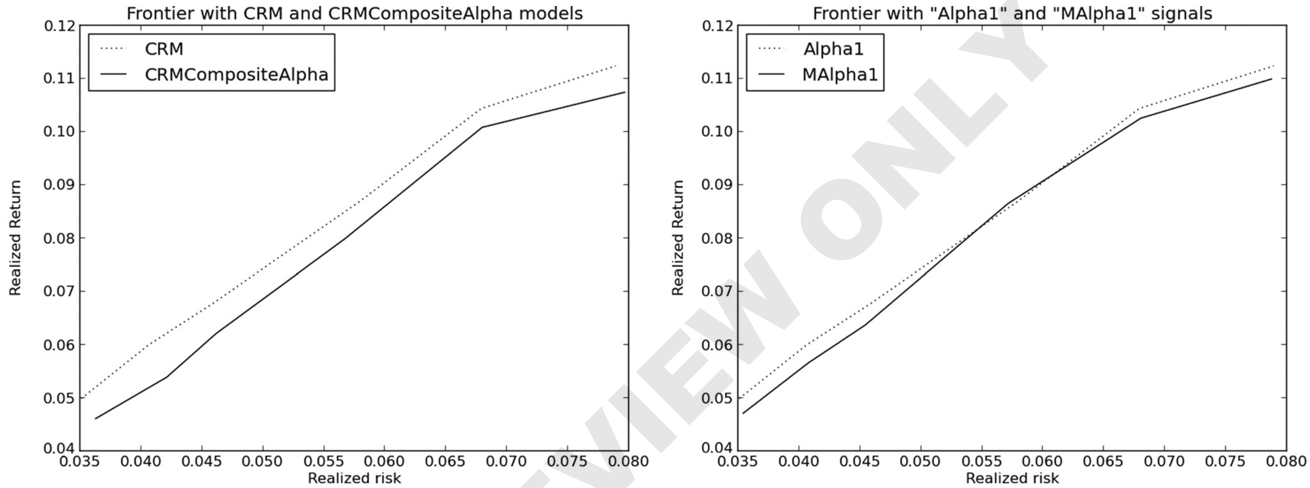
EXHIBIT 16

Frontier Summary for the *Alpha1-CRMCompositeAlpha* Portfolio

	Tracking Error					
	3%	**3.5%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	4.61%	5.39%	6.21%	8.00%	10.08%	10.75%
Avg Annualized Active Risk	3.62%	4.21%	4.61%	5.67%	6.80%	7.96%
Avg Turnover	16.32%	16.19%	16.15%	16.12%	16.03%	16.08%
Information Ratio	1.27	1.28	1.34	1.41	1.48	1.35

EXHIBIT 17

Realized Frontiers for Varying TE



(a) Compare *CRM* and *CRM CompositeAlpha*

(b) Compare *Alpha1* and *MAlpha1*

between *Alpha1* and *MAlpha1* is that the components EP-CP-REP-RCP and RSP-RBP have been replaced by MEP-CP-REP-RCP and MRSP-RBP, respectively. Not all the components of *MAlpha1* signal, especially MEP-CP-REP-RCP and MRSP-RBP, are represented as factors in the *CRM* model, potentially introducing a misalignment between this signal and the risk model. Exhibit 18 presents the frontier backtest summary. The right-hand panel of Exhibit 17 compares the realized frontiers of these two alpha signals. In general, the aligned alpha signal *Alpha1* is better able to push the frontier upward. Moreover, the portfolio generated with the aligned signal has a lower bias statistic when targeting a realized risk of 4%.

We generate three other alpha signals: *Alpha2*, *Alpha3*, and *Alpha4* from the 6 GLER attributes and Axioma's Growth factor as described in Exhibit 19.[2] Three more backtests are run with these alpha signals and the *CRM* to emphasize that the custom model is consistently delivering portfolios with high IR indepen-

EXHIBIT 18

Frontier Summary for the *MAlpha1-CRM* Portfolio

	Tracking Error					
	3%	**3.5%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	4.71%	5.66%	6.37%	8.65%	10.26%	11.00%
Avg Annualized Active Risk	3.54%	4.07%	4.54%	5.71%	6.80%	7.87%
Avg Turnover	16.37%	16.22%	16.15%	16.07%	16.09%	16.11%
Information Ratio	1.33	1.39	1.40	1.51	1.51	1.40

Exhibit 19

Attribute Weights in Alpha Signals

	Weight		
Attribute	**Alpha2**	**Alpha3**	**Alpha4**
EP-CP-REP-RCP	14.29%	9.25%	7.00%
RSP-RBP	17.9%	11.58%	11.91%
SP	21.53%	13.93%	9.59%
BP	16.62%	10.75%	12.58%
CTEF	12.03%	25.96%	19.93%
PM	17.62%	28.51%	20.78%
Growth	0.00%	0.00%	18.17%

Exhibit 20

Backtest Results with the *CRM* Model

	Backtest			
	Alpha1	**Alpha2**	**Alpha3**	**Alpha4**
Annualized Active Return	6.0%	6.21%	6.38%	6.05%
Avg Annualized Active Risk	4.06%	4.08%	4.14%	4.10%
Avg Turnover	16.21%	16.21%	16.16%	16.14%
Information Ratio	1.48	1.52	1.54	1.48

dently of the alpha signal that is used in the optimization. The backtests follow the original strategy outlined; we target a predicted TE of 3.5% to construct portfolios that have a realized risk of 4% over the backtest period. Note that all the components of these alpha signals are represented as factors in the *CRM* model, so the custom model is still aligned with all of these alpha signals. The summaries for these three backtests are given in Exhibit 20, where we have also included the summary for the equivalent backtest with the *Alpha1* signal for comparison. Exhibit 21 contains the average and return contributions for the backtest portfolios to some of the important style factors in the risk model. It indicates that the four portfolios are quite different. The three panels in Exhibit 22 plot the realized frontiers with the *CRM* and *BaseFund* models for the *Alpha2*, *Alpha3*, and *Alpha4* signals. We have not used the *BaseStat* model in these comparisons, as it severely underestimates the actual risk associated with these portfolios. The *CRM* frontier is above the *BaseFund* frontier for the *Alpha2* and *Alpha3* signals, although the distance between the realized frontiers is smaller than with the *Alpha1* signal. There is a smaller misalignment between the *Alpha4* and the *BaseFund* model, since the Growth factor in the *BaseFund* model is one of the components of the alpha signal with a large weight. In this case, although the *CRM* better corrects for risk underestimation, as can be seen from Exhibits 25 and 28, the realized frontier for the *BaseFund* model is above that for the *CRM* model for large realized risks. Exhibits 23, 24, 25, 26, 27, and 28 contain the frontier summaries for the three backtests with the *CRM* and *BaseFund* models.

The *BaseFund* portfolio has a larger allocation along $\alpha^I_{B^\perp}$, i.e., the portion of the implied alpha that is orthogonal to the factors in the risk model in each rebalancing period (since it has fewer factors than *CRM*), where it does not see any systematic risk. It is possible that some of these allocations, though uninformed, pay off handsomely, giving the portfolio a high IR.

We now look at how an aligned risk model is better able to allocate risk, which generally leads to improved IRs over long periods. Each of the risk models views the risk in the alpha factors differently. To examine how the three risk models are trading off the risk and the return of these factors, we construct FMPs for the six GLER factors at the end of September 2008. We then compute three covariance matrices (in the dimension of the FMPs)

$$\Theta_{ij} = (h^i)^T Q(h^j), \quad i, j = 1, \ldots, 6$$

where h^i is the FMP corresponding to factor *i,* and *Q* is *CRM*, *BaseFund*, and *BaseStat* in turn. Note that

$$w^T \Theta w = h^T Q h$$

where

$$h^* = \sum_{i=1}^{6} w_i h^i$$

is our optimal portfolio. We solve a simple mini-MVO problem for each risk model to determine the weights *w* used in the optimal portfolio. The mini-MVO maximizes the alpha signal that is given in Exhibit 7. It has only a risk constraint with a right-hand side of 4%. The upper triangular portions of the three covariance matrices are given in Equations (13), (14), and (15),

EXHIBIT 21
Average Exposures and Return Contributions of Portfolios to Important Factors

	Backtest—Return (Avg Exposure)			
Factor	**Alpha1**	**Alpha2**	**Alpha3**	**Alpha4**
BP	2.23% (135.03%)	2.11% (128.84%)	1.70% (105.36%)	1.52% (97.10%)
SP	1.96%(133.49%)	1.91% (129.97%)	1.59% (109.14%)	1.23% (80.24%)
EP-CP-REP-RCP	0.8% (65.60%)	0.73% (57.87%)	0.47% (36.35%)	0.54% (50.80%)
RSP-RBP	0.52% (45.55%)	0.45% (30.96%)	0.23% (1.67%)	0.42% (25.19%)
Growth	–0.30% (–14.01%)	–0.21% (–10.53%)	–0.14% (–6.37%)	1.03% (53.81%)
CTEF	0.17% (13.33%)	0.22% (21.11%)	0.33% (32.14%)	0.29% (31.25%)
PM	–0.22% (–13.54%)	–0.01% (7.53%)	0.33% (37.22%)	0.22% (31.10%)
Momentum	–0.36% (–5.42%)	0.23% (5.06%)	1.15% (22.71%)	0.97% (18.79%)

EXHIBIT 22
Realized Frontiers for Varying TE

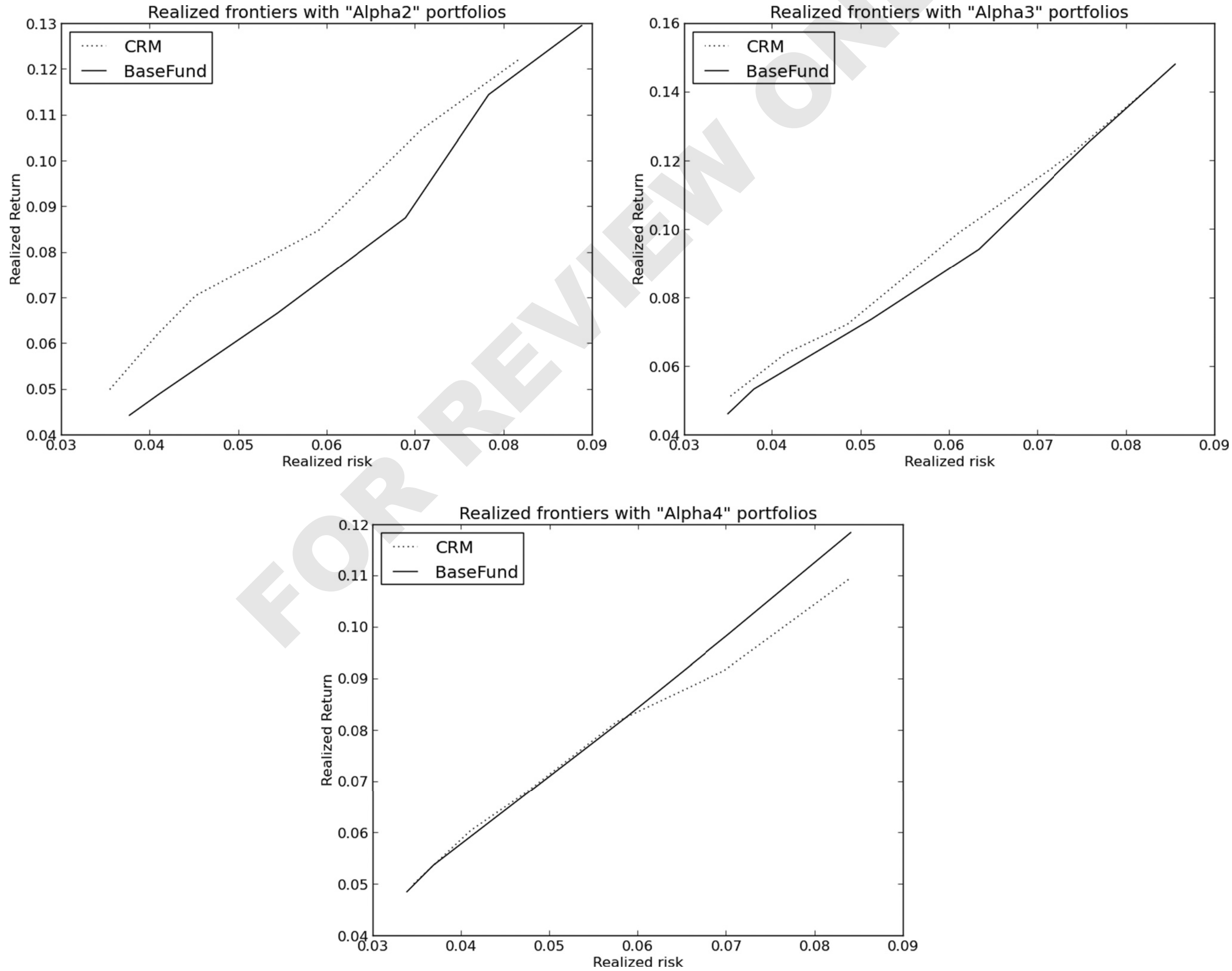


EXHIBIT 23
Frontier Summary for the *Alpha2-CRM* Portfolio

	Tracking Error					
	3%	**3.5%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	5.00%	6.21%	7.06%	8.50%	10.68%	12.22%
Avg Annualized Active Risk	3.54%	4.08%	4.51%	5.90%	7.06%	8.16%
Avg Turnover	16.34%	16.20%	16.15%	16.05%	16.06%	16.07%
Information Ratio	1.41	1.52	1.56	1.44	1.51	1.50

EXHIBIT 24
Frontier Summary for the *Alpha3-CRM* Portfolio

	Tracking Error					
	3%	**3.5%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	5.15%	6.38%	7.24%	9.91%	12.16%	14.30%
Avg Annualized Active Risk	3.52%	4.14%	4.84%	6.10%	7.36%	8.32%
Avg Turnover	16.27%	16.16%	16.11%	16.25%	16.01%	16.03%
Information Ratio	1.46	1.54	1.49	1.62	1.65	1.72

EXHIBIT 25
Frontier Summary for the *Alpha4-CRM* Portfolio

	Tracking Error					
	3%	**3.5%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	5.01%	6.05%	6.76%	8.18%	9.17%	10.97%
Avg Annualized Active Risk	3.45%	4.10%	4.70%	5.76%	6.97%	8.41%
Avg Turnover	16.24%	16.14%	16.10%	16.02%	16.03%	16.07%
Information Ratio	1.45	1.48	1.44	1.42	1.32	1.31

EXHIBIT 26
Frontier Summary for the *Alpha2-BaseFund* Portfolio

	Tracking Error					
	2.7%	**3%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	4.44%	4.90%	6.67%	8.76%	11.46%	12.97%
Avg Annualized Active Risk	3.76%	4.10%	5.43%	6.89%	7.82%	8.87%
Avg Turnover	16.84%	16.35%	16.12%	16.04%	16.06%	16.07%
Information Ratio	1.18	1.20	1.23	1.27	1.46	1.46

EXHIBIT 27
Frontier Summary for the *Alpha3-BaseFund* Portfolio

	Tracking Error					
	2.7%	**3%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	4.63%	5.35%	7.39%	9.44%	12.56%	14.83%
Avg Annualized Active Risk	3.50%	3.79%	5.12%	6.33%	7.58%	8.54%
Avg Turnover	16.79%	16.27%	16.13%	16.06%	16.00%	16.01%
Information Ratio	1.33	1.41	1.44	1.49	1.66	1.73

Exhibit 28

Frontier Summary for the *Alpha4-BaseFund* Portfolio

	Tracking Error					
	2.7%	**3%**	**4%**	**5%**	**6%**	**7%**
Annualized Active Return	4.86%	5.38%	6.92%	8.27%	9.87%	11.85%
Avg Annualized Active Risk	3.38%	3.68%	4.85%	5.86%	7.01%	8.40%
Avg Turnover	16.76%	16.29%	16.12%	16.05%	16.02%	16.04%
Information Ratio	1.44	1.46	1.43	1.41	1.41	1.41

where the diagonal entries contain the predicted annualized risk (multiplied by 100) and the off-diagonal entries contain the correlations.

$$\Theta^{CRM} = \begin{pmatrix} \text{EP-CP-REP-RCP} & \text{RSP-RBP} & \text{SP} & \text{BP} & \text{CTEF} & \text{PM} \\ 1.66 & -0.13 & 0.01 & 0.03 & 0.10 & 0.15 \\ & 1.87 & 0.08 & -0.24 & -0.06 & 0.27 \\ & & 2.16 & -0.03 & 0.01 & 0.12 \\ & & & 1.80 & -0.13 & 0.08 \\ & & & & 1.49 & 0.10 \\ & & & & & 1.98 \end{pmatrix} \tag{13}$$

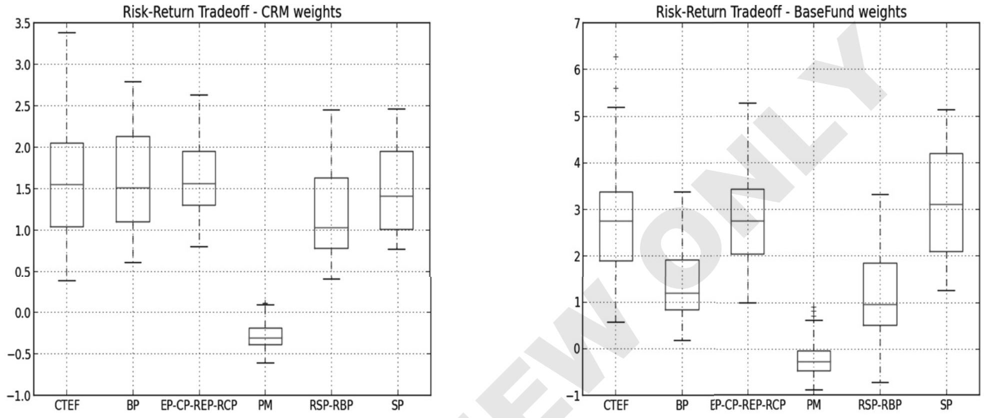
$$\Theta^{BaseFund} = \begin{pmatrix} \text{EP-CP-REP-RCP} & \text{RSP-RBP} & \text{SP} & \text{BP} & \text{CTEF} & \text{PM} \\ 1.18 & 0.43 & -0.08 & 0.23 & 0.32 & 0.39 \\ & 2.19 & 0.15 & 0.14 & 0.48 & 0.52 \\ & & 1.02 & -0.09 & 0.07 & 0.07 \\ & & & 1.60 & 0.13 & 0.08 \\ & & & & 0.72 & 0.23 \\ & & & & & 1.41 \end{pmatrix} \tag{14}$$

$$\Theta^{BaseStat} = \begin{pmatrix} \text{EP-CP-REP-RCP} & \text{RSP-RBP} & \text{SP} & \text{BP} & \text{CTEF} & \text{PM} \\ 1.50 & 0.47 & -0.23 & 0.12 & 0.23 & 0.38 \\ & 2.40 & -0.24 & -0.03 & 0.55 & 0.23 \\ & & 2.22 & 0.26 & -0.18 & -0.31 \\ & & & 1.47 & -0.15 & 0.24 \\ & & & & 0.96 & -0.12 \\ & & & & & 1.92 \end{pmatrix} \tag{15}$$

The optimal weights are given in the three rows of the matrix in Equation (16).

$$
\begin{array}{r} \\ \text{CRM} \\ \text{BaseFund} \\ \text{BaseStat} \end{array}
\begin{pmatrix}
\text{EP-CP-REP} & \text{RSP-RBP} & \text{SP} & \text{BP} & \text{CTEF} & \text{PM} \\
1.12 & 1.47 & 0.79 & 1.43 & 0.91 & -0.38 \\
1.46 & -0.35 & 3.24 & 0.86 & 1.44 & 0.03 \\
0.98 & 0.01 & 1.16 & 1.02 & 2.27 & 0.53
\end{pmatrix} \qquad (16)
$$

Exhibit 29
Factor Weights for the Mini-MVO Model with CRM and BaseFund



(a) Mini MVO weights for CRM (b) Mini MVO weights for BaseFund

Exhibit 30
Ex-Ante and Ex-Post Risk for Optimal Portfolios Generated from the Unconstrained Mini-MVO Model

		Ex-Post Risk		
Portfolio	**Ex-Ante Risk**	**BaseStat**	**BaseFund**	**CRM**
Target Portfolio	4.00%	7.26%	8.57%	5.26%

Clearly, the three risk models view the risk in the GLER factors differently, so the optimal weights in (16) are different as well. Let us highlight some of the prominent differences. Comparing the covariance matrices for *CRM* and *BaseFund*, we see that *BaseFund* believes that RSP-RBP is positively correlated with all the other factors, while *CRM* considers RSP-RBP to be positively correlated with PM and SP and negatively correlated with EP-CP-REP-RCP, BP, and CTEF. Moreover, *BaseFund* sees very little risk in the SP factor, while *CRM* considers SP to be the most risky factor, although both models consider SP to be relatively uncorrelated with the other factors. These different views explain the different weights assigned to RSP-RBP and SP by the optimizations that use these two models.

We repeat the mini-MVO optimization on the last trading of the month between January 2000 and November 2011. Exhibit 29 contains the box plot of the factor weights for the mini-MVO optimizations with the *CRM* and *BaseFund* models. Note that the weights from the *BaseFund* optimization are more leveraged than those obtained with *CRM*. Also, the *BaseFund* weights vary quite a bit, while those obtained with *CRM* are more stable with time. The realized risk of the three optimal portfolios generated from this optimization are given in Exhibit 30. Although there is risk underestimation in all of these leveraged unconstrained portfolios, the custom model best captures the risk associated with the alpha factors, so it best represents the alpha signal in the final portfolio.

EXHIBIT 31

PA Summaries for *Alpha1-BaseFund* with Fundamental and Custom Risk Models

Source of Return	BaseFund	CRM
Active Portfolio	4.15%	4.15%
Factor Contribution	1.44%	4.44%
Axioma Style	1.01%	–0.50%
GLER Style	0.00%	4.74%
Country	0.49%	0.39%
Industry	–0.06%	–0.16%
Currency	–0.01%	0.00%
Local	0.00%	0.01%
Market	0.00%	–0.02%
Specific Return	2.71%	–0.30%

EXHIBIT 32

Factors with the Largest Exposures in Active *Alpha1* Portfolios

Source of Return	Return Contribution	
Notable Factors	**BaseFund**	**CRM**
Style Contribution	1.01%	4.24%
GLER Style	0.00%	4.74%
BP	0.0%	1.96%
SP	0.0%	1.72%
EP-CP-REP-RCP	0.0%	0.68%
RSP-RBP	0.0%	0.45%
CTEF	0.0%	0.10%
PM	0.0%	–0.18%
Axioma Style	1.01%	–0.50%
Value	1.48%	0.0%
Short-Term Momentum	0.39%	0.0%
Size	0.25%	0.64%
Volatility	–0.26%	–0.25%
Medium-Term Momentum	–0.32%	–0.34%
Growth	–0.39%	–0.42%

Use of Custom Model in Ex-Post Performance Attribution

We would like to highlight the intuitive performance attribution(PA) summary that is given by the custom risk model in this section. We ran two performance attribution tasks on the *Alpha1-BaseFund* portfolio using the *BaseFund* and the *CRM* risk models in turn. Recall that this portfolio was generated with the *Alpha1* signal and the *BaseFund* risk model. Exhibit 31 reports the summaries from these PA tasks. Clearly, *BaseFund* PA summary assigns most of the active return of this portfolio to the specific returns, which is neither intuitive nor helpful to the portfolio manager. Moreover, this PA summary exaggerates the active returns associated with Axioma styles, countries, and industries. Exhibit 32 describes how the style returns are decomposed by the two PA tasks. Clearly, the PA summary with the *BaseFund* model is assigning some of the returns that are associated with the GLER attributes to some of the Axioma styles. As a result, the true returns associated with the Axioma styles are either exaggerated or diminished.

CONCLUSIONS

We used the GLER study to highlight the important role of custom risk models in addressing the *misalignment* problem resulting from the interactions between the alpha signal, the risk model, and the constraints in the MVO model. Our custom risk model includes all the components of the alpha signal as factors. We show that custom risk models:

1. Alleviate the risk-underestimation problem.
2. Represent the alpha signal in the portfolio in an optimal risk-adjusted fashion, thereby delivering portfolios with high IR, i.e., pushing the realized frontiers upward.
3. Generate a more intuitive and useful ex-post PA analysis of the portfolio.

ENDNOTES

[1]The global cap-weighted benchmark was provided to us by John Guerard, along with the other GLER attributes.

[2]Based on Guerard [2013], we generate *Alpha2* and *Alpha3* with larger weights to the CTEF and PM attributes, and *Alpha4*, which also contains the Axioma Growth factor.

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