Ten Lectures on Superconvergence in Finite Element Methods

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Preamble: The Way I Envision These Ten Lectures

a. There will be people in the audience with very little knowledge of superconvergence, and very little exposure to hard technical analysis in the context of mathematical numerical analysis of finite element methods. I shall start "slow" in order not to loose these people. (Towards the last three lectures I may start "cutting corners" in serious ways.)

b. After these ten lectures, such people as mentioned above will *not* be able to start research in superconvergence. But, if they have the interest to do so, after a semester of concentrated study they will, and I aim to have this series of lectures serve as a "motivational" study guide.

c. There will probably also be an audience of general "finite elementers" who aim to get an overview of the subject of superconvergence without necessarily having in mind to work in it later (unless something really strikes their fancy). These lectures should serve them well, I hope. (In fact, in my experience, such an audience is "no problem"; they are resilient to mistakes by the lecturer in pedagogical aims ...)

d. I also hope that even seasoned researchers in superconvergence will find a few points of interest in these lectures.

e. Let me next comment on the role between specific examples and general theory.

i) Superconvergence originated in observations in very specific examples.

ii) A practitioner, say an engineer doing stress analysis with three-dimensional tetrahedral quadratic elements, may well be best served by having a paper which explicitly tells him/her exactly what is going on in this particular case.

iii) A mathematical theory, however, must aim for general principles and elucidation of underlying fundamental reasons.

iv) *These lectures, while having a liberal sprinkling of specific examples* and historical comments, *will concentrate on general principles.* In the more open problems, we shall see the cycle of, first, specific examples, then general principles (yet unknown) emerge again ...

v) Reflecting the theory at present, I shall mainly treat the standard Galerkin finite element method in second order elliptic equations.

Now, for the actual lectures:

Lecture 1. Introduction and Overview

a. Excerpts from the exchange of notes between Stricklin and Filho (1966 and 1968): "What is it that we have discovered?"

b. Optimal rate of convergence and superconvergence; "Natural" superconvergence versus superconvergence by postprocessing: not a natural distinction, there is a "continuous" scale of things!

c. Low order simplicial elements: the interpolant close to the finite element solution by "chance". A Red Herring that throw research in the wrong direction for quite a while!

d. Outline of the rest of the lectures.

Lecture 2. One-Dimensional Example of Superconvergence

a. Superconvergence at meshpoints for continuous elements: The Green's function is (almost) in the finite element space (Douglas-Dupont [3]).

b. Continuous finite elements, superconvergence in the interior of mesh-intervals.

i) What makes this case so easy? The derivative of the finite element solution is (almost) the local L_2 projection of the derivative on each mesh-interval. First mention of the general technical tool of "superapproximation".

ii) All the interior points (Chen [2]).

c. Superconvergence in derivatives by averaging at meshpoints.

d. The case of general interelement degree of continuity: Description of a surprisingly neat picture (Richard Dunlap, Ph.D. Thesis, Cornell, 1995).

Lecture 3. The Fundamental Technical Tool: Local Pointwise A priori Error Estimates

a. Description of the sharp form given by Schatz 1998 [5].

b. Outline of the proof, the roles of weights and "superapproximation".

Lecture 4. Superconvergence in Tensor Product Elements

a. The first application of the fundamental technical tool. that sharpening the ideas of Douglas, Dupont and Wheeler [4].

b. The main theme: what happens in the one-dimensional cases tells what happens in their multi-dimensional tensor product situation.

c. "Superapproximation" again.

Lecture 5. Superconvergence by Local Symmetry Principles

a. Tensor product elements (at least in the interior) are explained by Lecture 4 above.

b. Simplicial elements (much more commonly used in practice for complicated geometries) proved a much harder nut to crack.

c. Recollection of the "Red Herring" for low accuracy elements (Lecture 1.c).

d. Description of the symmetry principle (1996): what is a "symmetry point"? Distinction between even and odd polynomial degree (superconvergence for function values at symmetry points in even degree; superconvergence of gradients at symmetry points in odd degree, if necessary after simple averaging).

e. Proof of the symmetry principle by the fundamental technical tool.

Lecture 6. Superconvergence by Difference Quotients

a. Recall "natural" versus "postprocessed" superconvergence again. For example, is it easier to evaluate the derivative at a Gauss point than to form short difference quotients based on the mesh-point values?

b. Description of the results on locally translation invariant meshes.

c. Proof of the results.

d. Generalization to meshes which are piecewise smooth mappings of translation invariant meshes.

Lecture 7. A Computer-Based Proof of Superconvergence

a. The main principles of their proof (including, the reliance on local a priori error estimates).

b. Comparison of their results in the case of simplicial elements with the symmetry principle. The symmetry principle explains *all* superconvergence points found for simplicial elements.

c. Some further discoveries by the computer-based proof [1] and their later theoretical justification ("tensor product", serendipity, and intermediate families, Zhang 1998 [6]).

Lecture 8. Use of Superconvergence for A Posteriori Error Estimation

a. Description of some general principles of a posteriori error estimation (recovered gradients and residual based estimators) and some of their properties; mainly, how local are they?

b. How to use superconvergence. This will now be almost self-evident.

c. The Zienkiewicz-Zhu estimator [7].

Lecture 9. Additional Topics (tentative)

a. Up to boundaries.

b. Mixed methods.

c. Singular perturbations.

Lecture 10. Concluding Lecture (more tentative)

a. Open problems (some already mentioned...).

b. Distilled Wisdom.

References

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[3] Douglas, J. Jr. and Dupont, T., Galerkin approximations for the two point boundary problem using continuous, piecewise polynomial space, *Numer. Math.* **22** (1974), pp.99-109.

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[7] Zienkiewicz O.C. and Zhu, J.Z., The superconvergence patch recovery (SPR) and adaptive finite element refinement, *Comput. Meth. Appl. Mech. Engrg.* **101** (1992), pp.207-224.