Hyperhomological Algebra with Applications to Commutative Rings 12 Dec 2006

Lars Winther Christensen

Hans-Bjørn Foxby

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, U.S.A.
E-mail address: winther@math.unl.edu

Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, DK-2100 København Ø, Denmark
E-mail address: foxby@math.ku.dk

Preface

Homological algebra is a well-established tool in ring theory and has been so for half a century. Hyperhomological algebra is a more powerful tool with important applications in ring theory. The use of hyperhomological methods has been growing steadily but slowly for the past 25 years. One reason for the low speed, no doubt, is the absence of an accessible introduction or reference to the theory and its applications. To be an effective practitioner of hyperhomological algebra one must be well-versed in a series of research articles and lecture notes, including unpublished ones. To get an overview of the applications of the theory the series grows further.

The purpose of the book is to remedy this deficiency. We make the case that hyperhomological methods provide stronger results and, in general, shorter and more transparent proofs than traditional homological algebra; this to an extent that far outweighs the effort it takes to master this tool.

The book is divided into three parts FOUNDATIONS, APPLICATIONS, and TECHNIQUES. In FOUNDATIONS we introduce the concepts and terminology of homological algebra and construct the derived category over a general ring. TECHNIQUES continues this systematic development of hyperhomological algebra. In APPLICATIONS we apply FOUNDATIONS and TECHNIQUES to the study of commutative noetherian rings.

This division serves several purposes. Readers familiar with the language of derived categories may skip FOUNDATIONS. TECHNIQUES is developed in a higher generality than needed for APPLICATIONS; we expect this to make TECHNIQUES a useful reference for researchers, not only in commutative algebra, but also in neighboring fields. APPLICATIONS can serve as an introduction to homological aspects of commutative algebra for graduate students in algebra and researchers in other fields.

The parts are ordered as follows: FOUNDATIONS – APPLICATIONS – TECHNIQUES. This order is chosen to get to applications of the theory fast. Thus, APPLICATIONS builds on technical constructions and results from TECHNIQUES and can be read after FOUNDATIONS, using TECHNIQUES as an appendix.

Contents

Introduction

The appearance in 1956 of the book "Homological Algebra" by Henri Cartan and Samuel Eilenberg initiated homological algebra. The first applications came the very same year with notable papers by David Buchsbaum, Maurice Auslander and Jean-Pierre Serre containing a homological characterization of regular local rings; this made proofs of the Krull conjectures possible. Homological algebra has ever since been an important tool in many areas in mathematics, in particular, in ring theory (commutative and non-commutative), algebraic geometry, algebraic topology, group theory, and Lie group theory to name a few.

Classical homological algebra studies the behavior of additive module functors by determining the behavior—notably vanishing—of its derived functors. To describe the construction of these derived functors, assume that R and S are rings, and that $T\colon \mathcal{M}(R) \to \mathcal{M}(S)$ is an additive functor from the category of modules over R and their homomorphisms to the same over S. Assume, for example, that it is covariant and that we want the ith left derived functor $\mathrm{L}_i T(-)$. Its value at a given module M is obtained as follows:

(1) Chooses any projective resolution $P_\bullet$ of M over R.
(2) Apply the functor T to the resolution to get a complex of S-modules $T(P_\bullet)$.
(3) Form the ith homology $\mathrm{H}_i\big(T(P_\bullet)\big)$; this is the desired value of $\mathrm{L}_i T$ at M. Actually, this module is only uniquely determined up isomorphisms, so one has to adjust for this.

This three-step procedure can be extended to homomorphisms.

Hyperhomological algebra studies—for given rings R and S—the behavior of additive functors T from the category $\mathcal{C}(R)$ of complexes $M_\bullet$ of R–modules and their morphisms into the corresponding category $\mathcal{C}(S)$ over S. Any R–module is viewed as a complex of R-modules, namely one that is concentrated in degree zero, and any homomorphism of R-modules is viewed as a morphism of the corresponding complexes. From this point of view, hyperhomological algebra becomes an extension of classical homological algebra.

If, say, T is covariant, then the value of the left derived functor $\mathbf{L}T$ at an R–complex $M_\bullet$ is obtained as follows:

(1) Choose any semiprojective resolution $P_\bullet$ of $M_\bullet$ (to be described later). If $M_\bullet$ is a module, then any usual projective resolution is a semiprojective one.
(2) Set $\mathbf{L}T(M_\bullet)$ equal to $T(P_\bullet)$. Actually, this complex of S–modules is only uniquely determined up to, so-called, quasi-isomorphisms (to be explained later). Adjustment for this is a procedure that involves the construction of derived categories.

This two step procedure can be extended to morphisms.

In hyperhomological algebra, the construction of a derived functor has **two** steps, while the construction in classical homological algebra has **three** steps. In the third step, valuable information is lost: One cannot retrieve a complex from its homology modules. While this is a technical point, it is also an important one; hyperhomological methods yield broader and stronger results. For example, many results in the theory of local flat homomorphisms have been extended to local homomorphisms of finite flat dimension, and new insight in the flat case has been gained in the process.

While hyperhomological algebra was mentioned and named already in the final chapter of "Homological algebra" by Cartan and Eilenberg, it was the work of Grothendieck that brought it to ring theory. Subsequent work by Iversen and Roberts demonstrated the utility of hyperhomological algebra in commutative ring theory, where it is now firmly established as a research tool.

The aim of this book is to provide a systematic development of hyperhomological algebra: This includes the construction of the derived category over an associative ring and a careful study of the functors of importance in ring theory. To demonstrate the utility of the theory and to motivate the choice of topics, the book includes a short course in homological aspects of commutative ring theory.

Synopsis

This synopsis caters for readers with some background in homological algebra. However, all notions discussed here will be defined in the main text, also those that belong to classical homological algebra.

The first section provides a brief introduction to hyperhomological algebra. **{** In bold braces some items are compared to the corresponding ones from classical homological algebra. **}** The remaining seven sections present applications of hyperhomological algebra **{** followed—when possible—by special cases that can be phrased within classical homological algebra **}**.

The organization of this synopsis does not follow that of the book, and no references to the main text are given.

Hyperhomological algebra

This section is a short introduction to hyperhomological algebra **{** pointing out how it extends classical homological algebra and mentioning the differences between the two versions of homological algebra **}**.

Hyperhomological algebra { versus classical homological algebra }. For given rings R and S, hyperhomological algebra studies derived functors of additive functors from the category $\mathcal{C}(R)$, of complexes $M_\bullet$ of R–modules[1] and their morphisms, into the corresponding category $\mathcal{C}(S)$. **{** Any R–module can be viewed as a complex of R–modules concentrated in degree zero, and any homomorphism of R–modules can be viewed as a morphism of the corresponding complexes. Thus, the category $\mathcal{M}(R)$ of R–modules and their homomorphisms is a full subcategory of $\mathcal{C}(R)$. Any additive module functor $T\colon \mathcal{M}(R) \to \mathcal{M}(S)$ extends to an additive functor $T\colon \mathcal{C}(R) \to \mathcal{C}(S)$, and it will follow, that hyperhomological algebra is an extension of classical homological algebra. **}**

If the functor T is, for example, covariant, then hyperhomological algebra determines the value of, for example, the *left derived* functor, $\mathbf{L}T$, of T at an R–complex $M_\bullet$ in two steps:

(1) Choose any *semiprojective resolution* $P_\bullet$ of $M_\bullet$. **{** If $M_\bullet$ is a module M, then any usual projective resolution of M is also a semiprojective resolution of M viewed as a complex **}**.
(2) Set $\mathbf{L}T(M_\bullet)$ equal to $T(P_\bullet)$; this complex is uniquely determined up to quasi-isomorphisms, so one has to adjust for these. This procedure involves the construction of the *derived category* $\mathcal{D}(R)$ over R.

{ In classical homological algebra, there is a third step:

[1] Module means left module.

(3) Take the ℓth homology module $\mathrm{H}_\ell\big(T(P_\bullet)\big)$, which is the desired module $\mathrm{L}_\ell T(M)$; it is uniquely determined up isomorphisms. **}**

The two-step procedure in hyperhomological algebra can be extended to morphisms: Any morphism $\alpha_\bullet\colon M_\bullet \to N_\bullet$ of R–complexes induces a morphism $\mathbf{L}T(\alpha_\bullet)\colon \mathbf{L}T(M_\bullet) \to \mathbf{L}T(N_\bullet)$ of S–complexes, and the latter is uniquely determined up to quasi-isomorphisms. **{** The three-step procedure in classical homological algebra can be extended to homomorphisms. Any homomorphism $\alpha\colon M \to N$ of R–modules yields a homomorphism $\mathrm{L}_\ell T(\alpha)\colon \mathrm{L}_\ell T(M) \to \mathrm{L}_\ell T(N)$ of S–modules, which is uniquely determined up to isomorphism. **}**

Comparison. The procedure in hyperhomological algebra has two steps **{** while the procedure in classical homological algebra has three steps, and in this extra step valuable information is lost: One cannot retrieve a complex from its homology modules. Examples of this are to follow. **}**

On the other hand, there are two items in hyperhomological algebra that are harder to take care of than the corresponding the ones in classical homological algebra. First, it requires more work to prove, say, the existence of a semiprojective resolution of a complex than to prove existence of a (classical) projective resolution of a module. Second, the derived category $\mathcal{D}(R)$ is structurally more complicated than the module category $\mathcal{M}(R)$. However, these issues need only be dealt with once and for all!

The derived category. The first step is to describe the category of R–complexes $\mathcal{C}(R)$ in further detail. An object M in $\mathcal{C}(R)$ is an R–complex, that is, a sequence of homomorphisms of R–modules

$$M = \cdots \longrightarrow M_{\ell+1} \xrightarrow{\partial^M_{\ell+1}} M_\ell \xrightarrow{\partial^M_\ell} M_{\ell-1} \longrightarrow \cdots$$

such that $\partial^M_\ell \partial^M_{\ell+1} = 0$ for all $\ell \in \mathbb{Z}$. The family $\{\partial^M_\ell\}_{\ell\in\mathbb{Z}}$ is the *differential* of M.

From now on, complexes are our primary objects of study, and we no longer indicate complexes by subscript dots (as used above, for example in $M_\bullet$). As indicated earlier, a complex M is identified with the module M_0, if and only if the complex M is concentrated in degree zero; that is, $M_\ell = 0$ for $\ell \neq 0$. A *morphism* $\alpha\colon M \to N$ in $\mathcal{C}(R)$ is a family $\big(\alpha_\ell\colon M_\ell \to N_\ell\big)_{\ell\in\mathbb{Z}}$ of R–linear maps such that $\partial^N_\ell \alpha_\ell = \alpha_{\ell-1}\partial^M_\ell$ for all $\ell \in \mathbb{Z}$. If M and N are R–modules, then α is a morphism in $\mathcal{C}(R)$ if and only if $\alpha_0\colon M_0 \to N_0$ is one in $\mathcal{M}(R)$, that is, a homomorphism of R–modules. Thus, $\mathcal{M}(R)$ is a full subcategory of $\mathcal{C}(R)$.

Any morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ induces for all $\ell \in \mathbb{Z}$ a homomorphism $\mathrm{H}_\ell(\alpha)\colon \mathrm{H}_\ell(M) \to \mathrm{H}_\ell(N)$, and α is said to be a *quasi-isomorphism*, when $\mathrm{H}_\ell(\alpha)$ is an isomorphism for all $\ell \in \mathbb{Z}$. The symbol $\simeq$ indicates quasi-isomorphisms, and we write $\alpha\colon M \xrightarrow{\simeq} N$ to signal that α is a quasi-isomorphism. The notation $M \simeq N$ has a (slightly) different meaning—to be described shortly.

The next step is to present the derived category $\mathcal{D}(R)$ over R. It has the same objects as $\mathcal{C}(R)$, that is, all R–complexes. In this synopsis, we shall not need the precise definition of the morphisms in $\mathcal{D}(R)$. It suffices to note the following facts:

(1) The objects of $\mathcal{D}(R)$ are exactly the R–complexes, that is, the classes of objects in the two categories $\mathcal{C}(R)$ and $\mathcal{D}(R)$ are identical.
(2) Any morphism α in $\mathcal{C}(R)$ is a morphism in $\mathcal{D}(R)$, and it is an isomorphism in $\mathcal{D}(R)$, if and only if it is a quasi-isomorphism in $\mathcal{C}(R)$.

(3) The symbol $\simeq$ is the sign for *isomorphism* in $\mathcal{D}(R)$ as well as a symbol that can be *attached* to quasi-isomorphisms in $\mathcal{C}(R)$.
(4) For any R–complexes M and N the following are equivalent.
 (i) M and N are isomorphic in $\mathcal{D}(R)$.
 (ii) There exist quasi-isomorphisms $\alpha\colon M \xrightarrow{\simeq} X$ and $\beta\colon N \xrightarrow{\simeq} X$.
 (iii) There exist quasi-isomorphisms $\gamma\colon Y \xrightarrow{\simeq} M$ and $\delta\colon Y \xrightarrow{\simeq} N$.
(5) If $M \simeq N$ (that is, M and N are isomorphic in $\mathcal{D}(R)$), then for each $\ell \in \mathbb{Z}$ there is an induced isomorphism $\mathrm{H}_\ell(M) \cong \mathrm{H}_\ell(N)$ in $\mathcal{M}(R)$.
(6) If α is a homomorphism of R–modules, then α is an isomorphism in $\mathcal{D}(R)$, if and only if it is an isomorphism in $\mathcal{M}(R)$.

Homomorphism functor. Any two R–complexes K and M induce a $\mathbb{Z}$–complex $\mathrm{Hom}_R(K,M)$ called their *homomorphism complex*; its ℓth module is

$$\mathrm{Hom}_R(K,M)_\ell = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(K_p, M_{p+\ell}),$$

and its differential is induced by those of K and M. For any R–complex K this yields a functor $\mathrm{Hom}_R(K,-)\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$. $\{$ If K and M are modules, then so is the complex $\mathrm{Hom}_R(K,M)$; it is the $\mathbb{Z}$–module of R–homomorphisms $K \to M$. $\}$

Resolutions. An R–complex P is said to be *semiprojective*, when the functor $\mathrm{Hom}_R(P,-)\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$ preserves surjective quasi-isomorphisms. That is, if $\alpha\colon M \to N$ is a surjective quasi-isomorphism, then the induced morphism $\mathrm{Hom}_R(P,\alpha)\colon \mathrm{Hom}_R(P,M) \to \mathrm{Hom}_R(P,N)$ is a surjective quasi-isomorphism. $\{$ If P is bounded below, that is, $P_\ell = 0$ for $\ell \ll 0$, then P is semiprojective, if and only if the R–module P_ℓ is projective for all ℓ. $\}$ It turns out that any R–complex K has a *semiprojective resolution*, that is, a quasi-isomorphism $\pi\colon P \to K$ with P semiprojective. $\{$ If M is an R–module, then any classical projective resolution P of M yields a semiprojective resolution of M viewed as an R–complex. $\}$ However, if P is a projective object in the category $\mathcal{C}(R)$, then $\mathrm{H}(T(P)) = 0$ for all additive functors $\mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$; thus, the object P is of no utility in homological algebra!

An R–complex I is said to be *semi-injective*, when $\mathrm{Hom}_R(-,I)\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$ takes *injective quasi-isomorphisms* into *surjective quasi-isomorphisms*.[2] $\{$ If I is bounded above, then I is semi-injective, if and only if I_ℓ is a injective for all ℓ. $\}$ It turns out that every R–complex M has a *semi-injective resolution*, i.e. a quasi-isomorphism $\iota\colon M \to I$ with I semi-injective. $\{$ If M is a module, then any classical injective resolution I of M yields a semi-injective resolution of the complex M. $\}$

Derived homomorphism functor. For any R–complex K, the covariant homomorphism functor $\mathrm{Hom}_R(K,-)\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$ has a derived functor $\mathbf{R}\mathrm{Hom}_R(K,-)\colon \mathcal{D}(R) \to \mathcal{D}(\mathbb{Z})$ defined on an R–complex M by

$$\mathbf{R}\mathrm{Hom}_R(M,K) = \mathrm{Hom}_R(K,I)$$

whenever $\iota\colon M \to I$ is a semi-injective resolution. Dually, for any R–complex M, the contravariant homomorphism functor $\mathrm{Hom}_R(-,M)\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$ has a derived functor $\mathbf{R}\mathrm{Hom}_R(-,M)\colon \mathcal{D}(R) \to \mathcal{D}(\mathbb{Z})$ defined on an R–complex K by

$$\mathbf{R}\mathrm{Hom}_R(K,M) = \mathrm{Hom}_R(P,M)$$

[2] Like the projective ones, injective objects in the category $\mathcal{C}(R)$ are not interesting from a homological viewpoint.

whenever $\pi\colon P \to K$ is semiprojective resolution. It turns out there are induced quasi-isomorphisms

$$\mathrm{Hom}_R(P,M) \xrightarrow{\simeq} \mathrm{Hom}_R(P,I) \xleftarrow{\simeq} \mathrm{Hom}_R(K,I)\,.$$

Thus, there is an isomorphism $\mathrm{Hom}_R(P,M) \simeq \mathrm{Hom}_R(K,I)$ in $\mathcal{D}(R)$, and hence $\mathbf{R}\mathrm{Hom}_R(-,-)$ can be derived from $\mathrm{Hom}_R(-,-)$ in either variable. $\{$Let K and M be modules. The bifunctors $\mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(K,M))$ and $\mathrm{Ext}^{\ell}_R(K,M)$ in K and M are isomorphic, in particular, so are $\mathrm{H}_0(\mathbf{R}\mathrm{Hom}_R(K,M))$ and $\mathrm{Hom}_R(K,M)$. If $\mathrm{Ext}^{\ell}_R(K,M) = 0$ for $\ell > 0$ then $\mathbf{R}\mathrm{Hom}_R(K,M) \simeq \mathrm{Hom}_R(K,M)$.$\}$

To define tensor products we need the *opposite ring* R°; it has the same addition as R and multiplication $R^\circ \times R^\circ \to R^\circ$ given by $(r,r') \mapsto r'r$.[3]

Tensor product functors. Let K be an R°–complex and M be an R–complex. The *tensor product* $K \otimes_R M$ is then a $\mathbb{Z}$–complex whose ℓth module is

$$(K \otimes_R M)_\ell = \coprod_{p\in\mathbb{Z}} K_p \otimes_R M_{\ell-p},$$

and whose differential is induced by those of K and M. $\{$If K and M are modules, then the tensor product complex is the $\mathbb{Z}$–module $K \otimes_R M$.$\}$

This construction yields an additive functor $K \otimes_R -\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$. The *derived tensor product functor*, denoted $K \otimes^{\mathbf{L}}_R -\colon \mathcal{D}(R) \to \mathcal{D}(\mathbb{Z})$, is defined an object M in $\mathcal{D}(R)$ as follows: Choose a semiprojective resolution $\pi\colon P \to M$, and set

$$K \otimes^{\mathbf{L}}_R M = K \otimes_R P.$$

It is uniquely determined up to isomorphism in $\mathcal{D}(\mathbb{Z})$. Furthermore, if $\xi\colon Q \to K$ is a semiprojective resolution of R°–modules, there turn out to be quasi-isomorphisms

$$Q \otimes_R M \xleftarrow{\simeq} Q \otimes_R P \xrightarrow{\simeq} K \otimes_R P.$$

Thus, there is an isomorphism $Q \otimes_R M \simeq K \otimes_R P$ in $\mathcal{D}(\mathbb{Z})$, so $-\otimes^{\mathbf{L}}_R -$ can be derived from $-\otimes_R -$ in either variable. $\{$If K and M are modules, then $\mathrm{H}_\ell(K \otimes^{\mathbf{L}}_R M) \cong \mathrm{Tor}^R_\ell(K,M)$ functorially in K and M, and hence $\mathrm{H}_0(K \otimes^{\mathbf{L}}_R M) \cong K \otimes_R M$. If $\mathrm{Tor}^R_\ell(K,M) = 0$ for $\ell > 0$, then $K \otimes^{\mathbf{L}}_R M \simeq K \otimes_R M$.$\}$

An R°–complex F is said to be *semiflat*, when the functor $F \otimes_R -\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$ preserves *injective quasi-isomorphisms*. $\{$If F is bounded below, then F is semiflat, if and only if the R°–module F_ℓ is flat for all ℓ.$\}$ If $\varphi\colon F \to K$ is a *semiflat resolution*, then the functors $K \otimes^{\mathbf{L}}_R -$ and $F \otimes_R -$ are isomorphic.

Application 1: Homological dimensions

Boundedness and Finiteness. For M in $\mathcal{D}(R)$ the supremum and infimum of the set $\{i \in \mathbb{Z} \mid \mathrm{H}_i(M) \neq 0\}$ are denoted $\sup M$ and $\inf M$. The category of bounded complexes, denoted $\mathcal{D}_\square(R)$, is the full subcategory of $\mathcal{D}(R)$ of complexes with $\sup M$ and $\inf M$ finite. A bounded R–complex M is said to be *finite*, when $\mathrm{H}_\ell(M)$ is finitely generated for all $\ell \in \mathbb{Z}$, and the full subcategory of $\mathcal{D}(R)$ consisting of these complexes is denoted $\mathcal{D}^f_\square(R)$.

[3] Some refer to R°–modules as right R–modules. This nomenclature is not used in this text; here module means left module.

Homological dimensions. For a complex $M \in \mathcal{D}_{\Box}(R)$ the *projective dimension* $\operatorname{pd}_R M$, the *flat dimension* $\operatorname{fd}_R M$, and the *injective dimension* $\operatorname{id}_R M$ are defined as follows.

$$\operatorname{pd}_R M = \inf\{\, s \in \mathbb{Z} \mid M \text{ has semiprojective resolution } P \text{ with } P_\ell = 0 \text{ for } \ell > s \,\}.$$
$$\operatorname{fd}_R M = \inf\{\, s \in \mathbb{Z} \mid M \text{ has semiflat resolution } F \text{ with } F_\ell = 0 \text{ for } \ell > s \,\}.$$
$$\operatorname{id}_R M = \inf\{\, i \in \mathbb{Z} \mid M \text{ has semiinjective resolution } I \text{ with } I_\ell = 0 \text{ for } -\ell > i \,\}.$$

These numbers belong to the extended integers $\mathbb{Z}^* = \mathbb{Z} \cup \{\, -\infty, \infty \,\}$. **{** If M is a module, then these numbers are the usual homological dimensions. **}**

Depth and Width. Let $(R, \mathfrak{m}, k)$ be a local ring, that is, R is a commutative Noetherian ring with a unique maximal ideal $\mathfrak{m}$, and residue field $k = R/\mathfrak{m}$. For $M \in \mathcal{D}(R)$ we define the next numbers in $\mathbb{Z}^*$:

$$\operatorname{depth}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(k, M) \qquad \text{and} \qquad \operatorname{width}_R M = \inf(k \otimes^{\mathbf{L}}_R M)\,.$$

It turns out that the following are equivalent for $M \in \mathcal{D}_{\Box}(R)$:

(i) $\operatorname{depth}_R M < \infty$; (ii) $\operatorname{width}_R M < \infty$; (iii) $\operatorname{depth}_R M + \operatorname{width}_R M \le \dim R$.

Furthermore, $\mathrm{H}(M) \neq 0$ if and only if $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for some $\mathfrak{p} \in \operatorname{Spec} R$.
{ Let M be an R–module. The above depth is then the usual concept, that is, $\operatorname{depth}_R M = \inf\{\, \ell \in \mathbb{N}_0 \mid \operatorname{Ext}^\ell_R(k, M) \neq 0 \,\}$. If there is an M–regular sequence $x_1, \dots, x_d \in \mathfrak{m}$ with $d = \dim R$ and $(x_1, \dots, x_d)M \neq M$, then $\operatorname{depth}_R M = d$. If M is finitely generated, then $\operatorname{depth}_R M$ is the maximal length n of an M–regular sequence $x_1, \dots, x_n$ in $\mathfrak{m}$. In classical homological algebra, the width of M is the number $\inf\{\, \ell \in \mathbb{N}_0 \mid \operatorname{Tor}^R_\ell(k, M) \neq 0 \}$; if M is finitely generated and non-zero, then $\operatorname{width}_R M = 0$. **}**

Auslander–Buchsbaum Equalities. If R is a local ring, M and N belong to $\mathcal{D}_{\Box}(R)$, and $\operatorname{fd}_R N < \infty$, then the next equality in $\mathbb{Z}^*$ hold.

$$\operatorname{depth}_R(M \otimes^{\mathbf{L}}_R N) = \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R.$$

If, in addition, $N \in \mathcal{D}^f_{\Box}(R)$ and $\mathrm{H}(N) \neq 0$, then

$$\operatorname{pd}_R N + \operatorname{depth}_R N = \operatorname{depth} R.$$

{ Let N be a non-zero finitely generated R–module with $\operatorname{pd}_R N$ finite. The latter formula is then the classical Auslander–Buchsbaum Equality. If M is a finitely generated R–module such that $\operatorname{Tor}^R_\ell(M, N) = 0$ for $\ell > 0$, then the former is the equality $\operatorname{depth}_R(M \otimes_R N) = \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth} R$. **}**

Gorenstein local rings. A local ring R is *Gorenstein*, if $\operatorname{id}_R R$ is finite.

The following are equivalent for a local ring R.

(i) R is Gorenstein.
(ii) For all $N \in \mathcal{D}_{\Box}(R)$, $\operatorname{fd}_R N$ is finite if and only if $\operatorname{id}_R N$ is finite.
(iii) There exists $N \in \mathcal{D}_{\Box}(R)$ such that $\operatorname{fd}_R N$, $\operatorname{id}_R N$, and $\operatorname{depth}_R N$ are finite.

Dimension. Let R be a commutative ring. Its Krull dimension, $\dim R$, is the supremum of the set of $n \in \mathbb{N}_0$ such that there exists a chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals in R. The *Krull dimension*, $\dim_R M$, of $M \in \mathcal{D}(R)$ is defined as

$$\dim_R M = \sup\{\, \dim(R/\mathfrak{p}) - \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}.$$

If $\operatorname{depth}_R M$ is finite, then there is an inequality $\operatorname{depth}_R M \leqslant \dim_R M$. **{** If M is an R–module, then $\dim_R M = \sup\{\, \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R M \,\}$ is the usual Krull dimension of a module. Here, $\dim(R/\mathfrak{p})$ is the Krull dimension of the ring $R/\mathfrak{p}$ (which equals the dimension $\dim_R(R/\mathfrak{p})$ of the R–module $R/\mathfrak{p}$). **}**

The dimension of a complex is given by the dimension of its homology modules:

$$\sup\{\,\dim_R \mathrm{H}_\ell(M) - \ell \mid \ell \in \mathbb{Z}\,\}.$$

If M and N are R–modules, then $\dim_R(M \otimes^{\mathbf{L}}_R N) = \dim_R(M \otimes_R N)$.

Application 2: Duality

In this section $(R, \mathfrak{m}, k)$ is a local ring. By definition, a complex D in $\mathcal{D}^f_\Box(R)$ is *dualizing* for R if the injective dimension $\mathrm{id}_R\, D$ is finite, and the homothety morphism $R \to \mathbf{R}\mathrm{Hom}_R(D,D)$ is an isomorphism in $\mathcal{D}(R)$. In this section we assume that R has a dualizing complex. **{** If D is a finitely generated R–module, then D is a dualizing complex, if and only if it is a dualizing module, which means, that $\mathrm{Ext}^\ell_R(D,D) = 0$ for $\ell > 0$, the homothety map $R \to \mathrm{Hom}_R(D,D)$ is an isomorphism, and $\mathrm{id}_R\, D < \infty$. If R admits a dualizing module, then it is Cohen–Macaulay. The ring R is a dualizing R–complex, if and only if it is Gorenstein. **}**

Duality morphism. There is a natural morphism

$$\varepsilon^M \colon M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,D),D)$$

called the *duality morphism*. **{** If R is artinian, then the injective hull $\mathrm{E}_R(k)$ is a dualizing module for R, and for an R–module M the morphism ε^M maps M to the double Matlis dual: $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathrm{E}_R(k)), \mathrm{E}_R(k))$. **}**

Duality Theorem. The duality morphism ε^M is an isomorphism in $\mathcal{D}(R)$ for all $M \in \mathcal{D}^f_\Box(R)$, and there is a duality:

$$\mathcal{D}^f_\Box(R) \underset{\mathbf{R}\mathrm{Hom}_R(-,D)}{\overset{\mathbf{R}\mathrm{Hom}_R(-,D)}{\rightleftarrows}} \mathcal{D}^f_\Box(R)\,.$$

Shift. For $M \in \mathcal{D}(R)$ and $n \in \mathbb{Z}$ the complex $\Sigma^n M \in \mathcal{D}(R)$ is defined by $(\Sigma^n M)_\ell = M_{\ell-n}$ and $\partial^{\Sigma^n M}_\ell = (-1)^n \partial^M_{\ell-n}$. The complex $\Sigma^n M$ is said to be M *shifted* n degrees (against the differential).

Existence and uniqueness of Dualizing complexes. The ring R possesses a dualizing complex, if and only if it is a homomorphic image of a Gorenstein local ring. An R–complex $D \in \mathcal{D}^f_\Box(R)$ is dualizing for R, if and only if $\mathbf{R}\mathrm{Hom}_R(k,D) \simeq \Sigma^m k$ for some $m \in \mathbb{Z}$. If D and D' are dualizing complexes over R, then there exists an $n \in \mathbb{Z}$, such that $D' \simeq \Sigma^n D$.

Dagger Duality. A dualizing complex D is said to be *normalized*, when $\mathbf{R}\mathrm{Hom}_R(k,D) \simeq k$. In that case $\inf D = \operatorname{depth} R$ and $\sup D = \dim R$. If R is Cohen–Macaulay, and C is a dualizing R–module, then $\Sigma^{\dim R} C$ is a normalized dualizing R–complex.

The *dagger dual* of $M \in \mathcal{D}(R)$ is defined as $M^\dagger = \mathbf{R}\mathrm{Hom}_R(M,D)$. By the Duality Theorem $M^{\dagger\dagger} \simeq M$ for $M \in \mathcal{D}^f_\Box(R)$. Furthermore, the following hold:

$$\sup M^\dagger = \dim_R M \qquad \text{and} \qquad \inf M^\dagger = \operatorname{depth}_R M.$$

Local Duality. The local section functor, with support on $\mathfrak{m}$, is defined as $\Gamma_{\mathfrak{m}}(-) = \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{m}^n, -)$. Its right derived functor is denoted $\mathbf{R}\Gamma_{\mathfrak{m}}(-)$. There is a natural isomorphism of functors

$$\mathbf{R}\Gamma_{\mathfrak{m}}(-) \simeq \operatorname{Hom}_R(-^{\dagger}, \mathrm{E}_R(k))$$

$\{$ If R is a Cohen–Macaulay ring, then the nth right derived functor of $\Gamma_{\mathfrak{m}}(-)$ is $\operatorname{Hom}_R(\operatorname{Ext}_R^{d-n}(-, D), \mathrm{E}_R(k))$. $\}$

Application 3: Intersection results

In this section $(R, \mathfrak{m}, k)$ is a local ring.

Intersection Theorem. If $M, N \in \mathcal{D}_{\square}^{f}(R)$ have non-zero homology, then

$$\dim_R M \leq \dim_R(M \otimes_R^{\mathbf{L}} N) + \operatorname{pd}_R N\,.$$

$\{$ If M and N are non-zero finitely generated R–modules, then $\dim_R M \leq \dim_R(M \otimes_R N) + \operatorname{pd}_R N$. In particular, if also $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \{\,\mathfrak{m}\,\}$, then $\dim_R M \leq \operatorname{pd}_R N$. This has been known as the Intersection Conjecture. $\}$

New Intersection Theorem. If $F = 0 \to F_s \to \cdots \to F_0 \to 0$ is a non-trivial complex of finitely generated free R–modules such that $\dim_R \mathrm{H}_{\ell}(F) \leq \ell$ for all ℓ, then $\dim R \leq s$.

Cohen–Macaulay–defect. The *Cohen–Macaulay–defect* of an R–complex M is $\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M$ $(\in \mathbb{Z}^*)$. If $M \in \mathcal{D}_{\square}(R)$ has finite $\operatorname{depth}_R M$, then it turns out that $\operatorname{depth}_R M \leq \dim_R M$, that is, $\operatorname{cmd}_R M \geq 0$. The Intersection Theorem above and the Auslander–Buchsbaum equality yield the next *Cohen–Macaulay–defect Inequality:*

$$\operatorname{cmd}_R M \leq \operatorname{cmd}_R(M \otimes_R^{\mathbf{L}} N)\,,$$

provided $M, N \in \mathcal{D}_{\square}^{f}(R)$, $\operatorname{pd}_R N$ is finite, and $\mathrm{H}(N) \neq 0$. For $M = R$ the inequality is $\operatorname{cmd}_R N \geq \operatorname{cmd}_R R$. $\{$ A finitely generated R–module N is said to be *Cohen–Macaulay*, when $\operatorname{cmd}_R N = 0$. It follows from the above, that R is Cohen–Macaulay, if it admits a Cohen–Macaulay module of finite projective dimension. $\}$

Amplitude. For $M \in \mathcal{D}(R)$ we set $\operatorname{amp} M = \sup M - \inf M$ $(\in \mathbb{Z}^*)$. $\{$ Thus, if M is a non-zero R–module, then $\operatorname{amp} M = 0$. $\}$ The Cohen–Macaulay–defect Inequality above implies the next *Amplitude Inequality:*

$$\operatorname{amp} M \leq \operatorname{amp}(M \otimes_R^{\mathbf{L}} N)\,,$$

provided $M, N \in \mathcal{D}_{\square}^{f}(R)$, $\operatorname{pd}_R N$ is finite, and $\mathrm{H}(N) \neq 0$. $\{$ If N is a non-zero finitely generated R–module with $\operatorname{pd}_R N$ finite, and if $x_1, \dots, x_n \in \mathfrak{m}$ is an N–regular sequence, then this sequence is also R–regular. This was known as Auslander's zero-divisor conjecture. $\}$

Note from Dagger Duality that $\operatorname{cmd}_R M = \operatorname{amp} M^{\dagger}$ for M in $\mathcal{D}_{\square}^{f}(R)$.

Intersection Theorem, Special Dual Version. If $N \in \mathcal{D}_{\square}^{f}(R)$ has $\operatorname{id}_R N$ finite and $\mathrm{H}(N) \neq 0$, then

$$\operatorname{cmd}_R R \leq \operatorname{amp} N\,.$$

$\{$ If N is a finitely generated module with $\operatorname{id}_R N$ finite, then R is Cohen–Macaulay. This was known as Bass' Conjecture. $\}$

Grade. The *grade* (or codimension) of $M \in \mathcal{D}(R)$ is given as

$$\operatorname{grade}_R M = -\sup \mathbf{R}\mathrm{Hom}_R(M, R).$$

{ If M is a finitely generated R–module, then $\operatorname{grade}_R M$ equals the largest n such that there exists an R–regular sequence $x_1, \dots, x_n$ in $\operatorname{Ann}_R M$. **}**

If the local ring R is equidimensional and catenary, then it turns out that the next equality holds for $M \in \mathcal{D}^f_\square(R)$ with $\mathrm{H}(M) \neq 0$ and $\operatorname{pd}_R M$ finite.

$$\dim_R R = \dim_R M + \operatorname{grade}_R M.$$

{ This provides a partial confirmation of Auslander's Codimension Conjecture. **}**

Regular local rings. Assume R is regular, that is, the maximal ideal $\mathfrak{m}$ can be generated by $\dim R$ elements. This is known to be tantamount to $\operatorname{pd}_R N$ being finite for all R–modules N. In this case, Serre proved the inequality

$$\dim_R M + \dim_R N \leq \dim R$$

for finitely generated R–modules M and N with $\dim_R(M \otimes_R N) = 0$. This yields the next inequalities for all $M, N \in \mathcal{D}^f_\square(R)$:

$$\dim_R M + \dim_R N \leq \dim_R(M \otimes^{\mathbf{L}}_R N) + \dim R\,;$$
$$\operatorname{amp} M + \operatorname{amp} N \leq \operatorname{amp}(M \otimes^{\mathbf{L}}_R N)\,.$$

Note that these strengthen the Intersection Theorem and the Amplitude Inequality.

Intersection Theorem, Infinite version. Assume that R is equicharacteristic, that is, R and k have the same characteristic. If $M \in \mathcal{D}^f_\square(R)$, $N \in \mathcal{D}_\square(R)$, $\operatorname{fd}_R N < \infty$, and $\mathrm{H}(N) \neq 0$, then the next inequalities hold.

$$\dim_R M \leq \dim_R(M \otimes^{\mathbf{L}}_R N) + \sup(k \otimes^{\mathbf{L}}_R N) \leq \dim_R(M \otimes^{\mathbf{L}}_R N) + \operatorname{fd}_R N\,.$$

Application 4: Bass and Betti numbers

In this section $(R, \mathfrak{m}, k)$ is a local ring. For $M \in \mathcal{D}^f_\square(R)$ the Bass and Betti numbers, $\mu^\ell_R(M)$ and $\beta^R_\ell(M)$ are non-negative integers defined as

$$\mu^\ell_R(M) = |\,\mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(k, M))|_k \in \mathbb{N}_0 \quad \text{and} \quad \beta^R_\ell(M) = |\,\mathrm{H}_\ell(k \otimes^{\mathbf{L}}_R M)|_k \in \mathbb{N}_0\,,$$

where $|-|_k$ means vector space dimension over the residue field k.

Bass Series and Poincaré Series. The ring of *formal power series* with integer coefficients is denoted $\mathbb{Z}[[t]]$; the ring of *formal Laurent series* $\mathbb{Z}([t]) = \mathbb{Z}[[t]][t^{-1}]$ is obtained by inverting t. Elements of the latter are of the form $\alpha = \Sigma_{\ell \in \mathbb{Z}}\, a_\ell t^\ell$ with $a_\ell \in \mathbb{Z}$ and $a_\ell = 0$ for $\ell \ll 0$. The subset $\mathbb{N}_0([t])$ of $\mathbb{Z}([t])$ consists of the series $\alpha = \Sigma_{\ell \in \mathbb{Z}}\, a_\ell t^\ell$ with $\alpha_\ell \geq 0$ for all ℓ; it is closed under addition and multiplication.

The *Bass series* and *Poincaré series* of $M \in \mathcal{D}^f_\square(R)$,

$$\mathrm{I}^M_R(t) = \Sigma_{\ell \in \mathbb{Z}}\, \mu^\ell_R(M) t^\ell \qquad \text{and} \qquad \mathrm{P}^R_M(t) = \Sigma_{\ell \in \mathbb{Z}}\, \beta^R_\ell(M) t^\ell\,.$$

belong to $\mathbb{N}_0([t])$. Their degree and order carry information about M:

$$\operatorname{id}_R M = \deg \mathrm{I}^M_R(t) \qquad \operatorname{depth}_R M = \operatorname{ord} \mathrm{I}^M_R(t)$$
$$\operatorname{pd}_R M = \deg \mathrm{P}^R_M(t) \qquad \inf M = \operatorname{ord} \mathrm{P}^R_M(t).$$

If D belongs to $\mathcal{D}^f_\Box(R)$, then D is a normalized dualizing R–complex, if and only if $\mathrm{I}^D_R(t) = 1$, and when this is the case, the next formulae hold for $M \in \mathcal{D}^f_\Box(R)$.

$$\mathrm{I}^{M^\dagger}_R(t) = \mathrm{P}^R_M(t) \qquad \text{and} \qquad \mathrm{I}^D_R(t) = \mathrm{P}^R_{M^\dagger}(t)\,.$$

Derived functors. For $M, N \in \mathcal{D}^f_\Box(R)$ there are equalities:

$$\begin{aligned}
\mathrm{P}^R_{M\otimes^{\mathbf{L}}_R N}(t) &= \mathrm{P}^R_M(t)\,\mathrm{P}^R_N(t)\\
\mathrm{I}^{\mathbf{R}\mathrm{Hom}_R(M,N)}_R(t) &= \mathrm{P}^R_M(t)\,\mathrm{I}^N_R(t)\\
\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) &= \mathrm{I}^M_R(t)\,\mathrm{I}^N_R(t^{-1}) \quad \text{if } \mathrm{id}_R\, M < \infty\\
\mathrm{I}^{M\otimes^{\mathbf{L}}_R N}_R(t) &= \mathrm{I}^M_R(t)\,\mathrm{P}^R_N(t^{-1}) \quad \text{if } \mathrm{pd}_R\, M < \infty.
\end{aligned}$$

Localization. For $M \in \mathcal{D}^f_\Box(R)$ and $\mathfrak{p}$ a prime ideal in R the next hold.

$$\mathrm{P}^{R_\mathfrak{p}}_{M_\mathfrak{p}}(t) \preceq \mathrm{P}^R_M(t) \quad \text{and} \quad \mathrm{I}^{M_\mathfrak{p}}_{R_\mathfrak{p}}(t)t^{\dim_R(R/\mathfrak{p})} \preceq \mathrm{I}^M_R(t)\,.$$

Here, $\Sigma_{\ell\in\mathbb{Z}}\, a_\ell t \preceq \Sigma_{\ell\in\mathbb{Z}}\, b_\ell t \iff a_\ell \le b_\ell$ for all $\ell \in \mathbb{Z}$. $\big\{$If M is a finitely generated R–module and $\mathfrak{p} \in \mathrm{Spec}$ has $n = \dim_R R/\mathfrak{p}$, then $\mu^\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) \le \mu^{\ell+n}_R(M)$; in particular, if $\mu^\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ then $\mu^{\ell+n}_R(M) \neq 0.\big\}$

Type. For $M \in \mathcal{D}^f_\Box(R)$ with $\mathrm{H}(M) \neq 0$ the number $\mu^{\dim_R M}_R(M)$ is called the *type* of M. If $\dim_R M = \dim_R \mathrm{H}_{\sup M}(M) - \sup M$ (as is the case, when M is a module) and M has type 1, then there is an ideal $\mathfrak{a}$ in R such that M is a dualizing complex over $R/\mathfrak{a}$. $\big\{$If $M \in \mathcal{M}^f(R)$ is non-zero, then $\mu^\ell_R(M) \ge 2$ for $\mathrm{depth}_R\, M < \ell < \mathrm{id}_R\, M$, and when M has type one, then M is a dualizing module for $R/\mathrm{Ann}_R\, M$. If the ring has $\mu^{\dim_R R}_R(R) = 1$, then R is Gorenstein.$\big\}$

Application 5: Auslander and Bass categories

In this section $(R, \mathfrak{m}, k)$ is a local ring with normalized dualizing complex D. For each $M \in \mathcal{D}(R)$ there are two natural morphisms

$$\alpha_M \colon M \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R M) \quad \text{and} \quad \beta_M \colon D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, M) \to M.$$

Two full subcategories $\mathcal{A}(R)$ and $\mathcal{B}(R)$ of $\mathcal{D}_\Box(R)$ are defined as follows:

$$M \in \mathcal{A}(R) \iff M \otimes^{\mathbf{L}}_R D \in \mathcal{D}_\Box(R) \text{ and } \alpha_M \text{ is an isomorphism.}$$
$$N \in \mathcal{B}(R) \iff \mathbf{R}\mathrm{Hom}_R(N, D) \in \mathcal{D}_\Box(R) \text{ and } \beta_N \text{ is an isomorphism.}$$

There is an equivalence of categories:

$$\mathcal{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathcal{B}(R)\,.$$

Auslander's G–dimension. To every finitely generated R–module M, Auslander associated a number $\mathrm{G\text{–}dim}_R\, M \in \mathbb{N}^*_0 \cup \{-\infty\}$, known as the *G-dimension* of M. This homological dimension can be extended from $\mathcal{M}^f(R)$ to $\mathcal{D}^f_\Box(R)$ as follows. There is a map $\mathrm{G\text{–}dim}_R \colon \mathcal{D}^f_\Box(R) \to \mathbb{Z}^*$ with the following properties.

(1) $\mathrm{G\text{–}dim}_R\, M \le \mathrm{pd}_R\, M$ with equality, if $\mathrm{pd}_R\, M$ is finite.
(2) The following are equivalent:
  (i) R is Gorenstein.

(*ii*) G–$\dim_R M$ is finite for all finitely generated R–modules M.
(*iii*) G–$\dim_R k$ is finite.

(3) The following are equivalent for $M \in \mathcal{D}^f_\square(R)$.
(*i*) G–$\dim_R M$ is finite.
(*ii*) M belongs to the Auslander category $\mathcal{A}(R)$.
(*iii*) G–$\dim_R M = \operatorname{depth} R - \operatorname{depth}_R M$.
(*iv*) $\mathbf{R}\mathrm{Hom}_R(M,R)$ is bounded and the canonical morphism $M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,R),R)$ is an isomorphism.

This G-dimension can be extended to all of $\mathcal{D}_\square(R)$ in the following two ways.

Gorenstein Projective Dimension.
There is a map $\mathrm{Gpd}_R\colon\ \mathcal{D}_\square(R) \to \mathbb{Z}^*$ with the following properties.

(0) If $M \in \mathcal{D}^f_\square(R)$, then $\mathrm{Gpd}_R M =$ G–$\dim_R M$.
(1) If $M \in \mathcal{D}_\square(R)$, then $\mathrm{Gpd}_R M \le \mathrm{pd}_R M$ with equality if $\mathrm{pd}_R M < \infty$.
(2) If $M \in \mathcal{D}_\square(R)$, then $\mathrm{Gpd}_R M < \infty$ if and only if $M \in \mathcal{A}(R)$.
(3) R is Gorenstein, if and only if $\mathrm{Gpd}_R M$ is finite for all $M \in \mathcal{D}_\square(R)$.

Gorenstein Flat Dimension.
There is a map $\mathrm{Gfd}_R\colon\ \mathcal{D}_\square(R) \to \mathbb{Z}^*$ with the following properties.

(0) If $M \in \mathcal{D}^f_\square(R)$, then $\mathrm{Gfd}_R M =$ G–$\dim_R M$.
(1) If $M \in \mathcal{D}_\square(R)$, then $\mathrm{Gfd}_R M \le \mathrm{fd}_R M$ with equality if $\mathrm{fd}_R M < \infty$.
(2) $\mathrm{Gfd}_R M \le \mathrm{Gpd}_R M$ for all $M \in \mathcal{D}_\square(R)$.
(3) For $M \in \mathcal{D}_\square(R)$ the following are equivalent.
(*i*) $\mathrm{Gfd}_R M$ is finite.
(*i′*) $\mathrm{Gpd}_R M$ is finite.
(*ii*) $M \in \mathcal{A}(R)$.
(4) R is Gorenstein, if and only if $\mathrm{Gfd}_R M$ is finite for all $M \in \mathcal{D}_\square(R)$.

There is also a dual notion.

Gorenstein Injective Dimension.
There is a map $\mathrm{Gid}_R\colon\ \mathcal{D}_\square(R) \to \mathbb{Z}^*$ with the following properties.

(1) If $N \in \mathcal{D}_\square(R)$, then $\mathrm{Gid}_R N \le \mathrm{id}_R N$ with equality if $\mathrm{id}_R N < \infty$.
(2) If $N \in \mathcal{D}_\square(R)$, then $\mathrm{Gid}_R N < \infty$ if and only if $N \in \mathcal{B}(R)$.
(3) R is Gorenstein, if and only if $\mathrm{Gid}_R N$ is finite for all $N \in \mathcal{D}_\square(R)$.
(4) $\mathrm{Gid}_R N + \inf N = \operatorname{depth} R$ if $N \in \mathcal{D}^f_\square(R)$ has $\mathrm{H}(N) \neq 0$.

Cyclic Modules. If R possess a dualizing complex and a non-zero cyclic module of finite Gorenstein injective dimension, then R is Gorenstein. **{** If there exists a non-zero cyclic R–module of finite injective dimension, then R is Gorenstein. **}**

Application 6: Local homomorphisms

In this section, $(R,\mathfrak{m},k)$ and $(S,\mathfrak{n},\ell)$ are local rings, and $\varphi\colon R \to S$ is a *local* homomorphism, that is, $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. The homomorphism gives S an R–module structure, and it is said to be *flat*, if S, with this structure, is a flat over R. Similarly, φ is *of finite flat dimension* (written $\mathrm{fd}\,\varphi < \infty$), and *of finite Gorenstein flat dimension* (written $\mathrm{Gfd}\,\varphi < \infty$), when $\mathrm{fd}_R S < \infty$ and $\mathrm{Gfd}_R S < \infty$, respectively.

Assume D is a normalized dualizing complex for R.

Base Change. If φ is of finite flat dimension, then there exists a formal Laurent series $\mathrm{I}(\varphi) \in \mathbb{N}_0([t])$ such that the next equality holds for all $M \in \mathcal{D}^f_\square(R)$:

$$(*) \qquad \mathrm{I}_S^{M \otimes^{\mathbf{L}}_R S}(t) = \mathrm{I}_R^M(t)\,\mathrm{I}(\varphi)\,.$$

If φ is of finite Gorenstein flat dimension, then there exists a formal Laurent series $\mathrm{I}(\varphi) \in \mathbb{N}_0([t])$ such that $(*)$ holds for all $M \in \mathcal{D}^f_\square(R)$.

In particular, $\mathrm{I}_S^S(t) = \mathrm{I}_R^R(t)\,\mathrm{I}(\varphi)$. Moreover, if $M \neq 0$ then

$$\operatorname{depth}_S(M \otimes^{\mathbf{L}}_R S) - \operatorname{depth}_R M = \operatorname{ord}\mathrm{I}(\varphi) = \operatorname{depth} S - \operatorname{depth} R$$

and $\mu_R^{n+\operatorname{depth} R}(M) \leq \mu_S^{n+\operatorname{depth} S}(M \otimes^{\mathbf{L}}_R S)$.

Gorenstein Local Homomorphisms. Let $\varphi\colon R \to S$ be a local homomorphism. It is said to be *quasi-Gorenstein*, respectively, *Gorenstein*, when $\mathrm{I}(\varphi) = t^c$ for some $c \in \mathbb{Z}$, and $\operatorname{Gfd}\varphi < \infty$, respectively, $\operatorname{fd}\varphi < \infty$. $\{$ If φ is flat, that is, S is a flat R–module, with closed fiber $R/\mathfrak{m}R$, then there is an equality $\mathrm{I}(\varphi) = \mathrm{I}_{R/\mathfrak{m}R}^{R/\mathfrak{m}R}(t)$; and hence the homomorphism φ is Gorenstein in the above sense, if and only if it is Gorenstein in the classical sense. $\}$

There are two *Gorenstein Ascent–Descent Theorems:*

$$R \text{ Gorenstein and } \varphi \text{ quasi-Gorenstein} \iff S \text{ Gorenstein and } \operatorname{Gfd}\varphi < \infty\,.$$
$$R \text{ Gorenstein and } \varphi \text{ Gorenstein} \iff S \text{ Gorenstein and } \operatorname{fd}\varphi < \infty\,.$$

$\{$ Assume φ is flat. The target S is then Gorenstein, if and only if both the source R and the closed fiber $R/\mathfrak{m}R$ are so. Assume furthermore that the formal fiber $k(\mathfrak{p}) \otimes_R \widehat{R}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec} R$. If the closed fiber $S/\mathfrak{m}S$ is Gorenstein, then the fiber $k(\mathfrak{p}) \otimes_R S$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Spec} R$. $\}$

Cohen–Macaulay Local Homomorphisms. Let $\varphi\colon R \to S$ be local homomorphism. It is said to be *quasi-Cohen–Macaulay*, when $\operatorname{ord}\mathrm{I}(\varphi) = \deg\mathrm{I}(\varphi)$ and $\operatorname{Gfd}\varphi < \infty$, and it is said to be *Cohen–Macaulay* when $\operatorname{ord}\mathrm{I}(\varphi) = \deg\mathrm{I}(\varphi)$ and $\operatorname{fd}\varphi < \infty$. $\{$ If φ is flat with closed fiber $R/\mathfrak{m}R$, then there is an equality $\mathrm{I}(\varphi) = \mathrm{I}_{R/\mathfrak{m}R}^{R/\mathfrak{m}R}(t)$; the homomorphism φ is Cohen–Macaulay in the above sense, if and only if it is Cohen–Macaulay in the classical sense. $\}$

There is a *Cohen–Macaulay Ascent–Descent Theorem:*

$$R \text{ CM and } \varphi \text{ quasi-CM} \longrightarrow S \text{ CM and } \operatorname{Gfd}\varphi < \infty\,.$$
$$S \text{ CM and } \operatorname{Gfd}\varphi < \infty \longrightarrow \varphi \text{ quasi-CM}\,.$$
$$R \text{ CM and } \varphi \text{ CM} \iff S \text{ CM and } \operatorname{fd}\varphi < \infty\,.$$

$\{$ Let φ be flat. The target S is then Cohen–Macaulay, if and only if both R and $R/\mathfrak{m}R$ are so. Furthermore, let the formal fiber $k(\mathfrak{p}) \otimes_R \widehat{R}$ be Cohen–Macaulay for all $\mathfrak{p} \in \operatorname{Spec} R$. If the closed fiber $S/\mathfrak{m}S$ is Cohen–Macaulay, then the fiber $k(\mathfrak{p}) \otimes_R S$ is Cohen–Macaulay for all $\mathfrak{p} \in \operatorname{Spec} R$. This answers a question of Grothendieck. $\}$

Frobenius Endomorphism. Let R be of prime characteristic, and consider the Frobenius endomorphism $\phi\colon R \to R$. For $n \in \mathbb{N}$ let R_n denote R viewed as an R–module via ϕ^n. If R_n has finite Gorenstein flat dimension for some n, then R is Gorenstein. $\{$ If R_n has finite flat dimension for some n, then R is regular. $\}$ If R_n has finite injective dimension for some n, then R is regular. If R is a homomorphic image of a Gorenstein ring, then R is Gorenstein, provided R_n has finite Gorenstein injective dimension for some n.

Application 7: Fundamental isomorphisms

In this section Q is a commutative ring with $1 \neq 0$, and R and S are associative Q–algebras. The category of R–S–bimodules and –bihomomorphisms is denoted $\mathcal{M}(R,S)$, while $\mathcal{D}(R,S)$ denotes the corresponding derived category.

Fundamental Isomorphisms. There are natural isomorphisms in $\mathcal{D}(Q)$:

(Comm) $L \otimes^{\mathbf{L}}_R M \simeq M \otimes^{\mathbf{L}}_{R^\circ} L$ for $L \in \mathcal{D}(R^\circ)$ and $M \in \mathcal{D}(R)$.

(Assoc) For $L \in \mathcal{D}(R^\circ)$, $M \in \mathcal{D}(R,S^\circ)$, and $N \in \mathcal{D}(S)$:

$$(L \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_S N \simeq L \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_S N).$$

(Adjun) For $L \in \mathcal{D}(R^\circ,S)$, $M \in \mathcal{D}(R)$, and $N \in \mathcal{D}(S)$:

$$\mathbf{R}\mathrm{Hom}_S(L \otimes^{\mathbf{L}}_R M, N) \simeq \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(L,N)).$$

(Swap) for $L \in \mathcal{D}(R)$, $M \in \mathcal{D}(S)$, and $N \in \mathcal{D}(R,S)$:

$$\mathbf{R}\mathrm{Hom}_R(L, \mathbf{R}\mathrm{Hom}_S(M,N)) \simeq \mathbf{R}\mathrm{Hom}_S(M, \mathbf{R}\mathrm{Hom}_R(L,N)).$$

{ Let L, M, and N be modules. The classical isomorphisms are then obtained from the four above by taking the 0 th homology. If $R = S$ and N is an injective R–module, then for all $\ell \in \mathbb{Z}$ (Swap) and (Adjun) yield isomorphisms $\mathrm{Ext}^\ell_R(L, \mathrm{Hom}_R(M,N)) \cong \mathrm{Ext}^\ell_R(M, \mathrm{Hom}_R(L,N)) \cong \mathrm{Hom}_R(\mathrm{Tor}^R_\ell(L,M),N)$. **}**

Evaluation Isomorphisms. For $L \in \mathcal{D}(R)$, $M \in \mathcal{D}(R,S^\circ)$, and $N \in \mathcal{D}(S)$ there is an $\mathbb{Z}$–morphism (*Tensor evaluation*)

$$\omega_{LMN} \colon \mathbf{R}\mathrm{Hom}_R(L,M) \otimes^{\mathbf{L}}_S N \to \mathbf{R}\mathrm{Hom}_R(L, M \otimes^{\mathbf{L}}_S N).$$

It is functorial in L, M, and N and an isomorphism if $L \in \mathcal{D}^f_\square(R)$, $M \in \mathcal{D}_\square(R,S^\circ)$, $N \in \mathcal{D}_\square(S)$, R is Noetherian (as an R–module), and either $\mathrm{pd}_R\, L$ or $\mathrm{fd}_R\, N$ finite.

Hom evaluation. For $L \in \mathcal{D}(R)$, $M \in \mathcal{D}(R,S)$, and $N \in \mathcal{D}(S)$ there is an $\mathbb{Z}$–morphism, functorial in L, M, and N,

$$\theta_{LMN} \colon L \otimes^{\mathbf{L}}_{R^\circ} \mathbf{R}\mathrm{Hom}_S(M,N) \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(L,M),N).$$

It is an isomorphism if $L \in \mathcal{D}^f_\square(R)$, if $M \in \mathcal{D}_\square(R,S)$, $N \in \mathcal{D}_\square(S)$, R° is Noetherian (as an R°–module), and either $\mathrm{pd}_R\, L$ or $\mathrm{id}_R\, N$ finite.

{ For modules L, M, and N one has the classical evaluation homomorphisms $\mathrm{Hom}_R(L,M) \otimes_S N \to \mathrm{Hom}_R(L, M \otimes_S N)$ and $L \otimes_{R^\circ} \mathrm{Hom}_S(M,N) \to \mathrm{Hom}_S(\mathrm{Hom}_R(L,M),N)$; these induce the two above. **}**

{ Let R be commutative and Noetherian, and let L, M, and N be R–modules such that L is finitely generated. If N is flat, then $\mathrm{Ext}^\ell_R(L,M) \otimes_R N \cong \mathrm{Ext}^\ell_R(L, M \otimes_R N)$ for all $\ell \in \mathbb{Z}$, and hence $\mathrm{id}_R(M \otimes_R N) \leq \mathrm{id}_R\, M$. If N is injective, then $\mathrm{Tor}^R_\ell(L, \mathrm{Hom}_R(M,N)) \cong \mathrm{Hom}_R(\mathrm{Ext}^\ell_R(L,M),N)$ for all $\ell \in \mathbb{Z}$, and hence $\mathrm{id}_R(\mathrm{Hom}_R(M,N)) \leq \mathrm{id}_R\, M$. **}**

CHAPTER 1

Modules and Homomorphisms

1.1. Basic concepts

1.2. Exact functors and special modules

Free modules

(1.2.1) **Definition.** An R-module L is *free* if it has a basis, i.e. there is a set Λ such that $L \cong R^{(\Lambda)}$.

Projective modules

(1.2.2) **Definition.** An R-module P is *projective* if the functor $\mathrm{Hom}_R(P, -)$ is exact.

Injective modules

(1.2.3) **Definition.** An R-module I is *injective* if the functor $\mathrm{Hom}_R(-, I)$ is exact.

Flat modules

(1.2.4) **Definition.** An R-module F is *flat* if the functor $- \otimes_R F$ is exact.

1.3. Canonical homomorphisms

Identities

For any R-module M there are natural isomorphisms

(1.3.0.1) $$M \xrightarrow{\cong} \mathrm{Hom}_R(R, M) \quad \text{and} \quad M \xrightarrow{\cong} R \otimes_R M.$$

Standard isomorphisms

(1.3.1) **Lemma (Commutativity).** *The (tensor) commutativity homomorphism*

$$\tau_{MN} \colon M \otimes_R N \longrightarrow N \otimes_R M$$

is given by

$$\tau_{MN}(m \otimes n) = (n \otimes m).$$

It is invertible, and it is natural in M and N.

(1.3.2) **Lemma (Associativity).** *The (tensor) associativity homomorphism*

$$\sigma_{LMN} \colon (L \otimes_R M) \otimes_R N \longrightarrow L \otimes_R (M \otimes_R N)$$

is given by

$$\sigma_{LMN}((l \otimes m) \otimes n) = l \otimes (m \otimes n).$$

It is invertible, and it is natural in L, M, and N.

(1.3.3) **Lemma (Adjointness).** *The (Hom-tensor) adjointness homomorphism*

$$\rho_{LMN}\colon \operatorname{Hom}_R(L\otimes_R M, N) \longrightarrow \operatorname{Hom}_R(L, \operatorname{Hom}_R(M,N))$$

is given by

$$\rho_{LMN}(\psi)(l)(m) = \psi(l\otimes m).$$

It is invertible, and it is natural in L, M, and N.

(1.3.4) **Lemma (Swap).** *The (Hom) swap homomorphism*

$$\varsigma_{LMN}\colon \operatorname{Hom}_R(L, \operatorname{Hom}_R(M,N)) \longrightarrow \operatorname{Hom}_R(M, \operatorname{Hom}_R(L,N))$$

is given by

$$\varsigma_{LMN}(\psi)(m)(l) = \psi(l)(m).$$

It is invertible, and it is natural in L, M, and N.

Proof. Straightforward to verify naturality. Invertible because map is own inverse. $\square$

Evaluation (iso)morphisms

(1.3.5) **Lemma (Tensor evaluation).** *The tensor evaluation homomorphism*

$$\omega_{LMN}\colon \operatorname{Hom}_R(L,M)\otimes_R N \longrightarrow \operatorname{Hom}_R(L, M\otimes_R N)$$

is given by

$$\omega_{LMN}(\psi\otimes n)(l) = \psi(l)\otimes n.$$

It is natural in L, M, and N. It is invertible under each of the next extra conditions:

(a) *L is finite and projective;*

(b) *L is finite and N is flat.*

Proof. Straightforward to verify naturality.

(a): For $L = R$ by inspection. For finite free modules and summands of such by additivity of functors.

(b): Choose presentation of L by finite free modules

$$F_1 \to F_0 \to L \to 0$$

Apply $\operatorname{Hom}_R(-, M)$ and $-\otimes_R N$ in succession to obtain top row in commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \operatorname{Hom}_R(L,M)\otimes_R N \arrow[r] \arrow[d] & \operatorname{Hom}_R(F_0,M)\otimes_R N \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_R(F_1,M)\otimes_R N \arrow[d, "\cong"] \\
0 \arrow[r] & \operatorname{Hom}_R(L, M\otimes_R N) \arrow[r] & \operatorname{Hom}_R(F_0, M\otimes_R N) \arrow[r] & \operatorname{Hom}_R(F_1, M\otimes_R N)
\end{tikzcd}$$

The vertical isomorphisms are by part (a). Conclude by the 5-lemma or diagram chase. $\square$

(1.3.6) **Lemma (Homomorphism evaluation).** *The homomorphism evaluation homomorphism*

$$\theta_{LMN}\colon L\otimes_R \operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(L,M),N)$$

is given by

$$\theta_{LMN}(l\otimes\psi)(\vartheta) = \psi\vartheta(l).$$

It is natural in L, M, and N. It is invertible under each of the next two extra conditions:

(a) *L is finite and projective; or*

(b) *L is finite and N is injective.*

Proof. Straightforward to verify naturality.

(a): For $L = R$ by inspection. For finite free modules and summands of such by additivity of functors.

(b): See (E 1.3.3). □

Exercises

(E 1.3.1) Prove that Hom-tensor adjointness(1.3.3) is a natural homomorphism of R-modules.

(E 1.3.2) Give an alternative proof of swap (1.3.4) based on Lemma (1.3.3).

(E 1.3.3) Prove Lemma (1.3.6)(b).

(E 1.3.4) Let F be a flat R-module and I an injective one. Prove that $\mathrm{Hom}_R(F, I)$ is an injective R-module.

CHAPTER 2

Complexes and Morphisms

2.1. Basic concepts

Complexes

(2.1.1) **Definition.** An *R-complex* M is a sequence of R-modules $(M_v)_{v\in\mathbb{Z}}$ together with R-linear maps $(\partial_v^M\colon M_v \to M_{v-1})_{v\in\mathbb{Z}}$,

$$M \;=\; \cdots \longrightarrow M_{v+1} \xrightarrow{\partial_{v+1}^M} M_v \xrightarrow{\partial_v^M} M_{v-1} \longrightarrow \cdots ,$$

such that $\partial_v^M\partial_{v+1}^M = 0$ for all $v \in \mathbb{Z}$.

The module M_v is called the *module in degree* v, and the map $\partial_v^M\colon M_v \to M_{v-1}$ is the *vth differential*. The *degree* of an element m is denoted by $|m|$, i.e.,

$$|m| = v \iff m \in M_v.$$

Forgetting about the differentials on M one gets a graded R-module denoted $M^\natural$.

Let $u \geqslant w$ be integers. A complex M is said to be *concentrated in degrees* $u,\dots,w$ if $M_v = 0$ when $v > u$ or $v < w$; it is written

$$M \;=\; 0 \longrightarrow M_u \longrightarrow M_{u-1} \longrightarrow \cdots \longrightarrow M_{w+1} \longrightarrow M_w \longrightarrow 0.$$

In particular, the zero complex is written 0.

A complex M is said to be *bounded above* if $M_v = 0$ for $v \gg 0$, *bounded below* if $M_v = 0$ for $v \ll 0$, and simply *bounded* if it is bounded above and below, i.e. $M_v = 0$ for $|v| \gg 0$.

(2.1.2) **Remark.** A complex M concentrated in degree 0 is identified with the module M_0. A module M is considered as a complex, namely

$$M \;=\; 0 \to M \to 0$$

concentrated in degree 0.

(2.1.3) **Definition.** A *morphism* $\alpha\colon M \to N$ of R-complexes is a sequence $\alpha = (\alpha_v)_{v\in\mathbb{Z}}$ of R-module homomorphisms $\alpha_v\colon M_v \to N_v$ such that

$$\partial_v^N\alpha_v = \alpha_{v-1}\partial_v^M$$

for all $v \in \mathbb{Z}$.

For an element $r \in R$ and an R-complex M the morphism $r^M\colon M \to M$ is the *homothety* given by multiplication by r. In line with this, we denote the *identity morphism* on M by 1^M.

(2.1.4) **Remark.** R-complexes and their morphisms form a category that we denote $\mathsf{C}(R)$.

(2.1.5) **Definition.** A morphism $\alpha\colon M \to N$ of R-complexes is said to be an *isomorphism* when there exists a morphism $\alpha^{-1}\colon N \to M$ such that $\alpha^{-1}\alpha = 1^M$ and $\alpha\alpha^{-1} = 1^N$. Isomorphisms are indicated by the symbol $\cong$ next to their arrows, and two complexes M and N are *isomorphic*, $M \cong N$ in symbols, if and only if there exists an isomorphism $M \xrightarrow{\cong} N$.

(2.1.6) **Remark.** It is clear that two modules are isomorphic as complexes if and only if they are so as modules.

It is an elementary exercise (E 2.1.2) to verify that a morphism $\alpha\colon M \to N$ of R-complexes is an isomorphism if and only if all the homomorphisms $\alpha_v\colon M_v \to N_v$ are isomorphisms of R-modules.

(2.1.7) **Definition.** A sequence $(K_v \subseteq M_v)_{v\in\mathbb{Z}}$ of submodules constitute a *subcomplex* of M if the differentials ∂_v^M restrict to homomorphisms between the submodules K_v.

If K is a subcomplex of M one can form the *quotient complex* M/K in the obvious way.

(2.1.8) **Definition.** A *short exact sequence* of R-complexes is a diagram in $\mathsf{C}(R)$

$$0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0,$$

where α' is injective, α is surjective, and $\operatorname{Im}\alpha' = \operatorname{Ker}\alpha$. Equivalently, $0 \longrightarrow M_v' \xrightarrow{\alpha_v'} M_v \xrightarrow{\alpha_v} M_v'' \longrightarrow 0$ is a short exact sequence of R-modules for each $v \in \mathbb{Z}$.

Homology

(2.1.9) **Definition.** For an R-complex M set

$$\begin{aligned}
\mathrm{Z}_v(M) &= \operatorname{Ker}\partial_v^M,\\
\mathrm{B}_v(M) &= \operatorname{Im}\partial_{v+1}^M,\\
\mathrm{C}_v(M) &= \operatorname{Coker}\partial_{v+1}^M, \text{ and}\\
\mathrm{H}_v(M) &= \mathrm{Z}_v(M)/\mathrm{B}_v(M).
\end{aligned}$$

Elements in $\mathrm{Z}_v(M)$ are called *cycles*, and elements in $\mathrm{B}_v(M)$ are called *boundaries*. For each v the condition $\partial_v^M\partial_{v+1}^M = 0$ ensures that $\mathrm{B}_v(M) \subseteq \mathrm{Z}_v(M)$. The complex M is *exact in degree* v if $\mathrm{B}_v(M) = \mathrm{Z}_v(M)$, equivalently $\mathrm{H}_v(M) = 0$. The complex is *exact* if it is exact in each degree. An exact complex is also called *acyclic*.

(2.1.10) **Remark.** The sequences $(\mathrm{B}_v(M))_{v\in\mathbb{Z}}$, $(\mathrm{Z}_v(M))_{v\in\mathbb{Z}}$, $(\mathrm{C}_v(M))_{v\in\mathbb{Z}}$, and $(\mathrm{H}_v(M))_{v\in\mathbb{Z}}$ form complexes with 0 differentials. These are denoted $\mathrm{B}(M)$, $\mathrm{Z}(M)$, $\mathrm{C}(M)$, and $\mathrm{H}(M)$, respectively.

(2.1.11) **Definition.** Let M be an R-complex. The *supremum*, *infimum*, and *amplitude* of M capture the homological position and size of M. These numbers are defined as follows:

$$\begin{aligned}
\sup M &= \sup\{\, v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0 \,\},\\
\inf M &= \inf\{\, v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0 \,\}, \quad \text{and}\\
\operatorname{amp} M &= \sup M - \inf M.
\end{aligned}$$

(2.1.12) **Remark.** It follows straight from the definition (2.1.3) that a morphism $\alpha\colon M \to N$ of R-complexes maps boundaries to boundaries and cycles to cycles; thus, it induces a morphism $\mathrm{H}(\alpha)$ in homology

(2.1.12.1)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}(M) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Z}(M) \arrow[r] \arrow[d, "\alpha"] & \mathrm{H}(M) \arrow[r] \arrow[d, "\mathrm{H}(\alpha)"] & 0 \\
0 \arrow[r] & \mathrm{B}(N) \arrow[r] & \mathrm{Z}(N) \arrow[r] & \mathrm{H}(N) \arrow[r] & 0.
\end{tikzcd}$$

(2.1.13) **Lemma.** *For every short exact sequence of R-complexes,*

$$0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0,$$

there is a long exact sequence of homology modules

$$\cdots \longrightarrow \mathrm{H}_v(M') \xrightarrow{\mathrm{H}_v(\alpha')} \mathrm{H}_v(M) \xrightarrow{\mathrm{H}_v(\alpha)} \mathrm{H}_v(M'') \xrightarrow{\delta} \mathrm{H}_{v-1}(M') \longrightarrow \cdots .$$

The connecting homomorphism δ is natural in the following sense: Given a commutative diagram of R-complexes

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\beta'"] & N' \arrow[r, "\beta"] & N'' \arrow[r] & 0,
\end{tikzcd}$$

there is a commutative diagram of R-modules

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_v(M') \arrow[r, "\mathrm{H}_v(\alpha')"] \arrow[d, "\mathrm{H}(\varphi')"] & \mathrm{H}_v(M) \arrow[r, "\mathrm{H}_v(\alpha)"] \arrow[d, "\mathrm{H}(\varphi)"] & \mathrm{H}_v(M'') \arrow[r, "\delta"] \arrow[d, "\mathrm{H}(\varphi'')"] & \mathrm{H}_{v-1}(M') \arrow[r] \arrow[d, "\mathrm{H}(\varphi')"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_v(N') \arrow[r, "\mathrm{H}_v(\beta')"] & \mathrm{H}_v(N) \arrow[r, "\mathrm{H}_v(\beta)"] & \mathrm{H}_v(N'') \arrow[r, "\delta"] & \mathrm{H}_{v-1}(N') \arrow[r] & \cdots .
\end{tikzcd}$$

Proof. The connecting homomorphism is constructed through three applications of the Snake Lemma. Chase a diagram to see that it is natural. □

(2.1.14) **Definition.** A morphism $\alpha\colon M \to N$ of R-complexes is called a *quasiisomorphism* if the induced map $\mathrm{H}(\alpha)\colon \mathrm{H}(M) \to \mathrm{H}(N)$ is an isomorphism. A quasi-isomorphisms is marked by a $\simeq$ next to the arrow.

(2.1.15) **Example.** Let M be an R-module and P a projective resolution of M. Considered as a morphism of complexes, the surjective homomorphism $P_0 \twoheadrightarrow M$ is a quasiisomorphism from P to M.

Given a quasiisomorphism $\alpha\colon M \to N$ there need not exist a morphism $\beta\colon N \to M$ such that $\mathrm{H}(\beta) = \mathrm{H}(\alpha)^{-1}$.

(2.1.16) **Example.** The projective resolution of $\mathbb{Z}/(2)$ over $\mathbb{Z}$ yields a quasiisomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "2"] & 0 \arrow[d] \\
& 0 \arrow[r] & \mathbb{Z}/(2) \arrow[r] & 0,
\end{tikzcd}$$

but there is not even a morphism in the opposite direction, as there are no homomorphisms from $\mathbb{Z}/(2)$ to $\mathbb{Z}$.

(2.1.17) **Example.** Set $R = k[X,Y]$. The complexes

$$M \;=\; 0 \longrightarrow R/(X) \xrightarrow{Y} R/(X) \longrightarrow 0$$
$$N \;=\; 0 \longrightarrow R/(Y) \xrightarrow{X} R/(Y) \longrightarrow 0$$

concentrated in degrees 1 and 0 have isomorphic homology complexes $\mathrm{H}(M) \cong k \cong \mathrm{H}(N)$, but there are no morphisms between them and hence no quasiisomorphism $M \xrightarrow{\simeq} N$.

(2.1.18) **Observation.** It is immediate from Definition (2.1.3) that a surjective morphism is surjective on boundaries. An application of the Snake Lemma to the diagram (2.1.12.1) shows that a surjective quasiisomorphism is surjective on cycles as well. On the other hand, a quasiisomorphism that is surjective on cycles is also surjective on boundaries and hence surjective by the diagrams

(2.1.18.1)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Z}_v(M) \arrow[r] \arrow[d, "\alpha"] & M_v \arrow[r] \arrow[d, "\alpha"] & \mathrm{B}_{v-1}(M) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \mathrm{Z}_v(N) \arrow[r] & N_v \arrow[r] & \mathrm{B}_{v-1}(N) \arrow[r] & 0.
\end{tikzcd}$$

That is, a quasiisomorphism is surjective if it is surjective on boundaries or cycles and only if it is surjective on boundaries and cycles.

(2.1.19) **Proposition.** *Assume R is semisimple. For every R-complex M there is a quasiisomorphism* $\mathrm{H}(M) \xrightarrow{\simeq} M$.

Proof. Every R-module is projective. For each v the surjective homomorphism $\mathrm{Z}_v(M) \twoheadrightarrow \mathrm{H}_v(M)$ has an inverse $\sigma_v\colon \mathrm{H}_v(M) \to \mathrm{Z}_v(M)$. Let α_v be the composite $\mathrm{H}_v(M) \xrightarrow{\sigma_v} \mathrm{Z}_v(M) \hookrightarrow M_v$. It is clear that $\partial^M\sigma = 0$, so α is a morphism. It is also clear that $\mathrm{H}(\alpha) = 1^{\mathrm{H}(M)}$. □

Homotopy

(2.1.20) **Definition.** A morphism of R-complexes $\alpha\colon M \to N$ is *null-homotopic* if there exists a sequence of homomorphisms $(\sigma_v\colon M_v \to N_{v+1})_{v\in\mathbb{Z}}$ such that $\alpha_v = \partial^N_{v+1}\sigma_v + \sigma_{v-1}\partial^M_v$.

Two morphisms $\alpha\colon M \to N$ and $\alpha'\colon M \to N$ are *homotopic,* in symbols $\alpha \sim \alpha'$, if the difference $\alpha - \alpha'$ is null-homotopic.

(2.1.21) **Remark.** If α is null-homotopic, then the induced map $\mathrm{H}(\alpha)$ is the 0 map. Because $\mathrm{H}(-)$ is a functor this means that homotopic morphisms induce the same morphism in homology.

(2.1.22) **Example.** Consider a complex

$$M \;=\; 0 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0.$$

It is immediate that $\mathrm{H}(1^M) = 0$ if and only if M is exact, while 1^M is null-homotopic if and only if M splits.

(2.1.23) **Definition.** A morphism of R-complexes $\alpha\colon M \to N$ is a *homotopy equivalence* if there exists a morphism $N \xrightarrow{\beta} M$ such that $1^M - \beta\alpha$ and $1^N - \alpha\beta$ are null-homotopic.

(2.1.24) **Remark.** It is straightforward to verify that

$$\begin{aligned} \alpha \text{ is an isomorphism} &\implies \alpha \text{ is a homotopy equivalence} \\ &\implies \alpha \text{ is a quasiisomorphism.} \end{aligned}$$

EXERCISES

(E 2.1.1) Let $\beta\colon K \to M$ and $\alpha\colon M \to N$ be morphisms of complexes; show that the composite $\alpha\beta\colon K \to N$ is a morphism.

(E 2.1.2) Let $\alpha\colon M \to N$ be a morphism of R-complexes. Show that α is an isomorphism if and only if all the homomorphisms $\alpha_v\colon M_v \to N_v$ are isomorphisms of R-modules.

(E 2.1.3) Assume R is semisimple. Prove that for every R-complex M there is a quasiisomorphism $M \xrightarrow{\simeq} \mathrm{H}(M)$.

(E 2.1.4) Show that an injective morphism is injective on cycles. Show that a quasiisomorphism is injective if it is injective on boundaries or cycles and only if it is injective on boundaries and cycles.

2.2. Basic constructions

SHIFT

(2.2.1) **Definition.** The *n-fold shift* of a complex M is the complex $\Sigma^n M$ given by

$$(\Sigma^n M)_v = M_{v-n} \qquad \text{and} \qquad \partial_v^{\Sigma^n M} = (-1)^n \partial_{v-n}^M.$$

(2.2.2) **Remark.** Note that $\Sigma -$ is a functor on $\mathsf{C}(R)$.

TRUNCATIONS

(2.2.3) **Definition.** Let M be an R-complex and n an integer. The *hard truncation above of M at n* is the complex $M_{\leqslant n}$ given by:

$$(M_{\leqslant n})_v = \begin{cases} 0 & v > n \\ M_v & v \leqslant n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\leqslant n}} = \begin{cases} 0 & v > n \\ \partial_v^M & v \leqslant n. \end{cases}$$

Similarly, the *hard truncation below of M at n* is the complex $M_{\geqslant n}$ given by:

$$(M_{\geqslant n})_v = \begin{cases} M_v & v \geqslant n \\ 0 & v < n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\geqslant n}} = \begin{cases} \partial_v^M & v \geqslant n \\ 0 & v < n. \end{cases}$$

(2.2.4) **Remark.** For every n there is a short exact sequence of complexes

$$0 \longrightarrow M_{\leqslant n} \longrightarrow M \longrightarrow M_{\geqslant n+1} \longrightarrow 0.$$

(2.2.5) **Definition.** Let M be an R-complex and n an integer. The *soft truncation above of M at n* is the complex $M_{\subset n}$ given by:

$$(M_{\subset n})_v = \begin{cases} 0 & v > n \\ \mathrm{C}_n(M) & v = n \\ M_v & v < n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\subset n}} = \begin{cases} 0 & v > n \\ \overline{\partial_n^M} & v = n \\ \partial_v^M & v \leqslant n, \end{cases}$$

where $\overline{\partial_n^M}\colon \mathrm{C}_n(M) \to M_{n-1}$ is the induced homomorphism. Similarly, the *soft truncation below of M at n* is the complex $M_{\supset n}$ given by:

$$(M_{\supset n})_v = \begin{cases} M_v & v > n \\ \mathrm{Z}_n(M) & v = n \\ 0 & v < n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\supset n}} = \begin{cases} \partial_v^M & v \geqslant n \\ 0 & v = n \\ 0 & v < n. \end{cases}$$

(2.2.6) **Remark.** There is a morphism of complexes

$$M \longrightarrow M_{\subset n},$$

and it induces an isomorphism in homology in degrees $\leqslant n$.

(2.2.7) **Remark.** There is a morphism of complexes

$$M_{\supset n} \longrightarrow M,$$

and it induces an isomorphism in homology in degrees $\geqslant n$.

Cone

(2.2.8) **Definition.** Let $\alpha\colon M \to N$ be a morphism of R-complexes. The *mapping cone* of α is given by

$$(\operatorname{Cone}\alpha)_v = \begin{matrix} N_v \\ \oplus \\ M_{v-1} \end{matrix} \qquad \text{and} \qquad \partial_v^{\operatorname{Cone}\alpha} = \begin{bmatrix} \partial_v^N & \alpha_{v-1} \\ 0 & -\partial_{v-1}^M \end{bmatrix}.$$

It is clear that $\operatorname{Cone}\alpha$ is an R-complex.

(2.2.9) **Observation.** For every morphism of R-complexes $\alpha\colon M \to N$ there is a short exact sequence of R-complexes

$$0 \longrightarrow N \xrightarrow{\iota} \operatorname{Cone}\alpha \xrightarrow{\pi} \Sigma M \longrightarrow 0. \tag{2.2.9.1}$$

(2.2.10) **Lemma.** *A morphism of R-complexes $\alpha\colon M \to N$ is a quasiisomorphism if and only if* $\operatorname{Cone}\alpha$ *is acyclic.*

Proof. The short exact sequence (2.2.9.1) induces a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_v(N) \xrightarrow{\mathrm{H}_v(\iota)} \mathrm{H}_v(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{H}_v(\pi)} \mathrm{H}_v(\Sigma M) \xrightarrow{\delta_{v-1}} \mathrm{H}_{v-1}(N) \longrightarrow .$$

The connecting homomorphism $\delta\colon \mathrm{H}_v(\Sigma M) = \mathrm{H}_{v-1}(M) \to \mathrm{H}_{v-1}(N)$ is $\mathrm{H}_{v-1}(\alpha)$ and the claim follows by inspection. $\square$

Exercises

(E 2.2.1) Prove that a morphism of R-complexes $\alpha\colon M \to N$ is null-homotopic if and only if the short exact sequence (2.2.9.1) splits in $\mathsf{C}(R)$.

(E 2.2.2) Let M be an R-complex. Show that $\partial^M\colon M \to \Sigma M$ is a morphism of complexes. Show that the long exact sequence of homology modules associated to (2.2.9.1) is a direct sum of short exact sequences.

2.3. Homomorphisms

(2.3.1) **Definition.** For R-complexes M and N the *homomorphism complex* $\mathrm{Hom}_R(M,N)$ is defined as follows:

$$\mathrm{Hom}_R(M,N)_v \;=\; \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+v})$$

and

$$\partial_v^{\mathrm{Hom}_R(M,N)}(\psi) = (\partial_{i+v}^N \psi_i - (-1)^v \psi_{i-1}\partial_i^M)_{i\in\mathbb{Z}}.$$

An element $\psi \in \mathrm{Hom}_R(M,N)_v$ is called a *homomorphism of degree v.*

(2.3.2) **Remark.** A morphism $\alpha\colon M \to N$ is a homomorphism of degree 0. The differential ∂^M is a homomorphism of degree -1. The family $(\sigma_v\colon M_v \to N_{v+1})_{v\in\mathbb{Z}}$ in Definition (2.1.20) is a homomorphism of degree 1.

(2.3.3) **Proposition.** *A homomorphism $\alpha\colon M \to N$ of degree 0 is a morphism if and only if it is a cycle in $\mathrm{Hom}_R(M,N)$ and null-homotopic if and only if it is a boundary. That is,*

$$\begin{aligned} \alpha \text{ is a morphism } &\iff \alpha \in \mathrm{Z}_0(\mathrm{Hom}_R(M,N)) \quad \text{and} \\ \alpha \text{ is null-homotopic } &\iff \alpha \in \mathrm{B}_0(\mathrm{Hom}_R(M,N)). \end{aligned}$$

Proof. For $\alpha \in \mathrm{Hom}_R(M,N)_0$ and $\sigma \in \mathrm{Hom}_R(M,N)_1$ the definition of the differential on $\mathrm{Hom}_R(M,N)$ yields

$$\begin{aligned} &\partial_0^{\mathrm{Hom}_R(M,N)}(\alpha) = (\partial_i^N \alpha_i - \alpha_{i-1}\partial_i^M)_{i\in\mathbb{Z}} \quad \text{and} \\ &\partial_1^{\mathrm{Hom}_R(M,N)}(\sigma) = (\partial_{i+1}^N \sigma_i + \sigma_{i-1}\partial_i^M)_{i\in\mathbb{Z}}. \end{aligned}$$ □

(2.3.4) **Definition.** A cycle in $\mathrm{Hom}_R(M,N)$ is called a *chain map*. Two chain maps $\gamma,\gamma'\colon M \to N$ are *homotopic*, written $\gamma \sim \gamma'$, if $\gamma - \gamma' \in \mathrm{B}(\mathrm{Hom}_R(M,N))$.

A morphism is a chain map of degree 0.

(2.3.5) **Observation.** Let $\psi\colon K \to M$ and $\zeta\colon M \to N$ be homomorphisms of degree m and n, respectively. The composite

$$\zeta\psi \;=\; (\zeta_{i+m}\psi_i)_{i\in\mathbb{Z}}$$

is a homomorphism of degree $m+n$, i.e. $\zeta\psi \in \mathrm{Hom}_R(K,N)_{m+n}$.

Moreover,

$$\begin{aligned}\partial^{\mathrm{Hom}_R(K,N)}(\zeta\psi) &= \partial^N\zeta\psi - (-1)^{m+n}\zeta\psi\partial^K \\ &= \partial^{\mathrm{Hom}_R(M,N)}(\zeta)\psi + (-1)^n\zeta\partial^M\psi - (-1)^{m+n}\zeta\psi\partial^K \\ &= \partial^{\mathrm{Hom}_R(M,N)}(\zeta)\psi + (-1)^n\zeta(\partial^M\psi - (-1)^m\psi\partial^K) \\ &= \partial^{\mathrm{Hom}_R(M,N)}(\psi)\zeta + (-1)^n\psi\partial^{\mathrm{Hom}_R(K,M)}(\zeta).\end{aligned}$$

In particular, the composite of two chain maps is a chain map.

Covariant Hom

The purpose of the next construction is to make covariant Hom a functor on complexes.

(2.3.6) **Construction.** Let M be a complex and $\zeta\colon X \to Y$ a homomorphism of R-complexes of degree m. For brevity write $[M,X] = \mathrm{Hom}_R(M,X)$ and $[M,Y] = \mathrm{Hom}_R(M,Y)$. The map

$$[M,\zeta]\colon [M,X] \longrightarrow [M,Y]$$

given by composition, that is $[M,\zeta](\psi) = \zeta\psi$, is a homomorphism of degree m.

(2.3.7) **Lemma.** *In the notation of (2.3.6), the differential on* $\mathrm{Hom}_R([M,X],[M,Y])$ *maps* $[M,\zeta]$ *to* $[M,\partial^{\mathrm{Hom}_R(X,Y)}(\zeta)]$. *That is,*

$$\partial^{\mathrm{Hom}_R([M,X],[M,Y])}([M,\zeta]) = [M,\partial^{\mathrm{Hom}_R(X,Y)}(\zeta)].$$

In particular,

$$\begin{aligned}\zeta \in \mathrm{Z}_m(\mathrm{Hom}_R(X,Y)) &\implies [M,\zeta] \in \mathrm{Z}_m(\mathrm{Hom}_R([M,X],[M,Y])) \quad \textit{and} \\ \zeta \in \mathrm{B}_m(\mathrm{Hom}_R(X,Y)) &\implies [M,\zeta] \in \mathrm{B}_m(\mathrm{Hom}_R([M,X],[M,Y])).\end{aligned}$$

Proof. First note that

$$\partial^{\mathrm{Hom}_R([M,X],[M,Y])}([M,\zeta]) = \partial^{[M,Y]}[M,\zeta] - (-1)^m[M,\zeta]\partial^{[M,X]}.$$

For every $\psi \in [M,X]$ one has

$$\begin{aligned}&\partial^{\mathrm{Hom}_R([M,X],[M,Y])}([M,\zeta])(\psi) \\ &\quad= \partial^{[M,Y]}[M,\zeta](\psi) - (-1)^m[M,\zeta]\partial^{[M,X]}(\psi) \\ &\quad= \partial^{[M,Y]}(\zeta\psi) - (-1)^m[M,\zeta](\partial^M\psi - (-1)^{|\psi|}\psi\partial^M) \\ &\quad= \partial^Y\zeta\psi - (-1)^{m+|\psi|}\zeta\psi]\partial^M - (-1)^m\zeta(\partial^M\psi - (-1)^{|\psi|}\psi\partial^M) \\ &\quad= \partial^Y\zeta\psi - (-1)^m\zeta\partial^M\psi \\ &\quad= (\partial^{\mathrm{Hom}_R(X,Y)}(\zeta))\psi \\ &\quad= [M,\partial^{\mathrm{Hom}_R(X,Y)}(\zeta)](\psi).\end{aligned}$$

This proves the desired formula, and the remaining assertions follow. □

(2.3.8) **Theorem.** *For every R-complex M, complex homomorphisms from M define a covariant functor,* $\mathrm{Hom}_R(M,–)$, *on* $\mathsf{C}(R)$. *That is,*

(a) *to every R-complex X it associates a unique R-complex* $\mathrm{Hom}_R(M,X)$*;*

(b) *to every morphism* $\alpha\colon X \to Y$ *it associates a unique morphism* $\operatorname{Hom}_R(M,\alpha)\colon \operatorname{Hom}_R(M,X) \to \operatorname{Hom}_R(M,Y)$*;*

(c) *the equality* $\operatorname{Hom}_R(M,\beta\alpha) = \operatorname{Hom}_R(M,\beta)\operatorname{Hom}_R(M,\alpha)$ *holds for every pair of morphisms* $\alpha\colon X \to Y$ *and* $\beta\colon Y \to Z$*;*

(d) *The equality* $\operatorname{Hom}_R(M,1^X) = 1^{\operatorname{Hom}_R(M,X)}$ *holds for every* R*-complex* X*.*

Proof. Part (a) is explained in Definition (2.3.1), part (b) follows from Lemma (2.3.7), and part (d) is immediate from Construction (2.3.6).

(c): Apply Construction (2.3.6) three times. □

(2.3.9) **Proposition.** *Let* M *be an* R*-complex and* $\zeta\colon X \to Y$ *be a chain map.*

(a) *If* $\zeta'\colon X \to Y$ *is a chain map homotopic to* ζ*, then also the induced chain maps* $\operatorname{Hom}_R(M,\zeta)$ *and* $\operatorname{Hom}_R(M,\zeta')$ *are homotopic.*

(b) *If* ζ *is a homotopy equivalence, then so is the induced morphism* $\operatorname{Hom}_R(M,\zeta)$*.*

Proof. (a): By assumption $\zeta - \zeta'$ belongs to $\mathrm{B}(\operatorname{Hom}_R(X,Y))$. By functoriality of $\operatorname{Hom}_R(M,\text{-})$ and by Lemma (2.3.7) also

$$\operatorname{Hom}_R(M,\zeta) - \operatorname{Hom}_R(M,\zeta') = \operatorname{Hom}_R(M,\zeta-\zeta')$$

is a boundary.

(b): There exists a $\vartheta\colon Y \to X$ such that $\vartheta\zeta \sim 1^X$ and $\zeta\vartheta \sim 1^Y$. Because $\operatorname{Hom}_R(M,\text{-})$ is a functor it follows from (a) that

$$\operatorname{Hom}_R(M,\zeta)\operatorname{Hom}_R(M,\vartheta) = \operatorname{Hom}_R(M,\zeta\vartheta) \sim \operatorname{Hom}_R(M,1^Y) = 1^{\operatorname{Hom}_R(M,Y)}.$$

A similar argument yields $\operatorname{Hom}_R(M,\vartheta)\operatorname{Hom}_R(M,\zeta) \sim 1^{\operatorname{Hom}_R(M,X)}$. □

(2.3.10) **Lemma.** *For* R*-complexes* M, X *and* $n \in \mathbb{Z}$ *there is an identity of complexes*

$$\operatorname{Hom}_R(M,\Sigma^n X) = \Sigma^n \operatorname{Hom}_R(M,X).$$

Proof. A straightforward inspection. □

(2.3.11) **Lemma.** *For an* R*-complex* M *and a morphism* $\alpha\colon X \to Y$ *there is an identity of complexes*

$$\operatorname{Cone}\operatorname{Hom}_R(M,\alpha) = \operatorname{Hom}_R(M,\operatorname{Cone}\alpha).$$

Proof. A straightforward inspection. □

Contravariant Hom

The purpose of the next construction is to make contravariant Hom a functor on complexes.

(2.3.12) **Construction.** Let N be a complex and $\zeta\colon X \to Y$ a homomorphism of R-complexes of degree m. For brevity write $[X,N] = \operatorname{Hom}_R(X,N)$ and $[Y,N] = \operatorname{Hom}_R(Y,N)$. The map

$$[\zeta,N]\colon [Y,N] \longrightarrow [X,N]$$

given by $[\zeta,N](\psi) = (-1)^{m|\psi|}\psi\zeta$ is a homomorphism of degree m.

(2.3.13) **Lemma.** *In the notation of (2.3.12), the differential on* $\operatorname{Hom}_R([Y,N],[X,N])$ *maps* $[\zeta,N]$ *to* $[\partial^{\operatorname{Hom}_R(X,Y)}(\zeta),N]$. *That is,*

$$\partial^{\operatorname{Hom}_R([Y,N],[X,N])}([\zeta,N]) = [\partial^{\operatorname{Hom}_R(X,Y)}(\zeta),N].$$

In particular,

$$\zeta \in \mathrm{Z}_m(\operatorname{Hom}_R(X,Y)) \implies [\zeta,N] \in \mathrm{Z}_m(\operatorname{Hom}_R([Y,N],[X,N])) \quad \textit{and}$$
$$\zeta \in \mathrm{B}_m(\operatorname{Hom}_R(X,Y)) \implies [\zeta,N] \in \mathrm{B}_m(\operatorname{Hom}_R([Y,N],[X,N])).$$

Proof. First note that

$$\partial^{\operatorname{Hom}_R([Y,N],[X,N])}([\zeta,N]) = \partial^{[X,N]}[\zeta,N] - (-1)^m[\zeta,N]\partial^{[Y,N]}.$$

For every $\psi \in [Y,N]$ one has

$$\begin{aligned}
\partial^{\operatorname{Hom}_R([Y,N],[X,N])}&([\zeta,N])(\psi) \\
&= \partial^{[X,N]}[\zeta,N](\psi) - (-1)^m[\zeta,N]\partial^{[Y,N]}(\psi) \\
&= \partial^{[X,N]}((-1)^{m|\psi|}\psi\zeta) - (-1)^m[\zeta,N](\partial^N\psi - (-1)^{|\psi|}\psi\partial^Y) \\
&= (-1)^{m|\psi|}(\partial^N\psi\zeta - (-1)^{m+|\psi|}\psi\zeta\partial^X) \\
&\qquad - (-1)^{m+m(|\psi|-1)}(\partial^N\psi - (-1)^{|\psi|}\psi\partial^Y)\zeta \\
&= (-1)^{m|\psi|+|\psi|}\psi\partial^Y\zeta - (-1)^{m|\psi|+m+|\psi|}\psi\zeta\partial^X \\
&= (-1)^{(m-1)|\psi|}\psi(\partial^Y\zeta - (-1)^m\zeta\partial^X) \\
&= [\partial^{\operatorname{Hom}_R(X,Y)}(\zeta),N](\psi).
\end{aligned}$$

This proves the desired formula, and the remaining assertions follow. □

(2.3.14) **Theorem.** *For every R-complex N, complex homomorphisms to N define a contravariant functor,* $\operatorname{Hom}_R(-,N)$,*on* $\mathsf{C}(R)$. *That is,*

(a) *to every R-complex X it associates a unique R-complex* $\operatorname{Hom}_R(X,N)$*;*
(b) *to every morphism $\alpha\colon X \to Y$ it associates a unique morphism* $\operatorname{Hom}_R(\alpha,N)\colon \operatorname{Hom}_R(Y,N) \to \operatorname{Hom}_R(X,N)$*;*
(c) *the equality* $\operatorname{Hom}_R(\beta\alpha,N) = \operatorname{Hom}_R(\alpha,N)\operatorname{Hom}_R(\beta,N)$ *holds for every pair of morphisms $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$;*
(d) *the equality* $\operatorname{Hom}_R(1^X,N) = 1^{\operatorname{Hom}_R(X,N)}$ *holds for every R-complex X.*

Proof. Part (a) is explained in Definition (2.3.1), part (b) follows from Lemma (2.3.13), and part (d) is immediate from Construction (2.3.12).
(c): Apply Construction (2.3.12) three times. □

(2.3.15) **Proposition.** *Let N be an R-complex and $\zeta\colon X \to Y$ be a chain map.*

(a) *If $\zeta'\colon X \to Y$ is a chain map homotopic to ζ, then also the induced chain maps* $\operatorname{Hom}_R(\zeta,N)$ *and* $\operatorname{Hom}_R(\zeta',N)$ *are homotopic.*
(b) *If ζ is a homotopy equivalence, then so is the induced morphism* $\operatorname{Hom}_R(\zeta,N)$.

Proof. Similar to the covariant case. □

(2.3.16) **Lemma.** *For R-complexes N, X and $n \in \mathbb{Z}$ there is an isomorphism of complexes*

$$\operatorname{Hom}_R(\Sigma^n X,N) \cong \Sigma^{-n}\operatorname{Hom}_R(X,N).$$

Proof. A straightforward inspection. $\square$

(2.3.17) **Lemma.** *For an R-complex N and a morphism $\alpha\colon X \to Y$ there is an isomorphism of complexes*

$$\operatorname{Cone}\operatorname{Hom}_R(\alpha, N) = \Sigma\operatorname{Hom}_R(\operatorname{Cone}\alpha, N).$$

Proof. A straightforward inspection. $\square$

BOUNDEDNESS AND FINITENESS

(2.3.18) **Observation.** Let M and N be R-complexes. Suppose there exist integers u and w such that $M_v = 0$ for $v < w$ and $N_v = 0$ for $v > u$. For each $v \in \mathbb{Z}$ the module $\operatorname{Hom}_R(M, N)_v$ is then a finite product

$$\operatorname{Hom}_R(M, N)_v = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(M_i, N_{i+v}) = \bigoplus_{i=w}^{u-v} \operatorname{Hom}_R(M_i, N_{i+v}). \tag{2.3.18.1}$$

(2.3.19) **Lemma.** *Let M and N be R-complexes. If M is bounded below and N is bounded above, then $\operatorname{Hom}_R(M, N)$ is a bounded above R-complex. More precisely, if $M_v = 0$ for $v < w$ and $N_v = 0$ for $v > u$, then*

(a) $\operatorname{Hom}_R(M, N)_v = 0$ *for* $v > u - w$,

(b) $\operatorname{Hom}_R(M, N)_{u-w} = \operatorname{Hom}_R(M_w, N_u)$, *and*

(c) $\operatorname{H}_{u-w}(\operatorname{Hom}_R(M, N)) \cong \operatorname{Hom}_R(\operatorname{H}_w(M), \operatorname{H}_u(N))$.

Proof. Parts (a) and (b) are immediate from Observation (2.3.18).

(c): By Lemmas (2.3.10) and (2.3.16) there is an isomorphism

$$\operatorname{H}_{u-w}(\operatorname{Hom}_R(M, N)) = \operatorname{H}_0(\Sigma^{w-u}\operatorname{Hom}_R(M, N)) \cong \operatorname{H}_0(\operatorname{Hom}_R(\Sigma^{-w}M, \Sigma^{-u}N)).$$

The complexes $\Sigma^{-w}M$ and $\Sigma^{-u}N$ are concentrated in non-negative and non-positive degrees, respectively, so by (E 4.4.2) there is an isomorphism

$$\begin{aligned}\operatorname{H}_0(\operatorname{Hom}_R(\Sigma^{-w}M, \Sigma^{-u}N)) &\cong \operatorname{Hom}_R(\operatorname{H}_0(\Sigma^{-w}M), \operatorname{H}_0(\Sigma^{-u}N))\\ &= \operatorname{Hom}_R(\operatorname{H}_w(M), \operatorname{H}_u(N)).\end{aligned}$$ $\square$

(2.3.20) **Lemma.** *If M and N are complexes of finitely generated R-modules, such that M is bounded below and N is bounded above, then $\operatorname{Hom}_R(M, N)$ is a complex of finitely generated R-modules and bounded above.*

Proof. For every $v \in \mathbb{Z}$ and $i \in \mathbb{Z}$ the module $\operatorname{Hom}_R(M_i, N_{i+v})$ is finitely generated. There exist integers u and w such that $M_v = 0$ for $v < w$ and $N_v = 0$ for $v > u$, so it follows from (2.3.18.1) that the module $\operatorname{Hom}_R(M, N)_v$ is finitely generated for every v, and by Lemma (2.3.19) it vanishes for $v > u - w$. $\square$

EXERCISES

(E 2.3.1) Prove that a chain map $M \to N$ of degree m is a morphism $M \to \Sigma^{-m}N$ and vice versa.

(E 2.3.2) For R-complexes M and N consider the three degree -1 homomorphisms $\partial^{\operatorname{Hom}_R(M,N)}$, $\operatorname{Hom}_R(\partial^M, N)$, and $\operatorname{Hom}_R(M, \partial^N)$ from the complex $\operatorname{Hom}_R(M, N)$ to itself. Prove the identity

$$\partial^{\operatorname{Hom}_R(M,N)} = \operatorname{Hom}_R(M, \partial^N) - \operatorname{Hom}_R(\partial^M, N).$$

2.4. Tensor product

(2.4.1) **Definition.** For R-complexes M and N the *tensor product complex* $M \otimes_R N$ is defined as follows:

$$(M \otimes_R N)_v \ = \coprod_{i \in \mathbb{Z}} M_i \otimes_R N_{v-i}$$

and

$$\partial_v^{M \otimes_R N}(m_i \otimes n_{v-i}) = \partial_i^M(m_i) \otimes n_{v-i} + (-1)^i m_i \otimes \partial_{v-i}^N(n_{v-i}).$$

(2.4.2) **Observation.** Let M and N be R-complexes and $\zeta \colon X \to Y$ be a homomorphism of R-complexes. The maps

$$M \otimes_R \zeta \colon M \otimes_R X \longrightarrow M \otimes_R Y \qquad \text{and} \qquad \zeta \otimes_R N \colon X \otimes_R N \longrightarrow Y \otimes_R N$$

with vth components given by

$$\begin{aligned} (M \otimes_R \zeta)_v(m_i \otimes x_{v-i}) &= (-1)^{i|\zeta|} m_i \otimes \zeta_{v-i}(x_{v-i}) \quad \text{and} \\ (\zeta \otimes_R N)_v(x_i \otimes n_{v-i}) &= \zeta_i(x_i) \otimes n_{v-i} \end{aligned}$$

are homomorphisms of degree $|\zeta|$.

(2.4.3) **Lemma.** *The differential on* $\operatorname{Hom}_R(M \otimes_R X, M \otimes_R Y)$ *maps the induced homomorphism* $M \otimes_R \zeta$ *to the homomorphism induced by* $\partial^{\operatorname{Hom}_R(X,Y)}(\zeta)$*. That is,*

$$\partial^{\operatorname{Hom}_R(M \otimes_R X, M \otimes_R Y)}(M \otimes_R \zeta) \ = \ M \otimes_R \partial^{\operatorname{Hom}_R(X,Y)}(\zeta).$$

In particular,

$$\begin{aligned} \zeta \in \mathrm{Z}_m(\operatorname{Hom}_R(X,Y)) &\implies M \otimes_R \zeta \in \mathrm{Z}_m(\operatorname{Hom}_R(M \otimes_R X, M \otimes_R Y)) \\ \zeta \in \mathrm{B}_m(\operatorname{Hom}_R(X,Y)) &\implies M \otimes_R \zeta \in \mathrm{B}_m(\operatorname{Hom}_R(M \otimes_R X, M \otimes_R Y)). \end{aligned}$$

Proof. Straightforward computation. □

(2.4.4) **Lemma.** *The differential on* $\operatorname{Hom}_R(X \otimes_R N, Y \otimes_R N)$ *maps the induced homomorphism* $\zeta \otimes_R N$ *to the homomorphism induced by* $\partial^{\operatorname{Hom}_R(X,Y)}(\zeta)$*. That is,*

$$\partial^{\operatorname{Hom}_R(X \otimes_R N, Y \otimes_R N)}(\zeta \otimes_R N) \ = \ \partial^{\operatorname{Hom}_R(X,Y)}(\zeta) \otimes_R N.$$

In particular,

$$\begin{aligned} \zeta \in \mathrm{Z}_m(\operatorname{Hom}_R(X,Y)) &\implies \zeta \otimes_R N \in \mathrm{Z}_m(\operatorname{Hom}_R(X \otimes_R N, Y \otimes_R N)) \quad \textit{and} \\ \zeta \in \mathrm{B}_m(\operatorname{Hom}_R(X,Y)) &\implies \zeta \otimes_R N \in \mathrm{B}_m(\operatorname{Hom}_R(X \otimes_R N, Y \otimes_R N)). \end{aligned}$$

Proof. Straightforward computation. □

(2.4.5) **Theorem.** *For every R-complex M, the tensor product defines a covariant functor $M \otimes_R$ – on $\mathsf{C}(R)$. That is,*

(a) *to every R-complex X it associates a unique R-complex $M \otimes_R X$;*
(b) *to every morphism $\alpha \colon X \to Y$ it associates a unique morphism $M \otimes_R \alpha \colon M \otimes_R X \to M \otimes_R Y$;*
(c) *the equality $M \otimes_R \beta\alpha = (M \otimes_R \beta)(M \otimes_R \alpha)$ holds for every pair of morphisms $\alpha \colon X \to Y$ and $\beta \colon Y \to Z$;*
(d) *the equality $M \otimes_R 1^X = 1^{M \otimes_R X}$ holds for every R-complex X.*

Proof. Follows from Observation (2.4.2) and Lemma (2.4.3). □

(2.4.6) **Proposition.** *Let M be an R-complex and $\zeta\colon X \to Y$ be a chain map.*

(a) *If $\zeta'\colon X \to Y$ is a chain map homotopic to ζ, then also the induced chain maps $M \otimes_R \zeta$ and $M \otimes_R \zeta'$ are homotopic.*
(b) *If ζ is a homotopy equivalence, then so is the induced morphism $M \otimes_R \zeta$.*

Proof. Straightforward verification. □

(2.4.7) **Theorem.** *For every R-complex N, the tensor product defines a covariant functor $- \otimes_R N$ on $\mathsf{C}(R)$. That is,*

(a) *to every R-complex X it associates a unique R-complex $X \otimes_R N$;*
(b) *to every morphism $\alpha\colon X \to Y$ it associates a unique morphism $\alpha \otimes_R N\colon X \otimes_R N \to Y \otimes_R N$;*
(c) *the equality $\beta\alpha \otimes_R N = (\beta \otimes_R N)(\alpha \otimes_R N)$ holds for every pair of morphisms $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$;*
(d) *the equality $1^X \otimes_R N = 1^{X \otimes_R N}$ holds for every R-complex X.*

Proof. Follows from Observation (2.4.2) and Lemma (2.4.4). □

(2.4.8) **Proposition.** *Let N be an R-complex and $\zeta\colon X \to Y$ be a chain map.*

(a) *If $\zeta'\colon X \to Y$ is a chain map homotopic to ζ, then also the induced chain maps $\zeta \otimes_R N$ and $\zeta' \otimes_R N$ are homotopic.*
(b) *If ζ is a homotopy equivalence, then so is the induced morphism $\zeta \otimes_R N$.*

Proof. Straightforward computation. □

(2.4.9) **Lemma.** *For R-complexes M, X and $n \in \mathbb{Z}$ there is an isomorphism of complexes*

$$M \otimes_R \Sigma^n X \;\cong\; \Sigma^n(M \otimes_R X).$$

Proof. A straightforward inspection. □

(2.4.10) **Lemma.** *For R-complexes X, N and $n \in \mathbb{Z}$ there is an identity of complexes*

$$\Sigma^n X \otimes_R N \;=\; \Sigma^n(X \otimes_R N).$$

Proof. A straightforward inspection. □

(2.4.11) **Lemma.** *For an R-complex M and a morphism $\alpha\colon X \to Y$ there is an isomorphism of complexes*

$$\operatorname{Cone}(M \otimes_R \alpha) \;\cong\; M \otimes_R \operatorname{Cone}\alpha.$$

Proof. A straightforward inspection. □

(2.4.12) **Lemma.** *For an R-complex N and a morphism $\alpha\colon X \to Y$ there is an isomorphism of complexes*

$$\operatorname{Cone}(\alpha \otimes_R N) \;\cong\; (\operatorname{Cone}\alpha) \otimes_R N.$$

Proof. A straightforward inspection. □

Boundedness and finiteness

(2.4.13) **Observation.** Let M and N be R-complexes. Suppose there exist integers w and t such that $M_v = 0$ for $v < w$ and $N_v = 0$ for $v < t$. For each $v \in \mathbb{Z}$ the module $(M \otimes_R N)_v$ is then a finite sum:

$$(2.4.13.1) \qquad (M \otimes_R N)_v = \coprod_{i \in \mathbb{Z}} M_i \otimes_R N_{v-i} = \bigoplus_{i=w}^{v-t} M_i \otimes_R N_{v-i}.$$

(2.4.14) **Lemma.** *If M and N are bounded below R-complexes, then $M \otimes_R N$ is a bounded below R-complex. More precisely, if $M_v = 0$ for $v < w$ and $N_v = 0$ for $v < t$, then*

(a) $(M \otimes_R N)_v = 0$ *for* $v < w + t$,

(b) $(M \otimes_R N)_{w+t} = M_w \otimes_R N_t$, *and*

(c) $\mathrm{H}_{w+t}(M \otimes_R N) \cong \mathrm{H}_w(M) \otimes_R \mathrm{H}_t(N)$.

Proof. Parts (a) and (b) are immediate from Observation (2.4.13). For part (c), first note that

$$\mathrm{H}_w(M) = \mathrm{C}_w(M), \quad \mathrm{H}_t(N) = \mathrm{C}_t(N), \quad \text{and}$$
$$\mathrm{H}_{w+t}(M \otimes_R N) = \mathrm{C}_{w+t}(M \otimes_R N) = \frac{M_w \otimes_R N_t}{\mathrm{B}_{w+t}(M \otimes_R N)},$$

where the equalities in the second line follow from (a) and (b). The module $\mathrm{B}_{w+t}(M \otimes_R N)$ is generated by elements $\partial^M_{w+1}(m') \otimes n$ and $m \otimes \partial^N_{t+1}(n')$. It is clear that

$$[m \otimes n] \longmapsto [m] \otimes [n] \qquad \text{and} \qquad [m] \otimes [n] \longmapsto [m \otimes n]$$

well-define inverse homomorphisms of R-modules. □

(2.4.15) **Lemma.** *If M and N are complexes of finitely generated R-modules and bounded below, then $M \otimes_R N$ is a complex of finitely generated R-modules and bounded below.*

Proof. For every $v \in \mathbb{Z}$ and $i \in \mathbb{Z}$ the module $M_i \otimes_R N_{v-i}$ is finitely generated. There exist integers w and t such that $M_v = 0$ for $v < w$ and $N_v = 0$ for $v < t$, so it follows from (2.4.13.1) that the module $(M \otimes_R N)_v$ is finitely generated for every v, and by Lemma (2.4.14) it vanishes for $v < w + t$. □

2.5. Canonical morphisms

Standard isomorphisms

First we establish the (tensor product) *commutativity* isomorphism.

(2.5.1) **Theorem.** *Let M and N be R-complexes. The assignment*

$$m \otimes n \longmapsto (-1)^{|m||n|} n \otimes m,$$

for $m \in M$ and $n \in N$, defines an isomorphism in $\mathsf{C}(R)$

$$\tau_{MN} \colon M \otimes_R N \xrightarrow{\cong} N \otimes_R M,$$

which is natural in M and N.

Proof. The computation

$$\begin{aligned}
\tau_{MN}(&\partial^{M\otimes_R N}(m\otimes n))\\
&= \tau_{MN}(\partial^M(m)\otimes n + (-1)^{|m|}m\otimes\partial^N(n))\\
&= (-1)^{(|m|-1)|n|}(n\otimes\partial^M(m)) + (-1)^{|m|+|m|(|n|-1)}(\partial^N(n)\otimes m)\\
&= (-1)^{|m||n|}(\partial^N(n)\otimes m + (-1)^{|n|}(n\otimes\partial^M(m)))\\
&= (-1)^{|m||n|}(\partial^{N\otimes_R M}(n\otimes m))\\
&= \partial^{N\otimes_R M}(\tau_{MN}(m\otimes n))
\end{aligned}$$

shows that τ_{MN} is a morphism of complexes. It is clear that it has an inverse, namely $(\tau_{MN})^{-1} = \tau_{NM}$.

Let $\alpha\colon M\to M'$ be a morphism of complexes. The following computation shows that τ_{MN} is natural in M.

$$\begin{aligned}
\tau_{M'N}((\alpha\otimes_R N)(m\otimes n)) &= \tau_{M'N}(\alpha(m)\otimes n)\\
&= (-1)^{|\alpha(m)||n|}n\otimes\alpha(m)\\
&= (-1)^{|m||n|}n\otimes\alpha(m)\\
&= (N\otimes_R\alpha)((-1)^{|m||n|}n\otimes m)\\
&= (N\otimes_R\alpha)(\tau_{MN}(m\otimes n))
\end{aligned}$$

A similar computation shows that τ_{MN} is natural in N. □

The next map is the (tensor) *associativity* isomorphism.

(2.5.2) **Theorem.** *Let K, M, and N be R-complexes. The assignment*

$$(k\otimes m)\otimes n \longmapsto k\otimes(m\otimes n),$$

for $k\in K$, $m\in M$, and $n\in N$, defines an isomorphism in $\mathsf{C}(R)$

$$\sigma_{KMN}\colon (K\otimes_R M)\otimes_R N \xrightarrow{\cong} K\otimes_R(M\otimes_R N),$$

which is natural in K, M, and N.

Proof. Straightforward verification similar to the next proof. □

The next map is the (Hom-tensor) *adjointness* isomorphism.

(2.5.3) **Theorem.** *Let K, M, and N be R-complexes. The assignment*

$$\psi\longmapsto [k\mapsto[m\mapsto\psi(k\otimes m)]],$$

for $k\in K$, $m\in M$, and $\psi\in\operatorname{Hom}_R(K\otimes_R M,N)$, defines is an isomorphism in $\mathsf{C}(R)$

$$\rho_{KMN}\colon \operatorname{Hom}_R(K\otimes_R M,N)\xrightarrow{\cong}\operatorname{Hom}_R(K,\operatorname{Hom}_R(M,N)),$$

which is natural in K, M, and N.

Proof. First note that

$$\begin{aligned}
\operatorname{Hom}_R(K \otimes_R M, N)_v &= \prod_{h\in\mathbb{Z}} \operatorname{Hom}_R((K \otimes_R M)_h, N_{h+v}) \\
&= \prod_{h\in\mathbb{Z}} \operatorname{Hom}_R(\coprod_{i\in\mathbb{Z}} K_i \otimes_R M_{h-i}, N_{h+v}) \\
&= \prod_{h\in\mathbb{Z}} \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(K_i \otimes_R M_{h-i}, N_{h+v}) \\
&= \prod_{i\in\mathbb{Z}} \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(K_i \otimes_R M_j, N_{i+j+v})
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Hom}_R(K, \operatorname{Hom}_R(M, N))_v &= \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(K_i, \operatorname{Hom}_R(M, N)_{i+v}) \\
&= \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(K_i, \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(M_j, N_{j+i+v})) \\
&= \prod_{i\in\mathbb{Z}} \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(K_i, \operatorname{Hom}_R(M_j, N_{i+j+v})).
\end{aligned}$$

Next note that $(\rho_{KMN})_v = (\rho_{K_i M_j N_{i+j+v}})_{i\in\mathbb{Z}, j\in\mathbb{Z}}$ is an isomorphism by Lemma (1.3.3).

The following computation shows that ρ_{KMN} is a morphism, and hence an isomorphism of complexes.

$$\begin{aligned}
&\rho_{KMN}(\partial^{\operatorname{Hom}_R(K\otimes_R M,N)}(\psi))(k)(m) \\
&\qquad = \rho_{KMN}(\partial^N \psi - (-1)^{|\psi|}\psi\partial^{K\otimes_R M})(k)(m) \\
&\qquad = \partial^N \psi(k\otimes m) - (-1)^{|\psi|}\psi\partial^{K\otimes_R M}(k\otimes m) \\
&\qquad = \partial^N \psi(k\otimes m) - (-1)^{|\psi|}\psi(\partial^K(k)\otimes m + (-1)^{|k|} k\otimes \partial^M(m))
\end{aligned}$$

$$\begin{aligned}
&\partial^{\operatorname{Hom}_R(K,\operatorname{Hom}_R(M,N))}(\rho_{KMN}(\psi))(k)(m) \\
&\quad = (\partial^{\operatorname{Hom}_R(M,N)}\rho_{KMN}(\psi) - (-1)^{|\rho_{KMN}(\psi)|}\rho_{KMN}(\psi)\partial^K)(k)(m) \\
&\quad = \left(\partial^N \rho_{KMN}(\psi)(k) - (-1)^{|\rho_{KMN}(\psi)(k)|}\rho_{KMN}(\psi)(k)\partial^M\right)(m) \\
&\qquad\quad - (-1)^{|\psi|}\psi(\partial^K(k))(m) \\
&\quad = \partial^N\psi(k\otimes m) - (-1)^{|\psi|+|k|}\psi(k\otimes\partial^M(m)) - (-1)^{|\psi|}\psi(\partial^K(k)\otimes m) \\
&\quad = \partial^N \psi(k\otimes m) - (-1)^{|\psi|}\psi(\partial^K(k)\otimes m + (-1)^{|k|} k\otimes \partial^M(m))
\end{aligned}$$

It is straightforward to verify that ρ_{KMN} is natural. □

The next map is the (Hom) *swap* isomorphism.

(2.5.4) **Theorem.** *Let K, M, and N be R-complexes. The assignment*

$$\psi \longmapsto [m \mapsto [k \mapsto (-1)^{|k||m|}\psi(k)(m)]],$$

for $k \in K$, $m \in M$, and $\psi \in \operatorname{Hom}_R(K, \operatorname{Hom}_R(M, N))$, defines an isomorphism in $\mathsf{C}(R)$

$$\varsigma_{KMN}\colon \operatorname{Hom}_R(K, \operatorname{Hom}_R(M, N)) \xrightarrow{\cong} \operatorname{Hom}_R(M, \operatorname{Hom}_R(K, N)),$$

which is natural in K, M, and N.

Proof. Straightforward verification similar to the proof of Theorem (2.5.3). □

Evaluation morphisms

(2.5.5) **Theorem.** *Let K, M, and N be R-complexes. The assignment*

$$\psi \otimes n \longmapsto [k \mapsto (-1)^{|k||n|}\psi(k) \otimes n],$$

for $k \in K$, $n \in N$, and $\psi \in \operatorname{Hom}_R(K,M)$, defines a morphism in $\mathsf{C}(R)$

$$\omega_{KMN}\colon \operatorname{Hom}_R(K,M) \otimes_R N \longrightarrow \operatorname{Hom}_R(K, M \otimes_R N),$$

which is natural in K, M, and N.

This tensor evaluation morphism *is an isomorphism under each of the following conditions*

(a) *K is bounded below and degree-wise finitely generated, M and N are bounded above, and K is a complex of projective modules or N is a complex of flat modules.*
(b) *K is bounded and degree-wise finitely generated, and K is a complex of projective modules or N is a complex of flat modules.*

Proof. For each v the module $(\operatorname{Hom}_R(K,M) \otimes_R N)_v$ is generated by symbols $\psi \otimes n$, so it suffices to define ω_{KMN} on such symbols. The assignment is clearly bilinear, so ω_{KMN} is a degree 0 homomorphism. It is straightforward to verify that ω_{KMN} is natural; it is a morphism of R-complexes as:

$$\begin{aligned}
&\partial^{\operatorname{Hom}_R(K,M\otimes_R N)}(\omega(\psi \otimes n))(k)\\
&\qquad= \partial^{M\otimes_R N}\omega(\psi \otimes n)(k) - (-1)^{|\omega(\psi\otimes n)|}\omega(\psi \otimes n)(\partial^K(k))\\
&\qquad= (-1)^{|k||n|}\partial^{M\otimes_R N}(\psi(k) \otimes n) - (-1)^{|\psi|+(|k|-1)|n|}\psi(\partial^K(k)) \otimes n\\
&\qquad= (-1)^{|k||n|}\left(\partial^M\psi(k) \otimes n + (-1)^{|\psi(k)|}\psi(k) \otimes \partial^N(n)\right)\\
&\qquad\qquad - (-1)^{|\psi|+(|k|-1)|n|}\psi(\partial^K(k)) \otimes n\\
&\qquad= (-1)^{|k||n|}\partial^M\psi(k) \otimes n + (-1)^{|k||n|+|\psi|+|k|}\psi(k) \otimes \partial^N(n)\\
&\qquad\qquad - (-1)^{|\psi|+(|k|-1)|n|}\psi(\partial^K(k)) \otimes n
\end{aligned}$$

and

$$\begin{aligned}
&\omega(\partial^{\operatorname{Hom}_R(K,M)\otimes_R N}(\psi \otimes n))(k)\\
&\qquad= \omega\left(\partial^{\operatorname{Hom}_R(K,M)}(\psi) \otimes n + (-1)^{|\psi|}\psi \otimes \partial^N(n)\right)(k)\\
&\qquad= \omega\left((\partial^M\psi - (-1)^{|\psi|}\psi\partial^K) \otimes n + (-1)^{|\psi|}\psi \otimes \partial^N(n)\right)(k)\\
&\qquad= (-1)^{|k||n|}\partial^M\psi(k) \otimes n - (-1)^{|\psi|+(|k|-1)|n|}\psi(\partial^K(k)) \otimes n\\
&\qquad\qquad + (-1)^{|\psi|+|k|(|n|-1)}\psi(k) \otimes \partial^N(n).
\end{aligned}$$

Note that

$$\begin{aligned}(\mathrm{Hom}_R(K,M)\otimes_R N)_v &= \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_R(K,M)_i \otimes_R N_{v-i}\\ &= \coprod_{i\in\mathbb{Z}} (\prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(K_j, M_{j+i}))\otimes_R N_{v-i}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_R(K, M\otimes_R N)_v &= \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(K_j, (M\otimes_R N)_{j+v})\\ &= \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(K_j, \coprod_{h\in\mathbb{Z}} M_h\otimes_R N_{j+v-h})\\ &= \prod_{j\in\mathbb{Z}} (\coprod_{i\in\mathbb{Z}} \mathrm{Hom}_R(K_j, M_{j+i}\otimes_R N_{v-i})).\end{aligned}$$

(a): Under the assumptions on K, M, and N there are integers t, u and w such that

$$K_j = 0 \text{ for } j < w, \quad M_{j+i} = 0 \text{ for } j+i > t, \quad \text{and} \quad N_{v-i} = 0 \text{ for } v-i > u.$$

Therefore,

$$\begin{aligned}(\mathrm{Hom}_R(K,M)\otimes_R N)_v &= \coprod_{i=v-u}^{t-w} (\prod_{j=w}^{t-i} \mathrm{Hom}_R(K_j, M_{j+i}))\otimes_R N_{v-i}\\ &= \bigoplus_{i=v-u}^{t-w} \bigoplus_{j=w}^{t-i} \mathrm{Hom}_R(K_j, M_{j+i})\otimes_R N_{v-i}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_R(K, M\otimes_R N)_v &= \prod_{j=w}^{t+u-v} (\coprod_{i=v-u}^{t-j} \mathrm{Hom}_R(K_j, M_{j+i}\otimes_R N_{v-i}))\\ &= \bigoplus_{i=v-u}^{t-w} \bigoplus_{j=w}^{t-i} \mathrm{Hom}_R(K_j, M_{j+i}\otimes_R N_{v-i}).\end{aligned}$$

Next note that

$$(\omega_{KMN})_v \;=\; (\omega_{K_j M_{j+i} N_{v-i}})_{v-u\leqslant i\leqslant t-w,\, w\leqslant j\leqslant t-i}$$

is an isomorphism by Lemma (1.3.5).

(b): Similar to (a), only easier. $\square$

(2.5.6) **Theorem.** *Let K, M, and N be R-complexes. The assignment*

$$k\otimes\psi \longmapsto [\vartheta\mapsto (-1)^{|k|(|\psi|+|\vartheta|)}\psi\vartheta(k)],$$

for $k\in K$, $\psi\in\mathrm{Hom}_R(M,N)$, and $\vartheta\in\mathrm{Hom}_R(K,M)$ defines a morphism in $\mathsf{C}(R)$

$$\theta_{KMN}\colon K\otimes_R \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K,M),N),$$

which is natural in K, M, and N.

This homomorphism evaluation morphism is an isomorphism under each of the following conditions

(a) *K is bounded below and degree-wise finitely generated, M is bounded above and N is bounded below, and K is a complex of projective modules or N is a complex of injective modules.*

(b) *K is bounded and degree-wise finitely generated, and K is a complex of projective modules or N is a complex of injective modules.*

Proof. Similar to the proof of Theorem (2.5.5). □

Exercises

(E 2.5.1) For R-complexes M and N consider the three degree -1 homomorphisms $\partial^{M\otimes_R N}$, $\partial^M \otimes_R N$, and $M \otimes_R \partial^N$ from the complex $M \otimes_R N$ to itself. Verify the identity

$$\partial^{M\otimes_R N} = \partial^M \otimes_R N + M \otimes_R \partial^N.$$

CHAPTER 3

Resolutions

3.1. Semifreeness

(3.1.1) **Definition.** An R-complex L is *semifree* if the graded R-module $L^\natural$ has a graded basis $E = \bigsqcup_{n\geqslant 0} E^n$ such that $\partial^L(E^n) \subseteq R(E^{n-1})$. Such a basis is called a *semibasis*.

(3.1.2) **Remark.** If L is semifree, then L_v is free for all v. If $E = \bigsqcup_{n\geqslant 0} E^n$ is a semibasis for L, then $R(E^0) \subseteq \mathrm{Z}(L)$.

(3.1.3) **Example.** An R-module L is semifree if and only if it is free.
If each module L_v is free and $\partial^L = 0$ then L is semifree.
If each module L_v is free and L is bounded below, then L is semifree.

(3.1.4) **Example.** Over $R = \mathbb{Z}/(4)$ consider the Dold complex

$$L \;=\; \cdots \longrightarrow \mathbb{Z}/(4) \xrightarrow{\;2\;} \mathbb{Z}/(4) \xrightarrow{\;2\;} \mathbb{Z}/(4) \longrightarrow \cdots$$

of free R-modules. It has no semibasis, as no basis for $L^\natural$ contains a cycle.

(3.1.5) **Definition.** A *semifree resolution* of an R-complex M is a semifree complex L and a quasiisomorphism $L \xrightarrow{\;\simeq\;} M$.

(3.1.6) **Theorem.** *Every R-complex has a semifree resolution $\lambda\colon L \xrightarrow{\;\simeq\;} M$, and λ can be chosen surjective.*

The proof relies on the construction described below.

(3.1.7) **Construction.** Given an R-complex M we construct by induction on $n \geqslant 0$ a sequence of inclusions of R-complexes

$$\cdots \hookrightarrow L^n \hookrightarrow L^{n+1} \hookrightarrow \cdots$$

and compatible morphisms $\lambda^n\colon L^n \to M$.

For $n = 0$ choose a set $Z^0 \subseteq \mathrm{Z}(M)$ whose classes generate $\mathrm{H}(M)$. Let $E^0 = \{e_z : |e_z| = |z|\}_{z\in Z^0}$ be a linearly independent set over R. Set

$$(L^0)^\natural = R(E^0) \qquad \text{and} \qquad \partial^{L^0} = 0;$$

this defines an R-complex L^0. The map $\lambda^0\colon L^0 \to M$ defined by

$$\lambda^0(e_z) = z$$

and extended by linearity, is a morphism of complexes.

Let $n > 0$ and assume L^{n-1} and $\lambda^{n-1}\colon L^{n-1} \to M$ have been constructed. Choose a set $Z^n \subseteq \mathrm{Z}(L^{n-1})$ whose classes generate $\operatorname{Ker} \mathrm{H}(\lambda^{n-1})$. Let $E^n = \{e_z :$

$|e_z| = |z| + 1\}_{z \in Z^n}$ be a linearly independent set over R. An R-complex L^n is defined by

$$(L^n)^\natural = (L^{n-1})^\natural \oplus R(E^n) \quad \text{and} \quad \partial^{L^n}(x + \sum_{z \in Z^n} r_z e_z) = \partial^{L^{n-1}}(x) + \sum_{z \in Z^n} r_z z.$$

For each $z \in Z^n$ there is an $m_z \in M$ such that $\lambda^{n-1}(z) = \partial^M(m_z)$. The map $\lambda^n \colon L^n \to M$ defined by

$$\lambda^n(x + \sum_{z \in Z^n} r_z e_z) = \lambda^{n-1}(x) + \sum_{z \in Z^n} r_z m_z$$

is a morphism of complexes that agrees with λ^{n-1} on the subcomplex L^{n-1}.

Proof of Theorem (3.1.6). Let $(L^n)_{n \in \mathbb{Z}}$ and $(\lambda^n \colon L^n \to M)_{n \in \mathbb{Z}}$ be the R-complexes and morphisms constructed in (3.1.7). Note that each L^n is semifree with semibasis $\bigsqcup_{i=0}^{i=n} E^i$. Set $L = \operatorname{colim}_n L^n$ and $\lambda = \operatorname{colim}_n \lambda^n \colon L \to M$. The complex L is semifree with semibasis $E = \bigsqcup_{n \geqslant 0} E^n$.

For each n there is a commutative diagram

$$\begin{tikzcd}
\mathrm{H}(L^0) \arrow[r] \arrow[dr, "\mathrm{H}(\lambda^0)"'] & \mathrm{H}(L^n) \arrow[r] \arrow[d, "\mathrm{H}(\lambda^n)"] & \mathrm{H}(L) \arrow[dl, "\mathrm{H}(\lambda)"] \\
& \mathrm{H}(M). &
\end{tikzcd}$$

By construction, $\mathrm{H}(\lambda^0)$ is surjective and hence so is $\mathrm{H}(\lambda)$. To see that $\mathrm{H}(\lambda)$ is injective, let $l \in \mathrm{Z}(L)$ and assume that $\mathrm{H}(\lambda)([l]) = 0$. We can choose an integer n such that $l \in L^n$; now

$$0 = \mathrm{H}(\lambda)([l]) = \mathrm{H}(\lambda(l)) = \mathrm{H}(\lambda^n(l)) = \mathrm{H}(\lambda^n)([l]),$$

so $[l] \in \operatorname{Ker} \mathrm{H}(\lambda^n)$. By choice of Z^{n+1} there exists a $y \in L^n \subseteq L^{n+1}$ such that

$$l = \sum_{z \in Z^{n+1}} r_z z + \partial^L(y).$$

Now

$$l = \partial^L(\sum_{z \in Z^{n+1}} r_z e_z + y)$$

so $[l] = 0$ in $\mathrm{H}(L^n) \subseteq \mathrm{H}(L)$. Thus λ is a quasiisomorphism.

If Z^0 generates $\mathrm{Z}(M)$ then λ^0 and therefore λ is surjective on cycles and hence surjective by Observation (2.1.18). $\square$

(3.1.8) **Proposition.** *Let L be a semifree R-complex. For every morphism $\beta \colon L \to N$ and every surjective quasiisomorphism $\alpha \colon M \xrightarrow{\simeq} N$ there exists a morphism γ that makes the the following diagram commutative*

$$\begin{tikzcd}
& M \arrow[d, two heads, "\alpha"', "\simeq"] \\
L \arrow[r, "\beta"] \arrow[ur, dotted, "\gamma"] & N.
\end{tikzcd}$$

Proof. Choose a semibasis $E = \bigsqcup_{i\geqslant 0} E^i$ for L. Let L^n denote the semifree subcomplex of L on the semibasis $\bigsqcup_{i=0}^{i=n} E^i$. By induction on n we construct morphisms $\gamma^n\colon L^n \to M$ compatible with the inclusions

$$0 \hookrightarrow L^0 \hookrightarrow \cdots \hookrightarrow L^{n-1} \hookrightarrow L^n \hookrightarrow \cdots .$$

Assume by induction that γ^{n-1} has been constructed. For each element e of E^n, there exists an $m_e \in M$ such that $\alpha(m_e) = \beta(e)$. First compare $\gamma^{n-1}\partial^L(e)$ with $\partial^M(m_e)$:

$$\begin{aligned}\alpha(\gamma^{n-1}\partial^L(e) - \partial^M(m_e)) &= \beta\partial^L(e) - \partial^N\alpha(m_e)\\ &= \partial^N(\beta(e) - \alpha(m_e))\\ &= 0.\end{aligned}$$

This means that $\gamma^{n-1}\partial^L(e) - \partial^M(m_e)$ is in $\operatorname{Ker}\alpha$; it is easy to see that it is also a cycle of M, and since α is a quasiisomorphism it must even be a boundary. Choose $m'_e \in M$ such that $\partial^M(m'_e) = \gamma^{n-1}\partial^L(e) - \partial^M(m_e)$. Note that $\partial^N\alpha(m'_e) = \alpha\partial^M(m'_e) = 0$ so $\alpha(m'_e) \in \mathrm{Z}(N)$ and there exists a $m''_e \in \mathrm{Z}(M)$ such that $\alpha(m''_e) = \alpha(m'_e)$. For an element $l = x + \sum_{e\in E^n} r_e e$ in L^n define

$$\gamma^n(l) = \gamma^{n-1}(x) + \sum_{e\in E^n} r_e(m_e + m'_e - m''_e).$$

By construction,

$$\alpha\gamma^n(l) = \alpha\gamma^{n-1}(x) + \sum_{e\in E^n} r_e\alpha(m_e + m'_e - m''_e) = \beta(x) + \sum_{e\in E^n} r_e\beta(e) = \beta(l),$$

and

$$\begin{aligned}\gamma^n\partial^L(l) - \partial^M\gamma^n(l) &= \sum_{e\in E^n} r_e(\gamma^{n-1}\partial^L(e) - \partial^M\gamma^n(e))\\ &= \sum_{e\in E^n} r_e(\gamma^{n-1}\partial^L(e) - \partial^M(m_e + m'_e - m''_e))\\ &= 0.\end{aligned}$$

Now $\gamma = \operatorname{colim}_n \gamma^n$ is the desired morphism. □

Boundedness and finiteness

(3.1.9) **Theorem.** *Let M be an R-complex. There exists a semifree resolution L of M with $L_v = 0$ for $v < \inf M$.*

Proof. If $\inf M = \infty$, then $M \simeq 0$ and the 0-complex has the desired properties. If $\inf M = -\infty$, then any semifree resolution will do. Suppose $\infty > \inf M > -\infty$ and set $w = \inf M$. By Remark (2.2.7) there is a quasiisomorphism $M_{\supset w} \xrightarrow{\simeq} M$. By Theorem (3.1.6) the complex $M_{\supset w}$ has a semifree resolution $L \xrightarrow{\simeq} M_{\supset w}$, and it follows from Construction (3.1.7) that $L_v = 0$ for $v < w$. The composite of the two quasiisomorphisms is the desired resolution $L \xrightarrow{\simeq} M$. □

(3.1.10) **Theorem.** *Every complex M with $\mathrm{H}(M)$ bounded below and degreewise finitely generated has a semifree resolution L with L_v finitely generated for all v and $L_v = 0$ for $v < \inf M$.*

Proof. Set $w = \inf M$ and apply Construction (3.1.7) to $M_{\supset w}$. The set

$$E^0 = \{e_z : |e_z| = |z|\}_{z \in Z^0}$$

contains only finitely many elements of any given degree v, and no elements of degree less than w. For $n > 1$ the set

$$E^n = \{e_z : |e_z| = |z| + 1\}_{z \in Z^n}$$

contains only finitely many elements of any given degree v and no elements of degree less than $w + n$. Thus, for any v the set of basis elements in degree v

$$E_v = \Big(\bigsqcup_{n \geqslant 0} E^n\Big)_v = \bigsqcup_{n=0}^{v} E^n_v$$

is finite. As in the proof of Theorem (3.1.9) the desired resolution is the composite $L \xrightarrow{\simeq} M_{\supset w} \xrightarrow{\simeq} M$. □

The requirement in the theorem that $\mathrm{H}(M)$ be bounded below cannot be relaxed.

(3.1.11) **Example.** Let $(R, \mathfrak{m}, k)$ be a singular local ring. Set $K = \coprod_{v \geqslant 0} \Sigma^{-v} k$. If K had a semifree resolution L with L_v finitely generated for all v, then

$$\mathrm{H}_0(k \otimes^{\mathbf{L}}_R K) \cong \mathrm{H}_0\Big(\coprod_{v \geqslant 0} \Sigma^{-v}\big(k \otimes^{\mathbf{L}}_R k\big)\Big) \cong \coprod_{v \geqslant 0} \mathrm{Tor}^R_v(k, k)$$

were finitely generated, and that is absurd.

Exercises

(E 3.1.1) Let L be a complex of free R-modules. Show that L is semifree if $\partial^L_v = 0$ for $v \ll 0$.

3.2. Semiprojectivity

(3.2.1) **Definition.** An R-complex P is *semiprojective* if the functor $\mathrm{Hom}_R(P, -)$ preserves surjective quasiisomorphisms.

A *semiprojective resolution* of an R-complex M is a semiprojective complex P and a quasiisomorphism $P \xrightarrow{\simeq} M$.

(3.2.2) **Theorem.** *If L is semifree, then $\mathrm{Hom}_R(L, -)$ preserves exact sequences and quasiisomorphisms.*

In particular, every semifree R-complex is semiprojective.

Proof. Let L be a semifree R-complex and $\alpha \colon M \to N$ a surjective morphism. The induced morphism $\mathrm{Hom}_R(L, \alpha,)$ is surjective because $L^\natural$ is free.

Let $\beta \colon M \to N$ be a quasiisomorphism; set $C = \mathrm{Cone}\,\beta$ and consider the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(L, N) \longrightarrow \mathrm{Hom}_R(L, C) \longrightarrow \mathrm{Hom}_R(L, \Sigma M) \longrightarrow 0.$$

To see that $\mathrm{Hom}_R(L, \beta)$ is a quasiisomorphism, it suffices to show that $\mathrm{H}(\mathrm{Hom}_R(L, C)) = 0$; see Lemma (2.2.10) and (2.3.11). Choose a semibasis

$E = \bigsqcup_{i \geqslant 0} E^i$ for L. Let L^n denote the semifree subcomplex of L on the semibasis $\bigsqcup_{i=0}^{i=n} E^i$. For each $n \geqslant 0$ there is an exact sequence

$$0 \longrightarrow L^{n-1} \longrightarrow L^n \longrightarrow R(E^n) \longrightarrow 0.$$

The complex $R(E^n)$ is a sum of shifts of R, so $\mathrm{H}(\mathrm{Hom}_R(R(E^n), C)) = 0$. Now it follows by induction that $\mathrm{H}(\mathrm{Hom}_R(L^n, C)) = 0$ for all $n \geqslant 0$. The maps in the inverse system $(\mathrm{Hom}_R(L^n, C) \to \mathrm{Hom}_R(L^{n-1}, C))_{n \geqslant 0}$ are surjective and, therefore,

$$\mathrm{H}(\mathrm{Hom}_R(L, C)) = \mathrm{H}(\mathrm{Hom}_R(\operatorname*{colim}_n L^n, C)) = \mathrm{H}(\lim_n \mathrm{Hom}_R(L^n, C)) = 0$$

e.g. by [7, thm. 3.5.8]. $\square$

The next corollary is now immediate in view of Theorem (3.1.6).

(3.2.3) **Corollary.** *Every R-complex M has a semiprojective resolution $\pi\colon P \to M$, and π can be chosen surjective.* $\square$

The next theorem gives useful characterizations of semiprojective complexes.

(3.2.4) **Theorem.** *The following are equivalent for an R-complex P.*

(*i*) *P is semiprojective.*

(*ii*) *$\mathrm{Hom}_R(P, -)$ preserves exact sequences and quasiisomorphisms.*

(*iii*) *Given a chain map $\alpha\colon P \to N$ and a surjective quasiisomorphism $\beta\colon M \to N$ there exists a chain map $\gamma\colon P \to M$ such that $\alpha = \beta\gamma$.*

(*iv*) *Every exact sequence $0 \to M' \to M \xrightarrow{\beta} P \to 0$ with $\mathrm{H}(M') = 0$ splits.*

(*v*) *P is a direct summand of some semifree R-complex L.*

(*vi*) *P is a complex of projective R-modules and $\mathrm{Hom}_R(P, -)$ preserves quasiisomorphisms.*

Proof. $(i) \Longrightarrow (iii)$: The induced morphism $\mathrm{Hom}_R(P, \beta)$ is a surjective quasiisomorphism. In particular, it is surjective on cycles, see Observation (2.1.18), so there exists a $\gamma \in \mathrm{Z}(\mathrm{Hom}_R(P, M))$ such that $\alpha = \mathrm{Hom}_R(P, \beta)(\gamma) = \beta\gamma$.

$(iii) \Longrightarrow (iv)$: By Lemma (2.1.13) the surjective morphism β is a quasiisomorphism, so there exists a morphism $\gamma\colon P \to M$ such that $\lambda\gamma = 1^P$.

$(iv) \Longrightarrow (v)$: By Theorem (3.1.6) there exists a semifree complex L and a surjective quasiisomorphism $\lambda\colon L \to P$.

$(v) \Longrightarrow (ii)$: Immediate by Theorem (3.2.2).

$(ii) \Longrightarrow (i)$: Clear.

$(vi) \Longrightarrow (ii)$: Use the lifting property of projective modules.

Finally, it is clear that (ii) and (v) imply (vi). $\square$

Up to homotopy, chain maps from semiprojective complexes factor through quasiisomorphisms.

(3.2.5) **Proposition.** *Let P be a semiprojective R-complex, $\alpha\colon P \to N$ a chain map, and $\beta\colon M \to N$ a quasiisomorphism. There exists a chain map $\gamma\colon P \to M$ such that $\alpha \sim \beta\gamma$ and $\gamma \sim \gamma'$ for any other chain map γ' with $\alpha \sim \beta\gamma'$.*

Proof. The induced map $\mathrm{Hom}_R(P, \beta)\colon \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ is a quasiisomorphism, so there exists a $\gamma \in \mathrm{Z}(\mathrm{Hom}_R(P, M))$ such that

$$[\alpha] = \mathrm{H}(\mathrm{Hom}_R(P, \beta))[\gamma] = [\beta\gamma],$$

that is, $\alpha - \beta\gamma \in \mathrm{B}(\mathrm{Hom}_R(P,N))$. For any other γ' with $\alpha \sim \beta\gamma'$, i.e. $[\alpha] = \mathrm{H}(\mathrm{Hom}_R(P,\beta))[\gamma']$, it follows that $[\gamma - \gamma'] = 0$ because $\mathrm{H}(\mathrm{Hom}_R(P,\beta))$ is an isomorphism. Thus, $\gamma - \gamma' \in \mathrm{B}(\mathrm{Hom}_R(P,M))$. □

(3.2.6) **Proposition.** *Let $\beta\colon M \to P$ be a morphism of R-complexes.*

(a) *If P is semiprojective and β is a quasiisomorphism, then there exists a quasiisomorphism $P \xrightarrow{\simeq} M$.*

(b) *If P and M are semiprojective, then β is a quasiisomorphism if and only if it is a homotopy equivalence.*

Proof. (a): By Proposition (3.2.5) there is a morphism $\gamma\colon P \to M$ such that $1^P \sim \beta\gamma$. In particular, $1^{\mathrm{H}(P)} = \mathrm{H}(\beta)\,\mathrm{H}(\gamma)$ and since $\mathrm{H}(\beta)$ is an isomorphism so is $\mathrm{H}(\gamma)$.

(b): Every homotopy equivalence is a quasiisomorphism. Assume β is a quasiisomorphism. By Proposition (3.2.5) there are morphisms $\gamma\colon P \to M$ and $\beta'\colon M \to P$ such that $1^P \sim \beta\gamma$ and $1^M \sim \gamma\beta'$, so β is a homotopy equivalence. (Indeed, $1^M \sim \gamma\beta' = \gamma 1^P \beta' \sim \gamma\beta\gamma\beta' \sim \gamma\beta 1^M = \gamma\beta$.) □

Exercises

(E 3.2.1) Prove that a bounded below complex of projective modules is semiprojective.

(E 3.2.2) Prove that the tensor product of two semiprojective R-complexes is semiprojective.

(E 3.2.3) Let P be a bounded below complex of projective modules. Prove that P is contractible if and only if it is acyclic.

3.3. Semiinjectivity

(3.3.1) **Definition.** An R-complex I is *semiinjective* if the functor $\mathrm{Hom}_R(-,I)$ converts injective quasiisomorphisms into surjective quasiisomorphisms.

A *semiinjective resolution* of an R-complex M is a semiinjective complex I and a quasiisomorphism $M \xrightarrow{\simeq} I$.

(3.3.2) **Theorem.** *Every R-complex M has a semiinjective resolution $\iota\colon M \to I$, and ι can be chosen injective.*

Proof. Omitted. □

The next theorem gives useful characterizations of semiinjective complexes.

(3.3.3) **Theorem.** *The following are equivalent for an R-complex P.*

(*i*) *I is semiinjective.*

(*ii*) $\mathrm{Hom}_R(-,I)$ *preserves exact sequences and quasiisomorphisms.*

(*iii*) *Given a chain map $\alpha\colon M \to I$ and an injective quasiisomorphism $\beta\colon M \to N$ there exists a chain map $\gamma\colon N \to I$ such that $\alpha = \gamma\beta$.*

(*iv*) *Every exact sequence $0 \to I \xrightarrow{\beta} M \to M'' \to 0$ with $\mathrm{H}(M'') = 0$ splits.*

(*v*) *P is a complex of injective R-modules and $\mathrm{Hom}_R(-,I)$ preserves quasiisomorphisms.*

Proof. Essentially dual to the semiprojective case. □

Up to homotopy, chain maps to semiinjective complexes factor through quasiisomorphisms.

(3.3.4) **Proposition.** *Let I be a semiinjective R-complex, $\alpha\colon M \to I$ a chain map, and $\beta\colon M \to N$ a quasiisomorphism. There exists a chain map $\gamma\colon N \to I$ such that $\alpha \sim \gamma\beta$ and $\gamma \sim \gamma'$ for any other chain map γ' with $\alpha \sim \gamma'\beta$.*

Proof. Dual to the semiprojective case. □

(3.3.5) **Proposition.** *Let $\beta\colon I \to M$ be a morphism of R-complexes.*

(a) *If I is semiinjective and β is a quasiisomorphism, then there exists a quasiisomorphism $M \xrightarrow{\simeq} I$.*
(b) *If I and M are semiinjective, then β is a quasiisomorphism if and only if it is a homotopy equivalence.*

Proof. Dual to the semiprojective case. □

(3.3.6) **Proposition.** *If P is semiprojective and I is semiinjective, then $\operatorname{Hom}_R(P,I)$ is semiinjective.*

Proof. Let $\alpha\colon M \to N$ be an injective quasiisomorphism. By the assumptions on P and I, the induced morphism $\operatorname{Hom}_R(\alpha,I)$ is a surjective quasiisomprhism, and $\operatorname{Hom}_R(P,\operatorname{Hom}_R(\alpha,I))$ is the same. There is a commutative diagram in $\mathsf{C}(R)$

$$\begin{tikzcd}
\operatorname{Hom}_R(N,\operatorname{Hom}_R(P,I)) \arrow[r, "\operatorname{Hom}_R(\alpha,\operatorname{Hom}_R(P,I))"] \arrow[d, "\varsigma_{NPI}"', "\cong"] & \operatorname{Hom}_R(M,\operatorname{Hom}_R(P,I)) \arrow[d, "\cong"', "\varsigma_{MPI}"] \\
\operatorname{Hom}_R(P,\operatorname{Hom}_R(N,I)) \arrow[r, "\operatorname{Hom}_R(P,\operatorname{Hom}_R(\alpha,I))"] & \operatorname{Hom}_R(P,\operatorname{Hom}_R(M,I)).
\end{tikzcd}$$

It shows that $\operatorname{Hom}_R(\alpha,\operatorname{Hom}_R(P,I))$ is a surjective quasiisomorphism, whence $\operatorname{Hom}_R(P,I)$ is semiinjective. □

Boundedness

(3.3.7) **Theorem.** *Let M be an R-complex. There exists a semiinjective resolution I of M with $I_v = 0$ for $v > \sup M$.*

Proof. Omitted. □

(3.3.8) **Lemma.** *A bounded above complex of injective R-modules is semiinjective.*

Proof. Let I be a bounded above complex of injective R-modules. It is sufficient to prove that $\operatorname{Hom}_R(-,I)$ preserves quasiisomorphisms. It straightforward to verify that $\operatorname{Hom}_R(C,I)$ is acyclic for every acyclic complex C; apply this to C, the cone of a quasiisomorphism. □

Exercises

(E 3.3.1) Let I be a bounded above complex of injective modules. Prove that I is contractible if and only if it is acyclic.

3.4. Semiflatness

(3.4.1) **Definition.** An R-complex F is *semiflat* if the functor $- \otimes_R F$ preserves injective quasiisomorphisms.

(3.4.2) **Remark.** An R-complex F is semiflat if and only if the functor $F \otimes_R -$ preserves injective quasiisomorphisms; this is immediate by commutativity of tensor products (2.5.1).

(3.4.3) **Theorem.** *Let E be a faithfully injective R-module. An R-complex F is semiflat if and only if $\operatorname{Hom}_R(F,E)$ is semiinjective.*

Proof. Let $\alpha\colon M \to N$ be an injective quasiisomorphism. There is a commutative diagram in $\mathsf{C}(R)$

$$\begin{tikzcd}
\operatorname{Hom}_R(N \otimes_R F, E) \arrow[r, "\operatorname{Hom}_R(\alpha \otimes_R F, E)"] \arrow[d, "\rho_{NFE}"', "\cong"] & \operatorname{Hom}_R(M \otimes_R F, E) \arrow[d, "\cong"', "\rho_{MFE}"] \\
\operatorname{Hom}_R(N, \operatorname{Hom}_R(F,E)) \arrow[r, "\operatorname{Hom}_R(\alpha, \operatorname{Hom}_R(F,E))"] & \operatorname{Hom}_R(M, \operatorname{Hom}_R(F,E)).
\end{tikzcd}$$

If F is semiflat, then $\alpha \otimes_R F$ is an injective quasiisomorphism and $\operatorname{Hom}_R(\alpha \otimes_R F, E)$ is then a surjective quasiisomorphism, as E is semiinjective. By commutativity of the diagram, $\operatorname{Hom}_R(\alpha, \operatorname{Hom}_R(F,E))$ is now a surjective quasiisomorphism, whence $\operatorname{Hom}_R(F,E)$ is semiinjective.

If $\operatorname{Hom}_R(F,E)$ is semiinjective, then $\operatorname{Hom}_R(\alpha, \operatorname{Hom}_R(F,E))$ is a surjective quasiisomorphism, and by commutativity of the diagram, $\operatorname{Hom}_R(\alpha \otimes_R F, E)$ is the same. By faithful injectivity of E this implies that $\alpha \otimes_R F$ is an injective quasiisomorphism. $\square$

The next corollary is immediate in view of Proposition (3.3.6).

(3.4.4) **Corollary.** *Every semiprojective complex is semiflat.* $\square$

The next theorem gives useful characterizations of semiflat complexes.

(3.4.5) **Theorem.** *The following are equivalent for an R-complex F.*

(*i*) *F is semiflat.*

(*ii*) *$- \otimes_R F$ preserves exact sequences and quasiisomorphisms.*

(*iii*) *F is a complex of flat R-modules and $- \otimes_R F$ preserves quasiisomorphisms.*

Proof. Immediate from Theorem (3.4.3) and adjointness (2.5.3). $\square$

A quasiisomorphism of semiflat R-complexes need not be a homotopy equivalence.

(3.4.6) **Example.** The $\mathbb{Z}$-module $\mathbb{Q}$ has a semifree resolution $\lambda\colon L \xrightarrow{\simeq} \mathbb{Q}$ with $L_v = 0$ for $v \neq 0,1$. Both $\mathbb{Z}$-complexes $\mathbb{Q}$ and L are semiflat. Suppose $\beta\colon \mathbb{Q} \to L$ were a homotopy inverse, then $\lambda\beta \sim 1^{\mathbb{Q}}$ and hence $\lambda\beta = 1^{\mathbb{Q}}$ as $\partial^{\mathbb{Q}} = 0$. This would make $\mathbb{Q}$ a direct summand of L_0 and hence a free $\mathbb{Z}$-module. Contradiction!

(3.4.7) **Proposition.** *Let $\alpha\colon F \to F'$ be a quasiisomorphism of semiflat R-complexes.*

(a) *For every R-complex M, the induced morphism $M \otimes_R \alpha$ is a quasiisomorphism.*

(b) *For every R-complex N, the induced morphism $\alpha \otimes_R N$ is a quasiisomorphism.*

Proof. (a): Let E be a faithfully injective R-module. By Theorem (3.4.3) the induced morphism $\operatorname{Hom}_R(\alpha, E)$ is a quasiisomorphism of semiinjective modules and hence a homotopy equivalence by Proposition (3.3.5). By Proposition (2.3.9) the top horizontal map in the diagram below is also a homotopy equivalence.

$$\begin{tikzcd}
\operatorname{Hom}_R(M, \operatorname{Hom}_R(F', E)) \arrow[r, "\operatorname{Hom}_R(M,\operatorname{Hom}_R(\alpha,E))", "\sim"'] \arrow[d, "\rho_{MF'E}"', "\cong"] & \operatorname{Hom}_R(M, \operatorname{Hom}_R(F, E)) \arrow[d, "\cong"', "\rho_{MFE}"] \\
\operatorname{Hom}_R(M \otimes_R F', E) \arrow[r, "\operatorname{Hom}_R(M\otimes_R\alpha,E)"] & \operatorname{Hom}_R(M \otimes_R F, E)
\end{tikzcd}$$

The diagram is commutative and shows that $\operatorname{Hom}_R(M \otimes_R \alpha, E)$ is a quasiisomorphism, and by faithful injectivity of E it follows that $M \otimes_R \alpha$ is a quasiisomorphism.

(b): Follows from (a) by commutativity (2.5.1). □

(3.4.8) **Lemma.** *Let F'' be a complex of flat R-modules and $0 \to F' \to F \to F'' \to 0$ a short exact sequence of R-complexes. If two of the complexes are semiflat, then so is the third.*

Proof. The class of flat R-modules is projectively resolving, so F' is a complex of flat modules if an only if F is so. Let $\alpha\colon M \to N$ be a quasiisomorphism. The conclusion follows by application of the Five Lemma and Lemma (2.1.13) to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F' \otimes_R M \arrow[r] \arrow[d, "F'\otimes_R\alpha"] & F \otimes_R M \arrow[r] \arrow[d, "F\otimes_R\alpha"] & F'' \otimes_R M \arrow[r] \arrow[d, "F''\otimes_R\alpha"] & 0 \\
0 \arrow[r] & F' \otimes_R N \arrow[r] & F \otimes_R N \arrow[r] & F'' \otimes_R N \arrow[r] & 0.
\end{tikzcd}$$ □

Boundedness

(3.4.9) **Lemma.** *A bounded below complex of flat R-modules is semiflat.*

Proof. Let E be a faithfully injective R-module. If F is a bounded below complex of flat R-modules, then $\operatorname{Hom}_R(F, E)$ is a bounded above complex of injective R-modules and hence semiinjective by Lemma (3.3.8). The claim now follow by Theorem (3.4.3). □

Exercises

(E 3.4.1) Prove that the tensor product of two semiflat R-complexes is semiflat.

CHAPTER 4

The Derived Category

4.1. Construction and properties

The construction of the derived category $\mathsf{D}(R)$ was one topic covered in Greg's class. Here is a recap.

(4.1.1) The *homotopy category* $\mathsf{K}(R)$ has the same objects as $\mathsf{C}(R)$, and the morphisms are homotopy equivalence classes of morphisms in $\mathsf{C}(R)$. That is, $\mathsf{K}(R)(M,N) = \mathrm{H}_0(\mathrm{Hom}_R(M,N))$.

A morphism in $\mathsf{K}(R)$ is a *quasiisomorphism* if the induced morphism in homology is an isomorphism. This makes sense, as null-homotopic morphisms in $\mathsf{C}(R)$ induce the 0-morphism in homology. The quasiisomorphisms in $\mathsf{K}(R)$ are exactly the classes of quasiisomorphisms in $\mathsf{C}(R)$.

The quasiisomorphisms in $\mathsf{K}(R)$ form a multiplicative system. The derived category $\mathsf{D}(R)$ is the localization of $\mathsf{K}(R)$ with respect to this system.

(4.1.2) From the construction outlined above, it is not clear that $\mathsf{D}(R)(M,N)$ is a set for given complexes M and N. However, restricted to semiprojective complexes, localization does nothing as every quasiisomorphism is already invertible in $\mathsf{K}(R)$, cf. Proposition (3.2.6). In view of Corollary (3.2.3) it follows that $\mathsf{K}(R)|_{\mathrm{semiproj}}$ is a model for $\mathsf{D}(R)$.

Objects and morphisms

(4.1.3) The objects in $\mathsf{D}(R)$ are the same as in $\mathsf{K}(R)$ and $\mathsf{C}(R)$, i.e. all R-complexes.

(4.1.4) Given two R-complexes, the morphisms $M \to N$ in $\mathsf{D}(R)$ are equivalence classes of pairs (α, υ), where

$$M \xrightarrow{\alpha_1} N_1' \xleftarrow[\simeq]{\upsilon_1} N \quad \text{and} \quad M \xrightarrow{\alpha_2} N_2' \xleftarrow[\simeq]{\upsilon_2} N$$

are equivalent, if there exists a commutative diagram in $\mathsf{K}(R)$

$$\begin{tikzcd}
 & N_1' \arrow[d] & \\
M \arrow[ur, "\alpha_1"] \arrow[r, "\alpha_3"] \arrow[dr, "\alpha_2"'] & N_3' & N \arrow[ul, "\upsilon_1{,}\simeq"'] \arrow[l, "\upsilon_3", "\simeq"'] \arrow[dl, "\upsilon_2{,}\simeq"] \\
 & N_2'. \arrow[u] &
\end{tikzcd}$$

That is, the diagram is commutative up to homotopy in $\mathsf{C}(R)$. The equivalence class of a pair (α, υ) is called a *fraction* and written α/υ.

(4.1.5) Under localization, a morphism $\alpha\colon M \to N$ of complexes corresponds to the fraction $\alpha/1^N$.

(4.1.6) Given two pairs (α, υ) and (β, τ) there exists a morphism α' and a quasiisomorphism τ', such that the diagram

(4.1.6.1)
$$\begin{tikzcd}
& & N'' & & \\
& M' \arrow[ur, "\alpha'"] & & N' \arrow[ul, "\tau'"'] & \\
K \arrow[ur, "\beta"] & & M \arrow[ul, "\tau"'] \arrow[ur, "\alpha"] & & N \arrow[ul, "\upsilon"']
\end{tikzcd}$$

is commutative. The composite of the corresponding fractions is (well-)defined by

$$\alpha/\upsilon \circ \beta/\tau = \alpha'\beta/\tau'\upsilon.$$

Isomorphisms

(4.1.7) **Definition.** Two R-complexes are *isomorphic in* $\mathsf{D}(R)$ if there exists an invertible morphism $\alpha/\upsilon\colon M \to N$ in $\mathsf{D}(R)$. The notation $M \simeq N$ means that there exists an isomorphism $M \xrightarrow{\simeq} N$ in $\mathsf{D}(R)$.

(4.1.8) **Remark.** If $\alpha\colon M \to N$ is a quasiisomorphism, then it is straightforward to verify that

$$\begin{aligned}
&\alpha/1^N \circ 1^N/\alpha = 1^N/1^N \quad \text{and} \\
&1^N/\alpha \circ \alpha/1^N = \alpha/\alpha = 1^M/1^M.
\end{aligned}$$

Thus, $\alpha/1^N$ is an isomorphism in $\mathsf{D}(R)$.

(4.1.9) **Lemma.** *If $\alpha/\upsilon\colon M \to N$ is an isomorphism in $\mathsf{D}(R)$, then α is a quasiisomorphism in $\mathsf{C}(R)$. That is, there are diagrams*

$$M \xrightarrow{\simeq} N' \xleftarrow{\simeq} N \qquad \text{and} \qquad M \xleftarrow{\simeq} N'' \xrightarrow{\simeq} N$$

in $\mathsf{C}(R)$. In particular, $\mathrm{H}(M) \cong \mathrm{H}(N)$.

Proof. If $\alpha/\upsilon\colon M \to N$ is an isomorphism in $\mathsf{D}(R)$, then there exists $\beta/\tau\colon N \to M$ such that

$$\alpha/\upsilon \circ \beta/\tau = \alpha'\beta/\tau'\upsilon = 1^N/1^N \quad \text{and} \quad \beta/\tau \circ \alpha/\upsilon = \beta'\alpha/\upsilon'\tau = 1^M/1^M.$$

That is, there are commutative diagrams

$$\begin{tikzcd}
& N'' \arrow[d] & \\
N \arrow[ur, "\alpha'\beta"] \arrow[r] \arrow[dr, "1^N"'] & N_3' & N \arrow[ul, "{\tau'\upsilon,\simeq}"'] \arrow[l, "\simeq"] \arrow[dl, "1^N"] \\
& N \arrow[u] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& M'' \arrow[d] & \\
M \arrow[ur, "\beta'\alpha"] \arrow[r] \arrow[dr, "1^M"'] & M_3' & M. \arrow[ul, "{\upsilon'\tau,\simeq}"'] \arrow[l, "\simeq"] \arrow[dl, "1^M"] \\
& M \arrow[u] &
\end{tikzcd}$$

It follows from the first diagram by comparison to (4.1.6.1) that $\mathrm{H}(\upsilon)^{-1}\,\mathrm{H}(\alpha)\,\mathrm{H}(\tau)^{-1}\,\mathrm{H}(\beta) = 1^{\mathrm{H}(N)}$; in particular $\mathrm{H}(\alpha)$ is surjective. It follows from the second diagram that $\beta'\alpha$ is a quasiisomorphism; in particular $\mathrm{H}(\alpha)$ is injective. Thus, there is a diagram $M \xrightarrow[\simeq]{\alpha} N' \xleftarrow[\simeq]{\upsilon} N$, and because the quasiisomorphisms constitute a multiplicative system, there is also a diagram $M \xleftarrow{\simeq} N'' \xrightarrow{\simeq} N$. □

(4.1.10) **Proposition.** *If P is semiprojective and $P \simeq M$ in $\mathsf{D}(R)$, then there is a quasiisomorphism $P \xrightarrow{\simeq} M$ in $\mathsf{C}(R)$.*

Proof. By the lemma there is a diagram $P \xrightarrow{\simeq} M' \xleftarrow{\simeq} M$. Apply Proposition (3.2.5) to it. □

(4.1.11) **Proposition.** *If I is semiinjective and $M \simeq I$ in $\mathsf{D}(R)$, then there is a quasiisomorphism $M \xrightarrow{\simeq} I$ in $\mathsf{C}(R)$.*

Proof. By the lemma there is a diagram $M \xleftarrow{\simeq} I'' \xrightarrow{\simeq} I$. Apply Proposition (3.3.4) to it. □

Complexes with isomorphic homology need not be isomorphic in the derived category.

(4.1.12) **Example.** Over the ring $\mathbb{Z}/(4)$ consider the complexes

$$M \;=\; 0 \longrightarrow \mathbb{Z}/(4) \xrightarrow{2} \mathbb{Z}/(4) \longrightarrow 0 \qquad \text{and}$$

$$N \;=\; 0 \longrightarrow \mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \longrightarrow 0.$$

It is clear that $\mathrm{H}(M) \cong N$, so the two complexes have isomorphic homology. The complex M is semiprojective, so if $M \simeq N$ in $\mathsf{D}(R)$, then there would be a quasiisomorphism $M \xrightarrow{\simeq} N$ in $\mathsf{C}(R)$. It is straightforward to verify that any morphism $M \to N$ in $\mathsf{C}(R)$ induces the 0-morphism in homology.

(4.1.13) For convenience we will often write a morphism in $\mathsf{D}(R)$ as $\alpha\colon M \to N$.

Triangles

(4.1.14) The derived category $\mathsf{D}(R)$ is triangulated. The distinguished triangles are

$$M \xrightarrow{\alpha} N \longrightarrow \operatorname{Cone}\alpha \longrightarrow \Sigma M,$$

where α is a morphism of complexes.

Boundedness and finiteness

(4.1.15) **Definition.** The full subcategories $\mathsf{D}_{\sqsubset}(R)$, $\mathsf{D}_{\sqsupset}(R)$, and $\mathsf{D}_{\square}(R)$ of $\mathsf{D}(R)$ are defined by specifying their objects as follows

$$\begin{aligned} M \in \mathsf{D}_{\sqsubset}(R) &\iff \sup M < \infty, \\ M \in \mathsf{D}_{\sqsupset}(R) &\iff \inf M > -\infty, \text{ and} \\ M \in \mathsf{D}_{\square}(R) &\iff \sup M < \infty \;\wedge\; \inf M > -\infty. \end{aligned}$$

Moreover, $\mathsf{D}^{\mathrm{f}}(R)$ denotes the full subcategory of complexes with degreewise finitely generated homology. The notation $\mathsf{D}^{\mathrm{f}}_{\sqsubset}(R)$ is used for the full subcategory $\mathsf{D}^{\mathrm{f}}(R) \cap \mathsf{D}_{\sqsubset}(R)$. The symbols $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ are defined similarly.

4.2. Derived Hom functor

(4.2.1) **Observation.** Let M be an R-complex; let $\pi\colon P \to M$ and $\varpi\colon Q \to M$ be semiprojective resolutions of M. By Proposition (3.2.6) there is a homotopy equivalence $\alpha\colon Q \to P$. Let $\zeta\colon X \to Y$ be a morphism of R-complexes; by Proposition (2.3.15) the induced morphisms $\mathrm{Hom}_R(\alpha, X)$ and $\mathrm{Hom}_R(\alpha, Y)$ are also homotopy equivalences, and there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(P,X) \arrow[r, "\mathrm{Hom}_R(P,\zeta)"] \arrow[d, "\mathrm{Hom}_R(\alpha,X)"', "\sim"] & \mathrm{Hom}_R(P,Y) \arrow[d, "\sim"', "\mathrm{Hom}_R(\alpha,Y)"] \\
\mathrm{Hom}_R(Q,X) \arrow[r, "\mathrm{Hom}_R(Q,\zeta)"] & \mathrm{Hom}_R(Q,Y).
\end{tikzcd}$$

Let M' be an R-complex with semiprojective resolutions $\pi'\colon P' \to M'$ and $\varpi'\colon Q' \to M'$; let $\beta\colon M' \to M$ be a morphism. By Proposition (3.2.6) there is a homotopy equivalence $\alpha'\colon Q' \to P'$. By Proposition (3.2.5) there are morphisms $\gamma^P\colon P' \to P$ and $\gamma^Q\colon Q' \to Q$ such that $\pi\gamma^P \sim \beta\pi'$ and $\varpi\gamma^Q \sim \beta\varpi'$. The upshot is that the next diagram is commutative up to homotopy.

$$\begin{tikzcd}
\mathrm{Hom}_R(P,X) \arrow[r, "\mathrm{Hom}_R(\gamma^P,X)"] \arrow[d, "\mathrm{Hom}_R(\alpha,X)"'] & \mathrm{Hom}_R(P',X) \arrow[d, "\mathrm{Hom}_R(\alpha',X)"] \\
\mathrm{Hom}_R(Q,X) \arrow[r, "\mathrm{Hom}_R(\gamma^Q,X)"] & \mathrm{Hom}_R(Q',X)
\end{tikzcd}$$

(4.2.2) **Definition.** For R-complexes M and N the *right derived homomorphism complex*, $\mathbf{R}\mathrm{Hom}_R(M,N)$, is $\mathrm{Hom}_R(P,N)$, where P is a semiprojective resolution of M. By Observation (4.2.1) this determines $\mathbf{R}\mathrm{Hom}_R(M,N)$ uniquely up to quasi-isomorphism.

(4.2.3) **Theorem.** *For every R-complex M, the right derived Hom functor $\mathbf{R}\mathrm{Hom}_R(M,-)$ is an exact covariant functor (defined up to isomorphism) on $\mathsf{D}(R)$. That is,*

(a) *to every $X \in \mathsf{D}(R)$ it associates an R-complex $\mathbf{R}\mathrm{Hom}_R(M,X)$ that is unique up to isomorphism in $\mathsf{D}(R)$;*

(b) *to every morphism $\alpha\colon X \to Y$ in $\mathsf{D}(R)$ it associates a morphism $\mathbf{R}\mathrm{Hom}_R(M,\alpha)\colon \mathbf{R}\mathrm{Hom}_R(M,X) \to \mathbf{R}\mathrm{Hom}_R(M,Y)$;*

(c) *the equality $\mathbf{R}\mathrm{Hom}_R(M,\beta\alpha) = \mathbf{R}\mathrm{Hom}_R(M,\beta)\,\mathbf{R}\mathrm{Hom}_R(M,\alpha)$ holds for every pair of morphisms $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ in $\mathsf{D}(R)$;*

(d) *the equality $\mathbf{R}\mathrm{Hom}_R(M,1^X) = 1^{\mathbf{R}\mathrm{Hom}_R(M,X)}$ holds for every $X \in \mathsf{D}(R)$;*

(e) *to every exact triangle $X \to Y \to Z \to \Sigma X$ it associates an exact triangle $\mathbf{R}\mathrm{Hom}_R(M,X) \to \mathbf{R}\mathrm{Hom}_R(M,Y) \to \mathbf{R}\mathrm{Hom}_R(M,Z) \to \Sigma\mathbf{R}\mathrm{Hom}_R(M,X)$.*

Proof. Properties (a)-(d) follow from Theorem (2.3.8) in view of Observation (4.2.1). It suffices to prove part (e) for distinguished triangles, so the claim follows from Lemma (2.3.11). $\square$

(4.2.4) **Theorem.** *For every R-complex N, the right derived Hom functor $\mathbf{R}\mathrm{Hom}_R(-,N)$ is an exact contravariant functor (defined up to isomorphism) on $\mathsf{D}(R)$. That is,*

(a) *to every $X \in \mathsf{D}(R)$ it associates an R-complex $\mathbf{R}\mathrm{Hom}_R(X,N)$ that is unique up to isomorphism in $\mathsf{D}(R)$;*
(b) *to every morphism $\alpha\colon X \to Y$ in $\mathsf{D}(R)$ it associates a morphism $\mathbf{R}\mathrm{Hom}_R(\alpha,N)\colon \mathbf{R}\mathrm{Hom}_R(Y,N) \to \mathbf{R}\mathrm{Hom}_R(X,N)$;*
(c) *the equality $\mathbf{R}\mathrm{Hom}_R(\beta\alpha,N) = \mathbf{R}\mathrm{Hom}_R(\alpha,N)\,\mathbf{R}\mathrm{Hom}_R(\beta,N)$ holds for every pair of morphisms $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ in $\mathsf{D}(R)$;*
(d) *the equality $\mathbf{R}\mathrm{Hom}_R(1^X,N) = 1^{\mathbf{R}\mathrm{Hom}_R(X,N)}$ holds for every $X \in \mathsf{D}(R)$;*
(e) *to every exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathsf{D}(R)$ it associates an exact triangle $\mathbf{R}\mathrm{Hom}_R(Z,N) \to \mathbf{R}\mathrm{Hom}_R(Y,N) \to \mathbf{R}\mathrm{Hom}_R(X,N) \to \Sigma\mathbf{R}\mathrm{Hom}_R(Z,N)$.*

Proof. Properties (a)-(d) follow from Theorem (2.3.14) in view of Observation (4.2.1). It suffices to prove part (e) for distinguished triangles, so the claim follows from Lemma (2.3.17). $\square$

(4.2.5) **Observation.** The arguments in Observation (4.2.1) dualize to show that a right derived Hom functor can be well-defined up to isomorphism in $\mathsf{D}(R)$ by $\mathbf{R}\mathrm{Hom}_R(M,N) = \mathrm{Hom}_R(M,I)$, where I is a semiinjective resolution of N.

To see that this definition agrees with the one given in Definition (4.2.2), let M and N be R-complexes, let $\pi\colon P \xrightarrow{\simeq} M$ be a semiprojective resolution of M and $\iota\colon N \xrightarrow{\simeq} I$ a semiinjective resolution of N.

Further, let $\beta\colon M' \to M$ be a morphism and $\pi'\colon P' \to M'$ be a semiprojective resolution. By Proposition (3.2.5) there is a morphism $\gamma\colon P' \xrightarrow{\simeq} P$ that lifts $\beta\pi'$ along π up to homotopy. The next diagram is commutative up to homotopy.

$$\begin{tikzcd}
\mathrm{Hom}_R(P,N) \arrow[r, "\mathrm{Hom}_R(\gamma,N)"] \arrow[d, "\mathrm{Hom}_R(P,\iota)"', "\simeq"] & \mathrm{Hom}_R(P',N) \arrow[d, "\simeq"', "\mathrm{Hom}_R(P',\iota)"] \\
\mathrm{Hom}_R(P,I) \arrow[r, "\mathrm{Hom}_R(\gamma,I)"] & \mathrm{Hom}_R(P',I) \\
\mathrm{Hom}_R(M,I) \arrow[r, "\mathrm{Hom}_R(\beta,I)"] \arrow[u, "\mathrm{Hom}_R(\pi,I)", "\simeq"'] & \mathrm{Hom}_R(M',I) \arrow[u, "\simeq", "\mathrm{Hom}_R(\pi',I)"']
\end{tikzcd}$$

It shows that the two definitions yield the same contravariant functor on $\mathsf{D}(R)$. A similar diagram takes care of the covariant functor.

(4.2.6) **Lemma.** *Let P be an R-module. The following are equivalent.*

(i) *P is projective.*
(ii) *$-\inf \mathbf{R}\mathrm{Hom}_R(P,T) \leqslant 0$ for every R-module T.*
(iii) *$\mathrm{H}_{-1}(\mathbf{R}\mathrm{Hom}_R(P,T)) = 0$ for every R-module T.*

Proof. Well-known as $\mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(P,T)) = \mathrm{Ext}_R^m(P,T)$ for $m \geqslant 0$. $\square$

(4.2.7) **Lemma.** *Let I be an R-module. The following are equivalent.*

(i) *I is injective.*

(ii) $-\inf \mathbf{R}\mathrm{Hom}_R(T,I) \leqslant 0$ *for every R-module T.*

(iii) $\mathrm{H}_{-1}(\mathbf{R}\mathrm{Hom}_R(T,I)) = 0$ *for every cyclic R-module T.*

Proof. Well-known as $\mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(T,I)) = \mathrm{Ext}_R^m(T,I)$ for $m \geqslant 0$. $\square$

Boundedness and finiteness

(4.2.8) **Lemma.** *Let M and N be R-complexes. If $M \in \mathsf{D}_{\sqsupset}(R)$ and $N \in \mathsf{D}_{\sqsubset}(R)$, then $\mathbf{R}\mathrm{Hom}_R(M,N) \in \mathsf{D}_{\sqsubset}(R)$. More precisely, if $\inf M = w$ and $\sup N = u$, then*

(a) $\sup \mathbf{R}\mathrm{Hom}_R(M,N) \leqslant u - w$ *and*

(b) $\mathrm{H}_{u-w}(\mathbf{R}\mathrm{Hom}_R(M,N)) \cong \mathrm{Hom}_R(\mathrm{H}_w(M), \mathrm{H}_u(N))$.

Proof. By Theorem (3.1.9) the complex M has a semifree resolution $L \xrightarrow{\simeq} M$ with $L_v = 0$ for $v < w$. Now $\mathbf{R}\mathrm{Hom}_R(M,N) \simeq \mathrm{Hom}_R(L, N_{\subset u})$, and the assertions follow by Lemma (2.3.19). $\square$

(4.2.9) **Lemma.** *If M belongs to $\mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$ and $N \in \mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$, then $\mathbf{R}\mathrm{Hom}_R(M,N)$ is in $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$.*

Proof. Set $\inf M = w$ and $\sup N = u$. By Theorem (3.1.10) the complex M has a semifree resolution $L \xrightarrow{\simeq} M$ with L_v finitely generated for all v and $L_v = 0$ for $v < w$. Set $N' = N_{\subset u}$, then $\mathrm{H}_v(\mathbf{R}\mathrm{Hom}_R(M,N)) \cong \mathrm{H}_v(\mathrm{Hom}_R(L,N'))$ vanishes for $v > u - w$ by Lemma (4.2.8). To see that each module $\mathrm{H}_v(\mathrm{Hom}_R(L,N'))$ is finitely generated, fix $v \in \mathbb{Z}$ and set $n = v + w - 2$. Now $\mathrm{H}_v(\mathrm{Hom}_R(L,N')) \cong \mathrm{H}_v(\mathrm{Hom}_R(L,N'_{\supset n}))$, cf. Definition (2.3.1) and (2.3.18.1), and $N'_{\supset n}$ is a bounded complex with finitely generated homology modules. By Theorem (3.1.10) and Remark (2.2.6) there is a bounded complex N'' of finitely generated modules such that $N'' \simeq N'_{\supset n}$; therefore

$$\mathrm{H}_v(\mathrm{Hom}_R(L,N')) \cong \mathrm{H}_v(\mathrm{Hom}_R(L,N'_{\supset n})) \cong \mathrm{H}_v(\mathrm{Hom}_R(L,N'')),$$

and this module is finitely generated by Lemma (2.3.20). $\square$

4.3. Derived tensor product functor

(4.3.1) **Observation.** Let M be an R-complex; let $\pi\colon P \to M$ and $\varpi\colon Q \to M$ be semiprojective resolutions of M. By Proposition (3.2.6) there is a homotopy equivalence $\alpha\colon P \to Q$. Let $\zeta\colon X \to Y$ be a morphism of R-complexes; by Proposition (2.4.8) the induced morphisms $\alpha \otimes_R X$ and $\alpha \otimes_R Y$ are also homotopy equivalences, and there is a commutative diagram

$$\begin{tikzcd}
P \otimes_R X \arrow[r, "P\otimes_R\zeta"] \arrow[d, "\alpha\otimes_R X"', "\sim"] & P \otimes_R Y \arrow[d, "\alpha\otimes_R Y", "\sim"'] \\
Q \otimes_R X \arrow[r, "Q\otimes_R\zeta"] & Q \otimes_R Y.
\end{tikzcd}$$

Let M' be an R-complex with semiprojective resolutions $\pi'\colon P' \to M'$ and $\varpi'\colon Q' \to M'$; let $\beta\colon M \to M'$ be a morphism. By Proposition (3.2.6) there is a homotopy equivalence $\alpha'\colon P' \to Q'$. By Proposition (3.2.5) there are morphisms

$\gamma^P\colon P \to P'$ and $\gamma^Q\colon Q \to Q'$ such that $\pi'\gamma^P \sim \beta\pi$ and $\varpi'\gamma^Q \sim \beta\varpi$. The upshot is that the next diagram is commutative up to homotopy

$$\begin{tikzcd} P \otimes_R X \arrow[r, "\gamma^P \otimes_R X"] \arrow[d, "\alpha\otimes_R X"', "\sim"] & P' \otimes_R X \arrow[d, "\sim"', "\alpha'\otimes_R X"] \\ Q \otimes_R X \arrow[r, "\gamma^Q \otimes_R X"] & Q' \otimes_R X. \end{tikzcd}$$

(4.3.2) **Definition.** For R-complexes M and N the *left derived tensor product complex*, $M \otimes^{\mathbf{L}}_R N$, is $P \otimes_R N$, where P is a semiprojective resolution of M. By Observation (4.3.1) this defines $M \otimes^{\mathbf{L}}_R N$ uniquely up to isomorphism in $\mathsf{D}(R)$.

(4.3.3) **Theorem.** *For every R-complex M, the left derived tensor product defines (up to isomorphism) an exact covariant functor $M \otimes^{\mathbf{L}}_R -$ on $\mathsf{D}(R)$. That is,*

(a) *to every X in $\mathsf{D}(R)$ it associates an R-complex $M \otimes^{\mathbf{L}}_R X$ that is unique up to isomorphism in $\mathsf{D}(R)$;*

(b) *to every morphism $\alpha\colon X \to Y$ in $\mathsf{D}(R)$ it associates a morphism $M \otimes^{\mathbf{L}}_R \alpha\colon M \otimes^{\mathbf{L}}_R X \to M \otimes^{\mathbf{L}}_R Y$;*

(c) *the equality $M \otimes^{\mathbf{L}}_R \beta\alpha = \big(M \otimes^{\mathbf{L}}_R \beta\big)\big(M \otimes^{\mathbf{L}}_R \alpha\big)$ holds for every pair of morphisms $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ in $\mathsf{D}(R)$;*

(d) *the equality $M \otimes^{\mathbf{L}}_R 1^X = 1^{M \otimes^{\mathbf{L}}_R X}$ holds for every X in $\mathsf{D}(R)$;*

(e) *to every exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathsf{D}(R)$ it associates an exact triangle $M \otimes^{\mathbf{L}}_R X \to M \otimes^{\mathbf{L}}_R Y \to M \otimes^{\mathbf{L}}_R Z \to \Sigma\big(M \otimes^{\mathbf{L}}_R X\big)$.*

Proof. Properties (a)-(d) follow from Theorem (2.4.5) in view of Observation (4.3.1). It suffices to prove part (e) for distinguished triangles, and in that case the claim follows from Lemma (2.4.11). $\square$

(4.3.4) **Theorem.** *For every R-complex N, the left derived tensor product defines (up to isomorphism) an exact covariant functor $- \otimes^{\mathbf{L}}_R N$ on $\mathsf{D}(R)$. That is,*

(a) *To every X in $\mathsf{D}(R)$ it associates an R-complex $X \otimes^{\mathbf{L}}_R N$ that is unique up to isomorphism in $\mathsf{D}(R)$.*

(b) *To every morphism $\alpha\colon X \to Y$ in $\mathsf{D}(R)$ it associates a morphism $\alpha \otimes^{\mathbf{L}}_R N\colon X \otimes^{\mathbf{L}}_R N \to Y \otimes^{\mathbf{L}}_R N$.*

(c) *The equality $\beta\alpha \otimes^{\mathbf{L}}_R N = \big(\beta \otimes^{\mathbf{L}}_R N\big)\big(\alpha \otimes^{\mathbf{L}}_R N\big)$ holds for every pair of morphisms $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ in $\mathsf{D}(R)$.*

(d) *The equality $1^X \otimes^{\mathbf{L}}_R N = 1^{X \otimes^{\mathbf{L}}_R N}$ holds for every X in $\mathsf{D}(R)$.*

(e) *To every exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathsf{D}(R)$ it associates an exact triangle $X \otimes^{\mathbf{L}}_R N \to Y \otimes^{\mathbf{L}}_R N \to Z \otimes^{\mathbf{L}}_R N \to \Sigma\big(X \otimes^{\mathbf{L}}_R N\big)$.*

Proof. Properties (a)-(d) follow from Theorem (2.4.7) in view of Observation (4.3.1). It suffices to prove part (e) for distinguished triangles, and in that case the claim follows from Lemma (2.4.12). $\square$

(4.3.5) **Observation.** The arguments in Observation (4.3.1) can be recycled to show that a left derived tensor product functor can be well-defined up to isomorphism in $\mathsf{D}(R)$ by $M \otimes_R N = M \otimes_R Q$, where Q is a semiprojective resolution of N.

To see that this definition agrees with the one given in Definition (4.3.2), let M and N be R-complexes, let $\pi\colon P \xrightarrow{\simeq} M$ be a semiprojective resolution of M and $\varpi\colon Q \xrightarrow{\simeq} N$ be a semiinjective resolution of N.

Furthermore, let $\beta\colon M \to M'$ be a morphism and $\pi'\colon P' \to M'$ be a semiprojective resolution. By Proposition (3.2.5) there is a morphism $\gamma\colon P \to P'$ that lifts $\beta\pi$ along π' up to homotopy. The next diagram is commutative up to homotopy.

$$\begin{tikzcd}
P \otimes_R N \arrow[r, "\gamma\otimes_R N"] & P' \otimes_R N \\
P \otimes_R Q \arrow[r, "\gamma\otimes_R Q"] \arrow[u, "P\otimes_R\varpi", "\simeq"'] \arrow[d, "\pi\otimes_R Q"', "\simeq"] & P' \otimes_R Q \arrow[u, "\simeq", "P'\otimes_R\varpi"'] \arrow[d, "\simeq"', "\pi'\otimes_R Q"] \\
M \otimes_R Q \arrow[r, "\beta\otimes_R Q"] & M' \otimes_R Q
\end{tikzcd}$$

It shows that the two definitions yield the same covariant functor $- \otimes^{\mathbf{L}}_R N$ on $\mathsf{D}(R)$. A similar diagram, or an application of commutativity (2.5.1), shows that the two definitions also define the same covariant functor $M \otimes^{\mathbf{L}}_R -$.

(4.3.6) **Observation.** Let M be an R-complex and $\zeta\colon X \to Y$ be a morphism of R-complexes. Let $\pi\colon P \to M$ be a semiprojective resolution of M, and let F be a semiflat R-complex such that $F \simeq M$ in $\mathsf{D}(R)$. By Proposition (4.1.10) there is a quasiisomorphism $\pi\colon P \xrightarrow{\simeq} F$. The next diagram is commutative.

$$\begin{tikzcd}
P \otimes_R X \arrow[r, "P\otimes_R\zeta"] \arrow[d, "\pi\otimes_R X"', "\simeq"] & P \otimes_R Y \arrow[d, "\simeq"', "\pi\otimes_R Y"] \\
F \otimes_R X \arrow[r, "F\otimes_R\zeta"] & F \otimes_R Y
\end{tikzcd}$$

The vertical maps are quasiisomorphisms by Proposition (3.4.7).

By Observation (4.3.5) and commutativity (4.4.1) it follow that, if F, G are semiflat complexes such that $M \simeq F$ and $N \simeq G$, then there are isomorphisms

$$F \otimes_R N \;\simeq\; M \otimes_R N \;\simeq\; M \otimes_R G,$$

which are natural in N and M, respectively.

(4.3.7) **Lemma.** *Let F be an R-module. The following are equivalent.*

- (i) *F is flat.*
- (ii) $\sup T \otimes^{\mathbf{L}}_R F \leqslant 0$ *for every R-module T.*
- (iii) $\mathrm{H}_1(T \otimes^{\mathbf{L}}_R F) = 0$ *for every cyclic R-module T.*

Proof. Well-known as $\mathrm{H}_m(T \otimes^{\mathbf{L}}_R F) = \mathrm{Tor}^R_m(T, F)$ for $m \geqslant 0$. □

Boundedness and finiteness

(4.3.8) **Lemma.** *If M and N belong to $\mathsf{D}_{\sqsupset}(R)$, then also $M \otimes^{\mathbf{L}}_R N$ is in $\mathsf{D}_{\sqsupset}(R)$. More precisely, if* $\inf M = w$ *and* $\inf N = t$*, then*

- (a) $\inf M \otimes^{\mathbf{L}}_R N \geqslant w + t$ *and*
- (b) $\mathrm{H}_{w+t}(M \otimes^{\mathbf{L}}_R N) \cong \mathrm{H}_w(M) \otimes_R \mathrm{H}_t(N)$.

Proof. By Theorem (3.1.9) the complex M has a semifree resolution $L \xrightarrow{\simeq} M$ with $L_v = 0$ for $v < w$. Now $M \otimes^{\mathbf{L}}_R N \simeq L \otimes_R N_{\sqsupset t}$, and the assertions follow by Lemma (2.4.14). □

(4.3.9) **Lemma.** *If M and N belong to $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$, then also $M \otimes^{\mathbf{L}}_R N$ is in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$.*

Proof. Set $\inf M = w$ and $\inf N = t$. By Theorem (3.1.10) the complexes M and N have semifree resolutions $L \xrightarrow{\simeq} M$ and $L' \xrightarrow{\simeq} N$ with L_v and L'_v finitely generated for all v, $L_v = 0$ for $v < w$, and $L'_v = 0$ for $v < t$. Now $M \otimes^{\mathbf{L}}_R N \simeq L \otimes_R L'$, and the assertions follow by Lemma (2.4.15). □

4.4. Standard (iso)morphisms

Standard isomorphisms

The first map is the (derived tensor product) *commutativity* isomorphism.

(4.4.1) **Theorem.** *For R-complexes M and N there is an isomorphism in $\mathsf{D}(R)$*

$$\tau_{MN} \colon M \otimes^{\mathbf{L}}_R N \xrightarrow{\simeq} N \otimes^{\mathbf{L}}_R M,$$

which is natural in M and N.

Proof. Let P be a semiprojective resolution of M. By Theorem (2.5.1) there is a natural isomorphism in $\mathsf{C}(R)$

$$\tau_{PN} \colon P \otimes_R N \xrightarrow{\cong} N \otimes_R P.$$

The claim now follows in view of Observation (4.3.5). □

The next map is the (derived tensor product) *associativity* isomorphism.

(4.4.2) **Theorem.** *For R-complexes K, M, and N there is an isomorphism in $\mathsf{D}(R)$*

$$\sigma_{KMN} \colon \big(K \otimes^{\mathbf{L}}_R M\big) \otimes^{\mathbf{L}}_R N \xrightarrow{\simeq} K \otimes^{\mathbf{L}}_R \big(M \otimes^{\mathbf{L}}_R N\big),$$

which is natural in K, M, and N.

Proof. A consequence of Theorem (2.5.2). □

The next map is the (derived Hom-tensor) *adjointness* isomorphism.

(4.4.3) **Theorem.** *For R-complexes K, M, and N there is an isomorphism in $\mathsf{D}(R)$*

$$\rho_{KMN} \colon \mathbf{R}\mathrm{Hom}_R(K \otimes^{\mathbf{L}}_R M, N) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_R(M, N)),$$

which is natural in K, M, and N.

Proof. A consequence of Theorem (2.5.3). □

The next map is the (derived Hom) *swap* isomorphism.

(4.4.4) **Theorem.** *For R-complexes K, M, and N there is an isomorphism in $\mathsf{D}(R)$*

$$\varsigma_{KMN} \colon \mathbf{R}\mathrm{Hom}_R(K, \mathbf{R}\mathrm{Hom}_R(M, N)) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(K, N)),$$

which is natural in K, M, and N.

Proof. A consequence of Theorem(2.5.4). □

Evaluation morphisms

The next map is the (derived) *tensor evaluation* morphism.

(4.4.5) **Theorem.** *For R-complexes K, M, and N there is a morphism in $\mathsf{D}(R)$*

$$\omega_{KMN}\colon \mathbf{R}\mathrm{Hom}_R(K,M)\otimes^{\mathbf{L}}_R N \longrightarrow \mathbf{R}\mathrm{Hom}_R(K,M\otimes^{\mathbf{L}}_R N),$$

which is natural in K, M, and N. It is an isomorphism under each of the following conditions

(a) $K\in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$, $M\in \mathsf{D}_{\sqsubset}(R)$, $N\in \mathsf{D}_{\sqsubset}(R)$, *and* $\mathrm{fd}_R N<\infty$.

(b) $K\in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ *and* $\mathrm{pd}_R K<\infty$.

Proof. Choose a semifree resolution $L\xrightarrow{\simeq} K$ and a semiflat complex F such that $F\simeq N$. Now ω_{LMF} is the desired morphism:

$$\mathrm{Hom}_R(L,M)\otimes_R F \xrightarrow{\omega_{LMF}} \mathrm{Hom}_R(L,M\otimes_R F).$$

(a): Under the assumptions on K and N, we can assume that F is bounded above and L is bounded below and degreewise finitely generated; see Theorem (5.1.9) and Theorem (3.1.10). After replacing it with a suitable truncation, we can assume that M is bounded above, and then ω_{LMF} is an isomorphism in $\mathsf{C}(R)$ by Theorem (2.5.5).

(b): Under the assumption on K, we can assume that L is bounded and degreewise finitely generated; see Theorem (3.1.10) and Theorem (5.1.3). □

The next map is the (derived) *homomorphism evaluation* morphism.

(4.4.6) **Theorem.** *For R-complexes K, M, and N there is a morphism in $\mathsf{D}(R)$*

$$\theta_{KMN}\colon K\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,N) \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(K,M),N),$$

which is natural in K, M, and N. It is an isomorphism under each of the following conditions

(a) $K\in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$, $M\in \mathsf{D}_{\sqsubset}(R)$, $N\in \mathsf{D}_{\sqsupset}(R)$, *and* $\mathrm{id}_R N<\infty$.

(b) $K\in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ *and* $\mathrm{pd}_R K<\infty$.

Proof. Similar to the proof of Theorem (2.5.5). □

Exercises

(E 4.4.1) Let M be an R-complex and assume $\mathrm{H}_v(M)=0$ for $v\neq 0$.

(a) Suppose $M_v=0$ for $v<0$ and prove that there is a quasiisomorphism $M\xrightarrow{\simeq}\mathrm{H}(M)$ in $\mathsf{C}(R)$.

(b) Suppose $M_v=0$ for $v>0$ and prove that there is a quasiisomorphism $\mathrm{H}(M)\xrightarrow{\simeq} M$ in $\mathsf{C}(R)$.

(c) Conclude that for every complex M' with $\mathrm{amp}\,M'=0$ there is an isomorphism $M'\simeq \mathrm{H}(M')$ in $\mathsf{D}(R)$.

You may solve (a) and (b) by solving the next exercise.

(E 4.4.2) Let $M=\cdots\to M_1\to M_0\to 0$ and $N=0\to N_0\to N_1\to\cdots$ be R-complexes concentrated in non-negative and non-positive degrees, respectively. Prove that there is an isomorphism of R-modules

$$\mathrm{Hom}_R(\mathrm{H}_0(M),\mathrm{H}_0(N))\;\cong\;\mathrm{H}_0(\mathrm{Hom}_R(M,N)).$$

CHAPTER 5

Homological Dimensions

5.1. Classical dimensions

PROJECTIVE DIMENSION

(5.1.1) **Definition.** For an R-complex M the *projective dimension* $\operatorname{pd}_R M$ is defined as

$$\operatorname{pd}_R M \;=\; \inf\left\{ n \;\middle|\; \begin{array}{c} \exists \text{ semiprojective } R\text{-complex } P \text{ such} \\ \text{that } P \simeq M \text{ and } P_v = 0 \text{ for all } v > n \end{array} \right\}.$$

(5.1.2) **Remark.** Let M be an R-complex and m an integer. It is immediate that

$$\begin{gathered} \operatorname{pd}_R M \;\geqslant\; \sup M; \\ \operatorname{pd}_R M = -\infty \iff M \simeq 0; \\ \operatorname{pd}_R \Sigma^m M \;=\; \operatorname{pd}_R M + m. \end{gathered}$$

(5.1.3) **Theorem.** *Let M be an R-complex and n an integer. The following are equivalent.*

(*i*) $\operatorname{pd}_R M \leqslant n$.

(*ii*) $\inf X - \inf \mathbf{R}\mathrm{Hom}_R(M,X) \leqslant n$ *for every* $X \not\simeq 0$ *in* $\mathsf{D}_{\sqsupset}(R)$.

(*iii*) $-\inf \mathbf{R}\mathrm{Hom}_R(M,T) \leqslant n$ *for every R-module T.*

(*iv*) $n \geqslant \sup M$ *and* $\mathrm{H}_{-(n+1)}(\mathbf{R}\mathrm{Hom}_R(M,T)) = 0$ *for every R-module T.*

(*v*) $n \geqslant \sup M$ *and the module* $\mathrm{C}_n(P)$ *is projective for every semiprojective R-complex $P \simeq M$.*

(*vi*) $n \geqslant \sup M$ *and for every semiprojective R-complex $P \simeq M$ the truncation $P_{\subset n}$ is a semiprojective resolution of M.*

(*vii*) *There is a semiprojective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ when $v > n$ or $v < \inf M$.*

Furthermore, there are equalities

$$\begin{aligned} \operatorname{pd}_R M &= \sup\{\inf X - \inf \mathbf{R}\mathrm{Hom}_R(M,X) \mid X \in \mathsf{D}_{\sqsupset}(R) \text{ and } X \not\simeq 0\} \\ &= \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(M,T) \mid T \text{ is an } R\text{-module}\}. \end{aligned}$$

Proof. Similar to the proof of Theorem (5.1.9). □

INJECTIVE DIMENSION

(5.1.4) **Definition.** For an R-complex M the *injective dimension* $\operatorname{id}_R M$ is defined as

$$\operatorname{id}_R M \;=\; \inf\left\{ n \;\middle|\; \begin{array}{c} \exists \text{ semiinjective } R\text{-complex } I \text{ such} \\ \text{that } I \simeq M \text{ and } I_v = 0 \text{ for all } v < -n \end{array} \right\}.$$

(5.1.5) **Remark.** Let M be an R-complex and m an integer. It is immediate that

$$\begin{aligned} \operatorname{id}_R M \ &\geqslant\ -\inf M; \\ \operatorname{id}_R M = -\infty \ &\iff\ M \simeq 0; \\ \operatorname{id}_R \Sigma^m M \ &=\ \operatorname{id}_R M - m. \end{aligned}$$

(5.1.6) **Theorem.** *Let M be an R-complex and n be an integer. The following are equivalent.*

- (*i*) $\operatorname{id}_R M \leqslant n$.
- (*ii*) $-\sup X - \inf \mathbf{R}\mathrm{Hom}_R(X, M) \leqslant n$ *for every* $X \not\simeq 0$ *in* $\mathsf{D}_{\sqsubset}(R)$.
- (*iii*) $-\inf \mathbf{R}\mathrm{Hom}_R(T, M) \leqslant n$ *for every cyclic R-module T.*
- (*iv*) $-n \leqslant \inf M$ *and* $\mathrm{H}_{-(n+1)}(\mathbf{R}\mathrm{Hom}_R(T, M)) = 0$ *for every cyclic R-module T.*
- (*v*) $-n \leqslant \inf M$ *and the module* $\mathrm{Z}_{-n}(I)$ *is injective for every semiinjective R-complex $I \simeq M$.*
- (*vi*) $-n \leqslant \inf M$ *and for every semiinjective R-complex $I \simeq M$ the truncation $I_{\supset -n}$ is a semiinjective resolution of M.*
- (*vii*) *There is a semiinjective resolution $I \xrightarrow{\simeq} M$ with $I_v = 0$ when $v < -n$ or $v > \sup M$.*

Furthermore, there are equalities

$$\begin{aligned} \operatorname{id}_R M &= \sup\{\, -\sup X - \inf \mathbf{R}\mathrm{Hom}_R(X, M) \mid X \in \mathsf{D}_{\sqsubset}(R) \text{ and } X \not\simeq 0 \,\} \\ &= \sup\{\, -\inf \mathbf{R}\mathrm{Hom}_R(T, M) \mid T \text{ is a cyclic } R\text{-module} \,\}. \end{aligned}$$

Proof. Similar to the proof of Theorem (5.1.9). □

Flat dimension

(5.1.7) **Definition.** For an R-complex M the *flat dimension* $\operatorname{fd}_R M$ is defined as

$$\operatorname{fd}_R M \ =\ \inf\left\{ n \ \middle|\ \begin{array}{c} \exists \text{ semiflat } R\text{-complex } F \text{ such that} \\ F \simeq M \text{ and } F_v = 0 \text{ for all } v > n \end{array} \right\}.$$

(5.1.8) **Remark.** Let M be an R-complex and m an integer. It is immediate that

$$\begin{aligned} \operatorname{pd}_R M \ \geqslant\ \operatorname{fd}_R M \ &\geqslant\ \sup M; \\ \operatorname{fd}_R M = -\infty \ &\iff\ M \simeq 0; \\ \operatorname{fd}_R \Sigma^m M \ &=\ \operatorname{fd}_R M + m. \end{aligned}$$

(5.1.9) **Theorem.** *Let M be an R-complex and n an integer. The following are equivalent.*

- (*i*) $\operatorname{fd}_R M \leqslant n$.
- (*ii*) $\sup X \otimes^{\mathbf{L}}_R M - \sup X \leqslant n$ *for every* $X \not\simeq 0$ *in* $\mathsf{D}_{\sqsubset}(R)$.
- (*iii*) $\sup T \otimes^{\mathbf{L}}_R M \leqslant n$ *for every cyclic R-module T.*
- (*iv*) $n \geqslant \sup M$ *and* $\mathrm{H}_{n+1}(T \otimes^{\mathbf{L}}_R M) = 0$ *for every cyclic R-module T.*
- (*v*) $n \geqslant \sup M$ *and the module* $\mathrm{C}_n(F)$ *is flat for every semiflat R-complex $F \simeq M$.*
- (*vi*) $n \geqslant \sup M$ *and for every semiflat R-complex $F \simeq M$ the truncation $F_{\subset n}$ is semiflat and isomorphic to M in $\mathsf{D}(R)$.*

(*vii*) *There exists a semiflat R-complex F such that $F \simeq M$ and $F_v = 0$ when $v > n$ or $v < \inf M$.*

Furthermore, there are equalities

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\, \sup X \otimes^{\mathbf{L}}_R M - \sup X \mid X \in \mathsf{D}_{\sqsubset}(R) \text{ and } X \not\simeq 0 \,\} \\ &= \sup\{\, \sup T \otimes^{\mathbf{L}}_R M \mid T \text{ is a cyclic } R\text{-module} \,\}.\end{aligned}$$

Proof. The proof is cyclic; the implications $(ii) \Longrightarrow (iii)$ and $(vii) \Longrightarrow (i)$ are trivial.

$(i) \Longrightarrow (ii)$: Choose a semiflat R-complex F such that $F \simeq M$ and $F_v = 0$ for all $v > n$. Set $s = \sup X$; there is an isomorphism $X \otimes^{\mathbf{L}}_R M \simeq X_{\subset s} \otimes^{\mathbf{L}}_R M$ in $\mathsf{D}(R)$; see Remark (2.2.6). In particular, $\sup X \otimes^{\mathbf{L}}_R M = \sup X_{\subset s} \otimes_R F$. For $v > n + s$ and $i \in \mathbb{Z}$, either $i > s$ or $v - i \geqslant v - s > n$, so the module

$$(X_{\subset s} \otimes_R F)_v \;=\; \coprod_{i \in \mathbb{Z}} (X_{\subset s})_i \otimes_R F_{v-i}$$

vanishes. In particular, $\mathrm{H}_v(X_{\subset s} \otimes_R F) = 0$ for $v > n + s$ so $\sup X \otimes^{\mathbf{L}}_R M \leqslant n + s$ as desired.

$(iii) \Longrightarrow (iv)$: Apply (iii) to $T = R$ to get $\sup M = \sup R \otimes_R M \leqslant n$; the rest is immediate.

$(iv) \Longrightarrow (v)$: Let F be a semiflat R-complex such that $F \simeq M$. Note that $F_{\geqslant n}$ is a semiflat R-complex by Corollary (3.4.9) and $F_{\geqslant n} \simeq \mathrm{C}_n(F)$ as $n \geqslant \sup M = \sup F$. Let T be an R-module. In view of Lemma (4.3.7) the next computation shows that $\mathrm{C}_n(F)$ is a flat R-module.

$$\begin{aligned}\mathrm{H}_{n+1}(T \otimes^{\mathbf{L}}_R M) &\cong \mathrm{H}_{n+1}(T \otimes_R F) \\ &= \mathrm{H}_{n+1}(T \otimes_R F_{\geqslant n}) \\ &= \mathrm{H}_1(\Sigma^{-n}(T \otimes_R F_{\geqslant n})) \\ &= \mathrm{H}_1(T \otimes_R \Sigma^{-n} F_{\geqslant n}) \\ &\cong \mathrm{H}_1(T \otimes^{\mathbf{L}}_R \mathrm{C}_n(F))\end{aligned}$$

$(v) \Longrightarrow (vi)$: Let F be a semiflat R-complex such that $F \simeq M$. The complexes $F_{\geqslant n}$ and $\Sigma^n \mathrm{C}_n(F)$ are semiflat by Corollary (3.4.9), and by Lemma (3.4.8) so is the kernel B of the morphism $F_{\geqslant n} \twoheadrightarrow \Sigma^n \mathrm{C}_n(F)$. Since B is also the kernel of $F \twoheadrightarrow F_{\subset n}$, the complex $F_{\subset n}$ is also semiflat; again by Lemma (3.4.8). Because $n \geqslant \sup M = \sup F$ there are isomorphisms $M \simeq F \simeq F_{\subset n}$ in $\mathsf{D}(R)$, cf. Remark (2.2.6).

$(vi) \Longrightarrow (vii)$: Choose by Theorem (3.1.9) a semifree resolution L of M with $L_v = 0$ for $v < \inf M$. By Theorem (3.2.2) and Corollary (3.4.4) the complex L is semiflat, so $L_{\subset n}$ is the desired complex.

Finally, the equalities follow from the equivalence of (i), (ii), and (iii). $\square$

Exercises

(E 5.1.1) Let M be an R-module. Show that the semiflat complex $F \simeq M$ constructed in the proof of Theorem (5.1.9)($(vi) \Longrightarrow (vii)$) is an ordinary flat resolution of M.

The next exercise explains why the concept "semiflat resolution of a complex" has not been introduced.

(E 5.1.2) Give an example of a complex M of finite flat dimension such that for any semiflat complex $F \simeq M$ with $n = \operatorname{fd}_R M$ as described in Theorem (5.1.9)(*vii*) there is no quasiisomorphism $F \xrightarrow{\simeq} M$.
Hint: Example (3.4.6).

(E 5.1.3) (Belongs in Section 5.1) Show that the equality $\operatorname{pd}_R M = \operatorname{fd}_R M$ holds for $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$, also when R is not local.

5.2. Koszul homology

Depth and width

(5.2.1) **Definition.** Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. The *width* of M is

$$\operatorname{width}_R M \;=\; \inf k \otimes^{\mathbf{L}}_R M.$$

(5.2.2) **Observation.** Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. By Lemma (2.4.14) there is an inequality

$$\operatorname{width}_R M \geqslant \inf M.$$

If $\inf M = w > -\infty$, then equality holds if and only if $\mathfrak{m}\operatorname{H}_w(M) \neq \operatorname{H}_w(M)$; in particular, equality holds by NAK if M is in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$.

(5.2.3) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be a local ring. For R-complexes M and N there is an equality*

$$\operatorname{width}_R M \otimes^{\mathbf{L}}_R N = \operatorname{width}_R M + \operatorname{width}_R N.$$

Proof. A straightforward computation that uses associativity (4.4.2), Proposition (2.1.19), and Lemma (2.4.14):

$$\begin{aligned}
\operatorname{width}_R M \otimes^{\mathbf{L}}_R N &= \inf k \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R N)\\
&= \inf (k \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_R N\\
&= \inf ((k \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_k k) \otimes^{\mathbf{L}}_R N\\
&= \inf (k \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_k (k \otimes^{\mathbf{L}}_R N)\\
&= \inf \operatorname{H}(k \otimes^{\mathbf{L}}_R M) \otimes_k \operatorname{H}(k \otimes^{\mathbf{L}}_R N)\\
&= \inf k \otimes^{\mathbf{L}}_R M + \inf k \otimes^{\mathbf{L}}_R N\\
&= \operatorname{width}_R M + \operatorname{width}_R N. \qquad \square
\end{aligned}$$

(5.2.4) **Definition.** Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. The *depth* of M is

$$\operatorname{depth}_R M \;=\; -\sup \mathbf{R}\operatorname{Hom}_R(k, M).$$

(5.2.5) **Observation.** Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex. By Lemma (2.3.19) there is an inequality

$$\operatorname{depth}_R M \geqslant -\sup M.$$

If $\sup M = s < \infty$, then equality holds if and only if $\mathfrak{m}$ is an associated prime of the top homology module $\operatorname{H}_s(M)$.

(5.2.6) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be a local ring. For R-complexes M and N there is an equality*

$$\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(M,N) = \operatorname{width}_R M + \operatorname{depth}_R N.$$

Proof. A straightforward computation that uses adjointness (4.4.3), Proposition (2.1.19), and Lemma (2.3.19):

$$\begin{aligned}
\operatorname{depth}_R \mathbf{R}\mathrm{Hom}_R(M,N) &= -\sup \mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(M,N)) \\
&= -\sup \mathbf{R}\mathrm{Hom}_R(k \otimes^{\mathbf{L}}_R M, N) \\
&= -\sup \mathbf{R}\mathrm{Hom}_R((k \otimes^{\mathbf{L}}_R M) \otimes^{\mathbf{L}}_k k, N) \\
&= -\sup \mathbf{R}\mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R M, \mathbf{R}\mathrm{Hom}_R(k,N)) \\
&= -\sup \mathrm{Hom}_k(\mathrm{H}(k \otimes^{\mathbf{L}}_R M), \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,N))) \\
&= -(\sup \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,N)) - \inf \mathrm{H}(k \otimes^{\mathbf{L}}_R M)) \\
&= \inf k \otimes^{\mathbf{L}}_R M - \sup \mathbf{R}\mathrm{Hom}_R(k,N) \\
&= \operatorname{width}_R M + \operatorname{depth}_R N. \qquad \square
\end{aligned}$$

(5.2.7) **Corollary.** *If M and N are R-modules, M is finitely generated, and $\mathrm{Ext}^{i>0}_R(M,N) = 0$, then $\operatorname{depth}_R \mathrm{Hom}_R(M,N) = \operatorname{depth}_R N$.* $\square$

(5.2.8) **Lemma.** *For R-complexes $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $N \in \mathsf{D}_{\sqsubset}(R)$ there is an equality:*

$$-\sup \mathbf{R}\mathrm{Hom}_R(M,N) = \inf\{\, \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} + \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}.$$

Proof. Set $s = \sup \mathbf{R}\mathrm{Hom}_R(M,N)$. For every $\mathfrak{p} \in \operatorname{Spec} R$ there is a series of (in)equalities

$$\begin{aligned}
-s &\leqslant \operatorname{depth}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(M,N)_\mathfrak{p} \\
&= \operatorname{depth}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \\
&= \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} + \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \\
&= \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} + \inf M_\mathfrak{p}.
\end{aligned}$$

Indeed, the inequality is by Observation (5.2.5), and equality holds if $\mathfrak{p}$ is an associated prime of $\mathrm{H}_s(\mathbf{R}\mathrm{Hom}_R(M,N))$. The equalities are by Lemma (6.1.7), Proposition (5.2.6), and Observation (5.2.2), respectively. $\square$

(5.2.9) **Observation.** Let H be a finitely generated R-module and $N \in \mathsf{D}_{\sqsubset}(R)$ an R-complex. By Lemma (5.2.8) there are equalities

$$\begin{aligned}
-\sup \mathbf{R}\mathrm{Hom}_R(H,N) &= \inf\{\, \operatorname{depth}_R N_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R H \,\} \\
&= \inf\{\, \operatorname{depth}_R N_\mathfrak{p} \mid \mathfrak{p} \supseteq \operatorname{Ann}_R H \,\} \\
&= -\sup \mathbf{R}\mathrm{Hom}_R(R/\operatorname{Ann}_R H, N).
\end{aligned}$$

(5.2.10) **Lemma.** *If $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $N \in \mathsf{D}_{\sqsubset}(R)$, then*

$$\sup \mathbf{R}\mathrm{Hom}_R(M,N) = \sup\{\, \sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(M), N) - n \mid n \in \mathbb{Z} \,\}.$$

Proof. Set $s = \sup \mathbf{R}\mathrm{Hom}_R(M,N)$ and $t = \sup\{\, \sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(M), N) - n \mid n \in \mathbb{Z} \,\}$.

To prove $s \leqslant t$ it suffices by Lemma (5.2.8) to prove that $\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} + \inf M_\mathfrak{p} \geqslant -t$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Given $\mathfrak{p}$, set $w = \inf M_\mathfrak{p}$ and note that because $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}_w(M)$, Observation (5.2.9) yields the inequality

$$\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} + \inf M_\mathfrak{p} = w + \operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} \geqslant w - \sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_w(M), N) \geqslant -t.$$

For the opposite inequality, let $n \in \mathbb{Z}$ be given. Assume $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(M), N)) \neq 0$ and choose by Observation (5.2.9) a prime ideal $\mathfrak{p}$ such that $-\sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(M), N) = \operatorname{depth}_R N_\mathfrak{p}$. Now

$$n - \sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(M), N) \geqslant \inf M_\mathfrak{p} + \operatorname{depth}_R N_\mathfrak{p} \geqslant -s. \qquad \square$$

(5.2.11) **Lemma.** *Let $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $N \in \mathsf{D}_{\sqsupset}(R)$ be R-complexes and H be a finitely generated R-module. There are equalities*

(a) $$\inf H \otimes^{\mathbf{L}}_R N = \inf R/\operatorname{Ann}_R H \otimes^{\mathbf{L}}_R N \quad \textit{and}$$

(b) $$\inf M \otimes^{\mathbf{L}}_R N = \inf\{\, n + \inf \mathrm{H}_n(M) \otimes^{\mathbf{L}}_R N \mid n \in \mathbb{Z} \,\}.$$

Proof. Let E be a faithfully injective R-module.

(a): Set $C = R/\operatorname{Ann}_R H$, then

$$\begin{aligned} \inf H \otimes^{\mathbf{L}}_R N &= -\sup \mathbf{R}\mathrm{Hom}_R(H \otimes^{\mathbf{L}}_R N, E) \\ &= -\sup \mathbf{R}\mathrm{Hom}_R(H, \mathbf{R}\mathrm{Hom}_R(N, E)) \\ &= -\sup \mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_R(N, E)) \\ &= \inf C \otimes^{\mathbf{L}}_R N, \end{aligned}$$

by Observation (5.2.9).

(b): A straightforward computation based on Lemma (5.2.10):

$$\begin{aligned} \inf M \otimes^{\mathbf{L}}_R N &= -\sup \mathbf{R}\mathrm{Hom}_R(M \otimes^{\mathbf{L}}_R N, E) \\ &= -\sup \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_R(N, E)) \\ &= -\sup\{\, \sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(M), \mathbf{R}\mathrm{Hom}_R(N, E)) - n \mid n \in \mathbb{Z} \,\} \\ &= \inf\{\, n - \sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_n(M) \otimes^{\mathbf{L}}_R N, E) \mid n \in \mathbb{Z} \,\} \\ &= \inf\{\, n + \inf \mathrm{H}_n(M) \otimes^{\mathbf{L}}_R N \mid n \in \mathbb{Z} \,\} \qquad \square \end{aligned}$$

Auslander–Buchsbaum Formula

(5.2.12) **Theorem.** *Let M and N be R-complexes in $\mathsf{D}_{\sqsubset}(R)$. If $\operatorname{fd}_R N < \infty$, then*

$$\begin{aligned} \operatorname{depth}_R M \otimes^{\mathbf{L}}_R N &= \operatorname{depth}_R M - \sup k \otimes^{\mathbf{L}}_R N \\ &= \operatorname{depth}_R M + \operatorname{depth}_R N - \operatorname{depth}_R R. \end{aligned}$$

In particular,

$$\operatorname{depth}_R N = \operatorname{depth}_R R - \sup k \otimes^{\mathbf{L}}_R N.$$

Proof. The first equality follows from the computation

$$\begin{aligned}
\operatorname{depth}_R M \otimes^{\mathbf{L}}_R N &= -\sup \mathbf{R}\mathrm{Hom}_R(k, M \otimes^{\mathbf{L}}_R N) \\
&= -\sup \big(\mathbf{R}\mathrm{Hom}_R(k, M) \otimes^{\mathbf{L}}_R N\big) \\
&= -\sup \big(\big(\mathbf{R}\mathrm{Hom}_R(k, M) \otimes^{\mathbf{L}}_k k\big) \otimes^{\mathbf{L}}_R N\big) \\
&= -\sup \big(\mathbf{R}\mathrm{Hom}_R(k, M) \otimes^{\mathbf{L}}_k \big(k \otimes^{\mathbf{L}}_R N\big)\big) \\
&= -\sup \left(\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k, M)) \otimes_k \mathrm{H}(k \otimes^{\mathbf{L}}_R N)\right) \\
&= -(\sup \mathbf{R}\mathrm{Hom}_R(k, M) + \sup \big(k \otimes^{\mathbf{L}}_R N\big)) \\
&= \operatorname{depth}_R M - \sup k \otimes^{\mathbf{L}}_R N,
\end{aligned}$$

where the first and last equalities use the definition of depth (5.2.4), the second is by tensor evaluation (4.4.5), the fourth is by associativity (4.4.2), and the fifth is by Proposition (2.1.19).

Applied to $M = R$, the first equality in the theorem yields the third,

$$\operatorname{depth}_R N = \operatorname{depth}_R R - \sup k \otimes^{\mathbf{L}}_R N,$$

and the second equality follows. $\square$

(5.2.13) **Theorem.** *Let $(R, \mathfrak{m}, k)$ be local. If $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$, then*

$$\operatorname{pd}_R M = -\inf \mathbf{R}\mathrm{Hom}_R(M, k) = \sup k \otimes^{\mathbf{L}}_R M = \operatorname{fd}_R M.$$

Proof. By Remark (5.1.8), Theorem (5.1.9), faithful injectivity of $\mathrm{E}_R(k)$, and Hom-tensor adjointness (2.5.3) there are inequalities

$$\begin{aligned}
\operatorname{pd}_R M &\geqslant \operatorname{fd}_R M \\
&\geqslant \sup k \otimes^{\mathbf{L}}_R M \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(k \otimes^{\mathbf{L}}_R M, \mathrm{E}_R(k)) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(M, \mathrm{Hom}_R(k, \mathrm{E}_R(k))) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(M, k).
\end{aligned}$$

Set $n = -\inf \mathbf{R}\mathrm{Hom}_R(M, k)$. To see that $n \geqslant \operatorname{pd}_R M$, choose by Theorem (3.1.10) a semifree resolution $L \xrightarrow{\simeq} M$ with L_v finitely generated for all v. For every $m \in \mathbb{Z}$ the covariant functor $\mathrm{H}_m(\mathrm{Hom}_R(L, -))$ takes finitely generated modules to finitely generated modules, and it is linear and half-exact. If T is a finitely generated R-module such that $\mathrm{H}_m(\mathrm{Hom}_R(L, T)) = \mathrm{H}_m(\mathbf{R}\mathrm{Hom}_R(M, T)) \neq 0$, then it follows by Lemma (A.2) that $\mathrm{H}_m(\mathrm{Hom}_R(L, k)) = \mathrm{H}_m(\mathbf{R}\mathrm{Hom}_R(M, k)) \neq 0$. The desired equality now follows from Theorem (5.1.3). $\square$

(5.2.14) **Auslander–Buchsbaum Formula.** *Let $(R, \mathfrak{m}, k)$ be a local ring. If $N \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ and $\operatorname{pd}_R N < \infty$, then*

$$\operatorname{pd}_R N = \operatorname{depth}_R R - \operatorname{depth}_R N.$$

Proof. Immediate from Theorem (5.2.12) and Theorem (5.2.13). $\square$

Exercises

(5.2.15) **Definition.** An R-complex M is *contractible* if the identity 1^M is null-homotopic.

(E 5.2.1) Let $(R, \mathfrak{m}, k)$ be local. Prove that under suitable conditions on the complexes, the width of $\mathbf{R}\mathrm{Hom}_R(M, N)$ can be computed in terms of depth and width of M, N, and R.

(E 5.2.2) Prove the Hom-vanishing Lemma.

CHAPTER 6

Supports and Dimensions

6.1. Localization

SUPPORT

(6.1.1) **Definition.** Let M be an R-complex. For a prime ideal $\mathfrak{p}$ of R, the *localization of M at $\mathfrak{p}$* is the complex $M_\mathfrak{p} = R_\mathfrak{p} \otimes_R M$.

The *support of M* is the set

$$\operatorname{Supp}_R M = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \not\simeq 0 \,\}.$$

(6.1.2) **Remark.** If two R-complexes M and M' are isomorphic in $\mathsf{D}(R)$, then so are their localizations at any $\mathfrak{p} \in \operatorname{Spec} R$. Indeed,

$$M_\mathfrak{p} = R_\mathfrak{p} \otimes_R M \simeq R_\mathfrak{p} \otimes_R M' = M'_\mathfrak{p}$$

as $R_\mathfrak{p}$ is a semiflat R-complex. In particular, $\operatorname{Supp}_R M = \operatorname{Supp}_R M'$.

(6.1.3) **Observation.** Let M be an R-complex and $\mathfrak{p}$ a prime ideal of R. It is clear that $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-complex; moreover, there are inequalities

(6.1.3.1) $$\sup M_\mathfrak{p} \leqslant \sup M \qquad \text{and} \qquad \inf M_\mathfrak{p} \geqslant \inf M.$$

By flatness of $R_\mathfrak{p}$ over R there is an isomorphism $\mathrm{H}(M_\mathfrak{p}) \cong \mathrm{H}(M) \otimes_R R_\mathfrak{p}$ of $R_\mathfrak{p}$-complexes, and it follows that

(6.1.3.2) $$\operatorname{Supp}_R M = \bigcup_{v \in \mathbb{Z}} \operatorname{Supp}_R \mathrm{H}_v(M) = \operatorname{Supp}_R \mathrm{H}(M).$$

In particular, $\operatorname{Supp}_R M$ is non-empty if and only if $M \not\simeq 0$.

(6.1.4) **Lemma.** *For R-complexes M and N, and $\mathfrak{p} \in \operatorname{Spec} R$ there are isomorphisms of $R_\mathfrak{p}$-complexes*

$$(M \otimes_R N)_\mathfrak{p} \;\cong\; M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \qquad \text{and} \qquad (M \otimes^{\mathbf{L}}_R N)_\mathfrak{p} \;\simeq\; M_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} N_\mathfrak{p}.$$

Proof. The first isomorphism follows by associativity (2.5.2) and commutativity (2.5.1):

$$\begin{aligned} (M \otimes_R N)_\mathfrak{p} &= R_\mathfrak{p} \otimes_R (M \otimes_R N) \\ &\cong (R_\mathfrak{p} \otimes_R M) \otimes_R N \\ &\cong \big(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}\big) \otimes_R N \\ &\cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}. \end{aligned}$$

The isomorphism in $\mathsf{D}(R_\mathfrak{p})$ follows from the first one, as the localization of a semiflat R-complex is a semiflat $R_\mathfrak{p}$-complex; cf. (E 3.4.1). $\square$

(6.1.5) **Proposition.** *For R-complexes M and N there is an inclusion*

$$\operatorname{Supp}_R M \otimes^{\mathbf{L}}_R N \subseteq \operatorname{Supp}_R M \cap \operatorname{Supp}_R N;$$

equality holds if both complexes belong to $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$.

Proof. The inclusion is immediate from the second isomorphism in Lemma (6.1.4).

Suppose M and N are in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$. If $\mathfrak{p}$ is in $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N$, then the infima $\inf M_\mathfrak{p} = w$ and $\inf N_\mathfrak{p} = t$ are finite. By Lemma (6.1.4) and Lemma (4.3.8) there is an isomorphism $\mathrm{H}_{w+t}((M \otimes^{\mathbf{L}}_R N)_\mathfrak{p}) \cong \mathrm{H}_w(M_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \mathrm{H}_t(N_\mathfrak{p})$; this module is non-zero by NAK, so $\mathfrak{p}$ is in $\operatorname{Supp}_R M \otimes^{\mathbf{L}}_R N$ by (6.1.3.2). $\square$

(6.1.6) **Lemma.** *Let M and N be R-complexes and $\mathfrak{p} \in \operatorname{Spec} R$. If*

- *M is bounded and degreewise finitely generated, or*
- *N is bounded above, M is bounded below and degreewise finitely generated,*

then there is an isomorphism of $R_\mathfrak{p}$-complexes

$$\operatorname{Hom}_R(M,N)_\mathfrak{p} \;\cong\; \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}).$$

Proof. Follows by commutativity (2.5.1), tensor evaluation (2.5.5), and adjointness (2.5.3):

$$\begin{aligned}
\operatorname{Hom}_R(M,N)_\mathfrak{p} &\cong \operatorname{Hom}_R(M,N) \otimes_R R_\mathfrak{p}\\
&\cong \operatorname{Hom}_R(M, N \otimes_R R_\mathfrak{p})\\
&\cong \operatorname{Hom}_R(M, N_\mathfrak{p})\\
&\cong \operatorname{Hom}_R(M, \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, N_\mathfrak{p}))\\
&\cong \operatorname{Hom}_{R_\mathfrak{p}}(M \otimes_R R_\mathfrak{p}, N_\mathfrak{p})\\
&\cong \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}).
\end{aligned}$$ $\square$

(6.1.7) **Lemma.** *Let $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $N \in \mathsf{D}_{\sqsubset}(R)$ be R-complexes. For every $\mathfrak{p} \in \operatorname{Spec} R$ there is an isomorphism of $R_\mathfrak{p}$-complexes*

$$\mathbf{R}\!\operatorname{Hom}_R(M,N)_\mathfrak{p} \;\simeq\; \mathbf{R}\!\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}).$$

Proof. The localization of a semifree R-complex at $\mathfrak{p}$ is a semifree $R_\mathfrak{p}$-complex. The claim now follows from the previous lemma in view of Theorem (3.1.10) and Remark (2.2.6). $\square$

Homological dimensions

(6.1.8) **Observation.** Let $\mathfrak{p}$ be a prime ideal in R. For a semiprojective R-complex P, the natural isomorphisms of functors on $\mathsf{C}(R_\mathfrak{p})$

$$\operatorname{Hom}_{R_\mathfrak{p}}(P_\mathfrak{p}, -) \cong \operatorname{Hom}_{R_\mathfrak{p}}(P \otimes_R R_\mathfrak{p}, -) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, -)) \cong \operatorname{Hom}_R(P, -)$$

show that $P_\mathfrak{p}$ is a semiprojective $R_\mathfrak{p}$-complex. It follows that for every R-complex M there are inequalities

(6.1.8.1) $$\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{pd}_R M \qquad \text{and} \qquad \operatorname{fd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{fd}_R M.$$

It is also straightforward to see that localization of a semifree/-flat R-complex at $\mathfrak{p}$ yields a semifree/-flat $R_\mathfrak{p}$-complex.

If I is a bounded above complex of injective R-modules (and hence semiinjective), then $I_\mathfrak{p}$ is a bounded above complex of injective $R_\mathfrak{p}$-modules, and hence semiinjective. It follows that for R-complexes $M \in \mathsf{D}_\sqsubset(R)$ there are inequalities

$$\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{id}_R M. \tag{6.1.8.2}$$

By Lemmas (6.1.15) and (6.1.19), flat and injective dimension can be computed locally. The next example shows that this fails for projective dimension, even for modules over a regular ring.

(6.1.9) **Example.** Let $\mathbb{P}$ denote the set of prime numbers. Let M be the $\mathbb{Z}$-submodule of $\mathbb{Q}$ generated by $\{ \frac{1}{p} \mid p \in \mathbb{P} \}$. For every $p \in \mathbb{P}$ the module $M_{(p)} = \mathbb{Z}_{(p)} \frac{1}{p}$ is a free $\mathbb{Z}_{(p)}$-module, and $M_0 = \mathbb{Q} = \mathbb{Z}_0$, so $M_\mathfrak{p}$ is free for every $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z} = \{ 0 \} \cup \{ (p) \mid p \in \mathbb{P} \}$, but M is not projective.

(6.1.10) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-complex in $\mathsf{D}^\mathrm{f}_\sqsubset(R)$. If m is an integer and $\mathfrak{p}$ a prime ideal such that $\mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$, then $\mu_R^{m+\dim R/\mathfrak{p}}(M) \neq 0$.*

Proof. Use induction on $n = \dim R/\mathfrak{p}$.

$n = 1$: Set $R' = R/\mathfrak{p}$. By assumption

$$\begin{aligned} 0 \neq \mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) &= \operatorname{rank}_{k(\mathfrak{p})} \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})) \\ &= \operatorname{rank}_{k(\mathfrak{p})} \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(R', M))_\mathfrak{p}, \end{aligned}$$

where the second equality uses Lemma (6.1.6) and flatness of $R_\mathfrak{p}$ over R. In particular, $\mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(R', M)) \neq 0$. Choose an element $x \in \mathfrak{m} \setminus \mathfrak{p}$. The short exact sequence $0 \longrightarrow R' \xrightarrow{x} R' \longrightarrow R'/(x) \longrightarrow 0$ yields an exact sequence

$$\mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(R', M)) \xrightarrow{x} \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(R', M)) \longrightarrow \mathrm{H}_{-m-1}(\mathbf{R}\mathrm{Hom}_R(R'/(x), M)),$$

and it follows by NAK that $\mathrm{H}_{-m-1}(\mathbf{R}\mathrm{Hom}_R(R'/(x), M)) \neq 0$. As $\operatorname{Supp}_R R'/(x) = \{ \mathfrak{m} \}$ it follows by Lemma (A.2) that also $\mathrm{H}_{-m-1}(\mathbf{R}\mathrm{Hom}_R(k, M)) \neq 0$, so $\mu_R^{m+1}(M)$, which is the rank of this k-space, is non-zero.

$n > 1$: Choose a maximal chain of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{m}$. Set $S = R_{\mathfrak{p}_1}$ and note that $M_{\mathfrak{p}_1}$ belongs to $\mathsf{D}^\mathrm{f}_\sqsubset(S)$. Set $\mathfrak{q} = \mathfrak{p}_{\mathfrak{p}_1}$ and note that $\mu^m_{S_\mathfrak{q}}((M_{\mathfrak{p}_1})_\mathfrak{q}) = \mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. As $\dim S/\mathfrak{q} = 1$ the induction base yields $\mu_S^{m+1}(M_{\mathfrak{p}_1}) \neq 0$, and since $\dim R/\mathfrak{p}_1 = n - 1$ it follows by the induction hypothesis that $\mu_R^{m+n}(M) \neq 0$. □

(6.1.11) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a local ring and $M \in \mathsf{D}^\mathrm{f}_\sqsubset(R)$. For every $\mathfrak{p} \in \operatorname{Spec} R$ there is an inequality*

$$\operatorname{depth}_R M \leqslant \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}.$$

Proof. Immediate from Definition (5.2.4) and Lemma (6.1.10). □

(6.1.12) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a local ring and $M \in \mathsf{D}^\mathrm{f}_\sqsubset(R)$. For every $\mathfrak{p} \in \operatorname{Spec} R$ there is an inequality*

$$\operatorname{id}_R M \geqslant \operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}.$$

Proof. Immediate from Lemma (6.1.10) and Lemma (6.1.19). □

(6.1.13) **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring. If $M \in \mathsf{D}^\mathrm{f}_\sqsubset(R)$, then*

$$\operatorname{id}_R M = -\inf \mathbf{R}\mathrm{Hom}_R(k, M).$$

Proof. Immediate by Lemmas (6.1.19) and (6.1.10). ☐

Chouinard Formulas

(6.1.14) **Definition.** Let $(R, \mathfrak{m}, k)$ be a local ring. For an R-complex M and $m \in \mathbb{Z}$ the *mth Betti number of M* is

$$\beta_m^R(M) \;=\; \operatorname{rank}_k \mathrm{H}_m(k \otimes_R^{\mathbf{L}} M).$$

(6.1.15) **Lemma.** *For every R-complex M there is an equality*

$$\operatorname{fd}_R M = \sup\{\, m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon \beta_m^{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0 \,\}.$$

Proof. The inequality "$\geqslant$" is immediate as

$$\beta_m^{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{rank}_{k(\mathfrak{p})} \mathrm{H}_m(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} M_\mathfrak{p}) = \operatorname{rank}_{k(\mathfrak{p})} \mathrm{H}_m((R/\mathfrak{p} \otimes_R^{\mathbf{L}} M))_\mathfrak{p}.$$

by Lemma (6.1.4) and flatness of $R_\mathfrak{p}$ over R.

For the opposite inequality, let $n \leqslant \operatorname{fd}_R M$ be given; it suffices to prove the existence of an integer $m \geqslant n$ and a prime ideal $\mathfrak{q}$ such that $\beta_m^{R_\mathfrak{q}}(M_\mathfrak{q}) \neq 0$. Set

$$\mathrm{F}(-) = \coprod_{m \geqslant n} \mathrm{H}_m(- \otimes_R^{\mathbf{L}} M),$$

this defines a half-exact functor on R-modules. For every finitely generated R-module T and every T-regular element x the short exact sequence $0 \longrightarrow T \xrightarrow{x} T \longrightarrow T/xT \longrightarrow 0$ induces exact sequences

$$\mathrm{H}_{v+1}(T/xT \otimes_R^{\mathbf{L}} M) \longrightarrow \mathrm{H}_v(T \otimes_R^{\mathbf{L}} M) \xrightarrow{x} \mathrm{H}_v(T \otimes_R^{\mathbf{L}} M)$$

which combine to yield an exact sequence

$$\mathrm{F}(T/xT) \longrightarrow \mathrm{F}(T) \xrightarrow{x} \mathrm{F}(T).$$

By Theorem (5.1.9) there exists a finitely generated R-module T such that $\mathrm{F}(T) \neq 0$, so by Lemma (A.3) there is a $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathrm{F}(R/\mathfrak{q})_\mathfrak{q} \neq 0$. Localization (tensor product) commutes with coproducts, so there is an $m \geqslant n$ such that

$$0 \neq \mathrm{H}_m(R/\mathfrak{q} \otimes_R^{\mathbf{L}} M)_\mathfrak{q} \cong \mathrm{H}_m(k(\mathfrak{q}) \otimes_{R_\mathfrak{q}}^{\mathbf{L}} M_\mathfrak{q}),$$

where the isomorphism is by flatness of $R_\mathfrak{q}$ over R and Lemma (6.1.4). ☐

(6.1.16) **Theorem.** *Let M be an R-complex; if $\operatorname{fd}_R M < \infty$, then*

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\, \sup\,(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R \,\}\\ &= \sup\{\, \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}.\end{aligned}$$

(6.1.17) **Proof.** The first and last equalities below are by Lemma (6.1.15) and Theorem (5.2.12), respectively.

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\, m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon \beta_m^{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0 \,\}\\ &= \sup\{\, m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\ \mathrm{H}_m(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} M_\mathfrak{p}) \neq 0 \,\}\\ &= \sup\{\, \sup\,(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R \,\}\\ &= \sup\{\, \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}.\end{aligned}$$ ☐

(6.1.18) **Definition.** Let $(R, \mathfrak{m}, k)$ be a local ring. For an R-complex M and $m \in \mathbb{Z}$ the *mth Bass number of M* is

$$\mu_R^m(M) \;=\; \operatorname{rank}_k \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(k, M)).$$

(6.1.19) **Lemma.** *For every R-complex $M \in \mathsf{D}_{\sqsubset}(R)$ there is an equality*

$$\mathrm{id}_R M = \sup\{\, m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon \mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0 \,\}.$$

Proof. The inequality "$\geqslant$" is immediate as

$$\begin{aligned}\mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) &= \operatorname{rank}_{k(\mathfrak{p})} \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}))\\ &= \operatorname{rank}_{k(\mathfrak{p})} \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M))_\mathfrak{p},\end{aligned}$$

by Lemma (6.1.6) and flatness of $R_\mathfrak{p}$ over R.

For the opposite inequality, let T be a finitely generated R-module and n an integer such that $\mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_R(T, M)) \neq 0$. It suffices to prove the existence of a prime ideal $\mathfrak{q}$ such that $\mu^n_{R_\mathfrak{q}}(M_\mathfrak{q}) \neq 0$. The functor $\mathrm{F}(-) = \mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_R(-, M))$ is contravariant, half-exact, and linear, so it follows by Lemma (A.3) that there exists a prime ideal $\mathfrak{q}$ such that $0 \neq \mathrm{F}(R/\mathfrak{q})_\mathfrak{q} \cong \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{q}}(k(\mathfrak{q}), M_\mathfrak{q}))$, where the isomorphism uses flatness of $R_\mathfrak{q}$ over R and Lemma (6.1.6). □

(6.1.20) **Theorem.** *Let M be an R-complex in $\mathsf{D}_{\sqsubset}(R)$; if $\mathrm{id}_R M < \infty$, then*

$$\begin{aligned}\mathrm{id}_R M &= \sup\{\, -\inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R \,\}\\ &= \sup\{\, \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}.\end{aligned}$$

(6.1.21) **Proof.** The first equality is by Lemma (6.1.19) and the last one is by (E 5.2.1):

$$\begin{aligned}\mathrm{id}_R M &= \sup\{\, m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon \mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0 \,\}\\ &= \sup\{\, m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\ \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})) \neq 0 \,\}\\ &= \sup\{\, -\inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R \,\}\\ &= \sup\{\, \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}.\end{aligned}$$ □

6.2. Formal invariants

(6.2.1) **Setup.** In this section $(R, \mathfrak{m}, k)$ is a local ring.

Poincaré series

(6.2.2) **Definition.** For an R-complex $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ the *Poincaré series* is

$$\mathrm{P}^R_M(t) = \sum_{i \in \mathbb{Z}} \beta^R_i(M) t^i.$$

(6.2.3) **Remark.** The Poincaré series of an R-complex $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ belongs to $\mathbb{Z}(\!|t|\!)$, the domain of formal Laurent series with integer coefficients.

(6.2.4) **Observation.** Let M be an R-complex in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$. By Theorem (5.2.13) and Observation (5.2.2) the degree and order of the Poincaré series $\mathrm{P}^R_M(t)$ are

$$\begin{aligned}&\deg \mathrm{P}^R_M(t) = \sup\{\, m \in \mathbb{Z} \mid \beta^R_m(M) \neq 0 \,\} = \mathrm{pd}_R M = \mathrm{fd}_R M \quad \text{and}\\ &\operatorname{ord} \mathrm{P}^R_M(t) = \inf\{\, m \in \mathbb{Z} \mid \beta^R_m(M) \neq 0 \,\} = \inf M.\end{aligned}$$

(6.2.5) **Lemma.** *For R-complexes M and N in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ there is an equality of formal Laurent series*

$$\mathrm{P}^R_{M\otimes^{\mathbf{L}}_R N}(t) = \mathrm{P}^R_M(t)\,\mathrm{P}^R_N(t).$$

In particular,

$$\mathrm{pd}_R\, M \otimes^{\mathbf{L}}_R N = \mathrm{pd}_R\, M + \mathrm{pd}_R\, N.$$

Proof. By commutativity (4.4.1), associativity (4.4.2), and Proposition (2.1.19) there are isomorphisms in $\mathsf{D}(k)$

$$k \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R N) \simeq (M \otimes^{\mathbf{L}}_R k) \otimes^{\mathbf{L}}_k (k \otimes^{\mathbf{L}}_R N) \simeq \mathrm{H}(M \otimes^{\mathbf{L}}_R k) \otimes_k \mathrm{H}(k \otimes^{\mathbf{L}}_R N).$$

For every $m \in \mathbb{Z}$ there are, therefore, equalities

$$\begin{aligned}\beta^R_m(M \otimes^{\mathbf{L}}_R N) &= \mathrm{rank}_k\, \mathrm{H}_m(k \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R N))\\ &= \mathrm{rank}_k\, \mathrm{H}_m(\mathrm{H}(M \otimes^{\mathbf{L}}_R k) \otimes_k \mathrm{H}(k \otimes^{\mathbf{L}}_R N))\\ &= \sum_{i\in\mathbb{Z}} \mathrm{rank}_k\, \mathrm{H}_i(k \otimes^{\mathbf{L}}_R M)\, \mathrm{rank}_k\, \mathrm{H}_{m-i}(k \otimes^{\mathbf{L}}_R N)\\ &= \sum_{i+j=m} \beta^R_i(M)\beta^R_j(N),\end{aligned}$$

and this is the degree m coefficient of the product series $\mathrm{P}^R_M(t)\,\mathrm{P}^R_N(t)$.

The statement about projective dimensions follows from Observation (6.2.4). □

The next corollary is immediate in view of (E 4.4.1).

(6.2.6) **Corollary.** *If M and N are finitely generated R-modules with $\mathrm{Tor}^R_{i>0}(M,N) = 0$, then $\mathrm{pd}_R\, M \otimes_R N = \mathrm{pd}_R\, M + \mathrm{pd}_R\, N$.* □

Bass series

(6.2.7) **Definition.** For an R-complex $M \in \mathsf{D}^{\mathrm{f}}_{\sqsubset}(R)$ the *Bass series* is

$$\mathrm{I}^M_R(t) = \sum_{i\in\mathbb{Z}} \mu^i_R(M)t^i.$$

(6.2.8) **Remark.** The Bass series of an R-complex $M \in \mathsf{D}^{\mathrm{f}}_{\sqsubset}(R)$ belongs to $\mathbb{Z}(\!|t|\!)$, the domain of formal Laurent series with integer coefficients.

(6.2.9) **Observation.** Let M be an R-complex in $\mathsf{D}^{\mathrm{f}}_{\sqsubset}(R)$. By Theorem (6.1.13) and Definition (5.2.4) the degree and order of the Bass series $\mathrm{I}^M_R(t)$ are

$$\begin{aligned}\deg \mathrm{I}^M_R(t) &= \sup\{\, m \in \mathbb{Z} \mid \mu^m_R(M) \neq 0 \,\} = \mathrm{id}_R\, M \quad \text{and}\\ \mathrm{ord}\, \mathrm{I}^M_R(t) &= \inf\{\, m \in \mathbb{Z} \mid \mu^m_R(M) \neq 0 \,\} = \mathrm{depth}_R\, M.\end{aligned}$$

(6.2.10) **Lemma.** *For R-complexes $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $N \in \mathsf{D}^{\mathrm{f}}_{\sqsubset}(R)$ there is an equality of formal Laurent series*

$$\mathrm{I}^{\mathbf{R}\mathrm{Hom}_R(M,N)}_R(t) = \mathrm{P}^R_M(t)\,\mathrm{I}^N_R(t).$$

In particular,

$$\mathrm{id}_R\, \mathbf{R}\mathrm{Hom}_R(M,N) = \mathrm{pd}_R\, M + \mathrm{id}_R\, N.$$

Proof. By adjointness (4.4.3), commutativity (4.4.1), and Proposition (2.1.19) there are isomorphisms in $\mathsf{D}(k)$:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(M,N)) &\simeq \mathbf{R}\mathrm{Hom}_R(k \otimes^{\mathbf{L}}_R M, N)\\ &\simeq \mathbf{R}\mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R M, \mathbf{R}\mathrm{Hom}_R(k,N))\\ &\simeq \mathrm{Hom}_k(\mathrm{H}(k \otimes^{\mathbf{L}}_R M), \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,N))).\end{aligned}$$

In particular, for every $m \in \mathbb{Z}$ there are equalities

$$\begin{aligned}\mu^m_R(\mathbf{R}\mathrm{Hom}_R(M,N)) &= \mathrm{rank}_k\, \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(M,N)))\\ &= \mathrm{rank}_k\, \mathrm{H}_{-m}(\mathrm{Hom}_k(\mathrm{H}(k \otimes^{\mathbf{L}}_R M), \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,N))))\\ &= \sum_{i\in\mathbb{Z}} \mathrm{rank}_k\, \mathrm{H}_i(k \otimes^{\mathbf{L}}_R M)\, \mathrm{rank}_k\, \mathrm{H}_{i-m}(\mathbf{R}\mathrm{Hom}_R(k,N))\\ &= \sum_{i+j=m} \beta^R_i(M)\mu^j_R(N),\end{aligned}$$

and this is the degree m coefficient of the product series $\mathrm{P}^R_M(t)\,\mathrm{I}^N_R(t)$.

The statement about homological dimensions follows from Observations (6.2.4) and (6.2.9). □

The next corollary is immediate in view of (E 4.4.1).

(6.2.11) **Corollary.** *If M and N are finitely generated R-modules with $\mathrm{Ext}^{i>0}_R(M,N) = 0$, then $\mathrm{id}_R\,\mathrm{Hom}_R(M,N) = \mathrm{pd}_R\,M + \mathrm{id}_R\,N$.* □

(6.2.12) **Lemma.** *Let M and N be R-complexes in $\mathsf{D}^{\mathrm{f}}_{\square}(R)$. If $\mathrm{id}_R\,N < \infty$, then there is an equality of formal Laurent series:*

$$\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = \mathrm{I}^M_R(t)\,\mathrm{I}^N_R(t^{-1}).$$

In particular,

$$\begin{aligned}\mathrm{pd}_R\,\mathbf{R}\mathrm{Hom}_R(M,N) &= \mathrm{id}_R\,M - \mathrm{depth}_R\,N \quad \textit{and}\\ \inf \mathbf{R}\mathrm{Hom}_R(M,N) &= \mathrm{depth}_R\,M - \mathrm{id}_R\,N.\end{aligned}$$

Proof. The next chain of isomorphisms in $\mathsf{D}(k)$ are by homomorphism evaluation (4.4.6), adjointness (4.4.3), commutativity (4.4.1), and Proposition (2.1.19).

$$\begin{aligned}k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,N) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k,M),N)\\ &\simeq \mathbf{R}\mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_R(k,M), \mathbf{R}\mathrm{Hom}_R(k,N))\\ &\simeq \mathrm{Hom}_k(\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,M)), \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,N)))\end{aligned}$$

For every $m \in \mathbb{Z}$ this gives equalities:

$$\begin{aligned}&\beta^R_m(\mathbf{R}\mathrm{Hom}_R(M,N))\\ &\quad= \mathrm{rank}_k\, \mathrm{H}_m(k \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,N))\\ &\quad= \mathrm{rank}_k\, \mathrm{H}_m(\mathrm{Hom}_k(\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,M)), \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,N))))\\ &\quad= \sum_{i\in\mathbb{Z}} \mathrm{rank}_k\, \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k,M))\, \mathrm{rank}_k\, \mathrm{H}_{-i+m}(\mathbf{R}\mathrm{Hom}_R(k,N))\\ &\quad= \sum_{i+j=m} \mu^i_R(M)\mu^{-j}_R(N),\end{aligned}$$

which is the degree m coefficient in the product series $\mathrm{I}_R^M(t)\,\mathrm{I}_R^N(t^{-1})$.

The last two statements follow by Observations (6.2.4) and (6.2.9):

$$\begin{aligned}
\operatorname{pd}_R \mathbf{R}\mathrm{Hom}_R(M,N) &= \deg \mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(M,N)}(t)\\
&= \deg \mathrm{I}_R^M(t) + \deg \mathrm{I}_R^N(t^{-1})\\
&= \deg \mathrm{I}_R^M(t) - \operatorname{ord} \mathrm{I}_R^N(t)\\
&= \operatorname{id}_R M - \operatorname{depth}_R N,
\end{aligned}$$

and

$$\begin{aligned}
\inf \mathbf{R}\mathrm{Hom}_R(M,N) &= \operatorname{ord} \mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(M,N)}(t)\\
&= \operatorname{ord} \mathrm{I}_R^M(t) + \operatorname{ord} \mathrm{I}_R^N(t^{-1})\\
&= \operatorname{ord} \mathrm{I}_R^M(t) - \deg \mathrm{I}_R^N(t)\\
&= \operatorname{depth}_R M - \operatorname{id}_R N. \qquad \square
\end{aligned}$$

(6.2.13) **Bass Formula.** *Let M be an R-complex in $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$. If $\operatorname{id}_R M < \infty$, then*

$$\operatorname{id}_R M = \operatorname{depth} R - \inf M.$$

Proof. By Lemma (6.2.12) there are equalities:

$$\inf M = \inf \mathbf{R}\mathrm{Hom}_R(R,M) = \operatorname{depth} R - \operatorname{id}_R M. \qquad \square$$

(6.2.14) **Lemma.** *Let M and N be R-complexes in $\mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$. If $\operatorname{pd}_R M < \infty$, then there is an equality of formal Laurent series:*

$$\mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = \mathrm{P}^R_M(t^{-1})\,\mathrm{P}^R_N(t).$$

In particular,

$$\begin{aligned}
\operatorname{pd}_R \mathbf{R}\mathrm{Hom}_R(M,N) &= \operatorname{pd}_R N - \inf M \quad \textit{and}\\
\inf \mathbf{R}\mathrm{Hom}_R(M,N) &= \inf N - \operatorname{pd}_R M.
\end{aligned}$$

Proof. See (E 6.2.2). $\square$

(6.2.15) **Lemma.** *Let M and N be R-complexes in $\mathsf{D}^{\mathsf{f}}_{\square}(R)$. If $\operatorname{pd}_R N < \infty$, then there is an equality of formal Laurent series:*

$$\mathrm{I}_R^{M\otimes^{\mathbf{L}}_R N}(t) = \mathrm{I}_R^M(t)\,\mathrm{P}^R_N(t^{-1}).$$

In particular,

$$\begin{aligned}
\operatorname{id}_R M \otimes^{\mathbf{L}}_R N &= \operatorname{id}_R M - \inf N \quad \textit{and}\\
\operatorname{depth}_R M \otimes^{\mathbf{L}}_R N &= \operatorname{depth}_R M - \operatorname{pd}_R N.
\end{aligned}$$

Proof. The next chain of isomorphisms in $\mathsf{D}(k)$ are by tensor evaluation (4.4.5), associativity (4.4.2), and Proposition (2.1.19).

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(k, M\otimes^{\mathbf{L}}_R N) &\simeq \mathbf{R}\mathrm{Hom}_R(k,M)\otimes^{\mathbf{L}}_R N\\
&\simeq \mathbf{R}\mathrm{Hom}_R(k,M)\otimes^{\mathbf{L}}_k (k\otimes^{\mathbf{L}}_R N)\\
&\simeq \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,M))\otimes_k \mathrm{H}(k\otimes^{\mathbf{L}}_R N)
\end{aligned}$$

For every $m \in \mathbb{Z}$ this gives equalities:

$$\begin{aligned}
\mu_R^m(M \otimes_R^{\mathbf{L}} N) &= \operatorname{rank}_k \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(k, M \otimes_R^{\mathbf{L}} N)) \\
&= \operatorname{rank}_k \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(k, M)) \otimes_k \mathrm{H}(k \otimes_R^{\mathbf{L}} N) \\
&= \sum_{i \in \mathbb{Z}} \operatorname{rank}_k \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k, M)) \operatorname{rank}_k \mathrm{H}_{-m+i}(k \otimes_R^{\mathbf{L}} N) \\
&= \sum_{i+j=m} \mu_R^i(M) \beta_{-j}^R(N),
\end{aligned}$$

which is the degree m coefficient in the product series $\mathrm{I}_R^M(t)\,\mathrm{P}_N^R(t^{-1})$.

The last two statements follow by Observations (6.2.4) and (6.2.9):

$$\begin{aligned}
\operatorname{id}_R M \otimes_R^{\mathbf{L}} N &= \deg \mathrm{I}_R^{M \otimes_R^{\mathbf{L}} N}(t) \\
&= \deg \mathrm{I}_R^M(t) + \deg \mathrm{P}_N^R(t^{-1}) \\
&= \deg \mathrm{I}_R^M(t) - \operatorname{ord} \mathrm{P}_N^R(t) \\
&= \operatorname{id}_R M - \inf N,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{depth}_R M \otimes_R^{\mathbf{L}} N &= \operatorname{ord} \mathrm{I}_R^{M \otimes_R^{\mathbf{L}} N}(t) \\
&= \operatorname{ord} \mathrm{I}_R^M(t) + \operatorname{ord} \mathrm{P}_N^R(t^{-1}) \\
&= \operatorname{ord} \mathrm{I}_R^M(t) - \deg \mathrm{P}_N^R(t) \\
&= \operatorname{depth}_R M - \operatorname{pd}_R N.
\end{aligned}$$ □

EXERCISES

(E 6.2.1) Let $M \in \mathsf{D}_{\sqsupset}^{\mathrm{f}}(R)$. Show that if $\operatorname{pd}_R M < \infty$, then

$$\beta_m^R(\mathbf{R}\mathrm{Hom}_R(M, R)) = \beta_{-m}^R(M)$$

for all $m \in \mathbb{Z}$.

(E 6.2.2) Prove Lemma (6.2.14).

(E 6.2.3) Derive the Auslander–Buchsbaum and Bass formulas (5.2.14) and (6.2.13) from the Chouinard formulas (6.1.16) and (6.1.20).

6.3. Krull dimension

(6.3.1) **Definition.** The (Krull) *dimension* of an R-complex M is

$$\dim_R M \;=\; \sup\{\, \dim R/\mathfrak{p} - \inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R M \,\}.$$

(6.3.2) **Remark.** If M is an R-module, then

$$\begin{aligned}
\dim_R M &= \sup\{\, \dim R/\mathfrak{p} - \inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R M \,\} \\
&= \sup\{\, \dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M \,\}
\end{aligned}$$

is the usual Krull dimension.

(6.3.3) **Observation.** For every R-complex M there are inequalities

(6.3.3.1) $$-\inf M \leqslant \dim_R M \leqslant \dim R - \inf M.$$

Indeed, $\dim_R M \geqslant \sup_{\mathfrak{p}}\{-\inf M_{\mathfrak{p}}\} = -\inf_{\mathfrak{p}}\{\inf M_{\mathfrak{p}}\} = -\inf M$ and $\dim_R M \leqslant \sup_{\mathfrak{p}}\{\dim R - \inf M_{\mathfrak{p}}\} = \dim R - \inf M$.

It is also clear from the definition and Observation (6.1.3) that

$$\dim_R M = -\infty \iff \operatorname{Supp}_R M = \emptyset \iff \mathrm{H}(M) = 0. \tag{6.3.3.2}$$

(6.3.4) **Lemma.** *Let M be an R-complex. For every $\mathfrak{q} \in \operatorname{Spec} R$ there is an inequality*

$$\dim_R M \geqslant \dim_{R_{\mathfrak{q}}} M_{\mathfrak{q}} + \dim R/\mathfrak{q}.$$

Proof. A straightforward computation:

$$\begin{aligned}
&\dim_{R_{\mathfrak{q}}} M_{\mathfrak{q}} + \dim R/\mathfrak{q} \\
&\qquad = \sup\{\dim R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}} - \inf (M_{\mathfrak{q}})_{\mathfrak{p}_{\mathfrak{q}}} + \dim R/\mathfrak{q} \mid \mathfrak{p} \in \operatorname{Supp}_R M \text{ and } \mathfrak{p} \subseteq \mathfrak{q}\} \\
&\qquad \leqslant \sup\{\dim R/\mathfrak{p} - \inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R M \text{ and } \mathfrak{p} \subseteq \mathfrak{q}\} \\
&\qquad \leqslant \dim_R M. \qquad \square
\end{aligned}$$

(6.3.5) **Lemma.** *For every R-complex M there are equalities*

$$\dim_R M = \dim_R \mathrm{H}(M) = \sup\{\dim_R \mathrm{H}_n(M) - n \mid n \in \mathbb{Z}\}.$$

Proof. The first equality is immediate from the definition and (6.1.3.2). For the second equality, we may assume that $\mathrm{H}(M)$ is bounded below and not zero.

"$\leqslant$": Let $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}(M)$ and set $n = \inf \mathrm{H}(M)_{\mathfrak{p}}$. Since $\mathfrak{p}$ is in the support of the module $\mathrm{H}_n(M)$, there is an inequality $\dim R/\mathfrak{p} - \inf \mathrm{H}(M)_{\mathfrak{p}} \leqslant \dim_R \mathrm{H}_n(M) - n$. The desired inequality follows from this one.

"$\geqslant$": Let $n \in \mathbb{Z}$. If $\mathrm{H}_n(M) = 0$ the inequality

$$\dim R/\mathfrak{p} - \inf \mathrm{H}(M)_{\mathfrak{p}} \geqslant \dim_R \mathrm{H}_n(M) - n \tag{1}$$

holds for every $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}(M)$. If $\mathrm{H}_n(M) \neq 0$, choose $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}_n(M)$ such that $\dim R/\mathfrak{p} = \dim_R \mathrm{H}_n(M)$, then (1) holds as $\inf \mathrm{H}(M)_{\mathfrak{p}} \leqslant n$. $\square$

Cohen–Macaulay defect

(6.3.6) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be local and M be an R-complex in $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$. If $\mathrm{H}(M) \neq 0$, then there is an inequality*

$$\dim_R M \geqslant \operatorname{depth}_R M.$$

Proof. Set $s = \sup M$ and choose $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}_s(M)$ such that $\dim R/\mathfrak{p} = \dim_R \mathrm{H}_s(M)$. It follows that the maximal ideal of $R_{\mathfrak{p}}$ is associated to $\mathrm{H}_s(M_{\mathfrak{p}})$, so $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = -s$ by Observation (5.2.5). Lemma (6.1.11) and Lemma (6.3.5) now yield

$$\operatorname{depth}_R M \leqslant \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} = \dim_R \mathrm{H}_s(M) - s \leqslant \dim_R M. \qquad \square$$

(6.3.7) **Definition.** For an R-complex M the *Cohen–Macaulay defect* is

$$\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M.$$

(6.3.8) **Observation.** Let M be an R-complex in $\mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$. If $\mathrm{H}(M) \neq 0$, then it follows from Lemma (6.3.6) that the Cohen–Macaulay defect is non-negative, i.e. $\operatorname{cmd}_R M \geqslant 0$.

(6.3.9) **Proposition.** *Let $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $N \in \mathsf{D}_{\sqsupset}(R)$ be R-complexes and H a finitely generated R-module. There are equalities*

(a) $$\dim_R H \otimes^{\mathbf{L}}_R N = \dim_R R/\operatorname{Ann}_R H \otimes^{\mathbf{L}}_R N \quad \textit{and}$$

(b) $$\dim_R M \otimes^{\mathbf{L}}_R N = \sup\{\, \dim_R \mathrm{H}_n(M) \otimes^{\mathbf{L}}_R N - n \mid n \in \mathbb{Z} \,\}.$$

Proof. (a): A straightforward computation based on Lemma (5.2.11):

$$\begin{aligned} \dim_R H \otimes^{\mathbf{L}}_R N &= \sup\{\, \dim R/\mathfrak{p} - \inf\,(H \otimes^{\mathbf{L}}_R N)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\ &= \sup\{\, \dim R/\mathfrak{p} - \inf\,(C \otimes^{\mathbf{L}}_R N)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\ &= \dim_R H \otimes^{\mathbf{L}}_R N. \end{aligned}$$

(b): A straightforward computation based on Lemma (5.2.11):

$$\begin{aligned} \dim_R M \otimes^{\mathbf{L}}_R N &= \sup\{\, \dim R/\mathfrak{p} - \inf\,(M \otimes^{\mathbf{L}}_R N)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\ &= \sup\{\, \dim R/\mathfrak{p} - \inf M_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\ &= \sup\{\, \dim R/\mathfrak{p} - \inf\{\, n + \inf \mathrm{H}_n(M_{\mathfrak{p}}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid n \in \mathbb{Z} \,\} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\ &= \sup\{\, \dim R/\mathfrak{p} + \sup\{\, -\inf \mathrm{H}_n(M_{\mathfrak{p}}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} - n \mid n \in \mathbb{Z} \,\} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\ &= \sup\{\, \dim R/\mathfrak{p} - \inf \mathrm{H}_n(M_{\mathfrak{p}}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} - n \mid \mathfrak{p} \in \operatorname{Spec} R,\ n \in \mathbb{Z} \,\} \\ &= \sup\{\, \dim_R \mathrm{H}_n(M) \otimes^{\mathbf{L}}_R N - n \mid n \in \mathbb{Z} \,\}. \qquad \square \end{aligned}$$

Exercises

(E 6.3.1) For finitely generated R-modules M and N, prove that

$$\dim_R M \otimes_R N = \dim_R M \otimes^{\mathbf{L}}_R N.$$

(E 6.3.2) Let $(R, \mathfrak{m}, k)$ be a local ring. For an R-complex $M \in \mathsf{D}^{\mathrm{f}}_{\sqsubset}(R)$, show that if $\mathrm{H}(M) \neq 0$, then

$$\operatorname{depth}_R M + \sup M \leqslant \dim R.$$

(E 6.3.3) Let $(R, \mathfrak{m}, k)$ be a local ring. Under suitable assumptions on $M \in \mathsf{D}(R)$, show that

$$\operatorname{depth}_R M + \operatorname{width}_R M \leqslant \dim R.$$

Hint: Consider the complex $K \otimes_R M$, where K is a Koszul complex on a system of parameters.

6.4. Small support

(6.4.1) **Definition.** Let M be an R-complex. The *small support of* M is the set

$$\operatorname{supp}_R M = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R M) \neq 0 \,\}.$$

(6.4.2) **Observation.** For every R-complex M and every prime ideal $\mathfrak{p}$ associativity (4.4.2) and Lemma (6.1.4) yield isomorphisms

(6.4.2.1) $$k(\mathfrak{p}) \otimes^{\mathbf{L}}_R M \cong (R/\mathfrak{p} \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}} \cong k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}.$$

In particular,

(6.4.2.2) $$\operatorname{supp}_R M = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \exists\, m \in \mathbb{Z}\colon \beta^{R_{\mathfrak{p}}}_m(M_{\mathfrak{p}}) \neq 0 \,\},$$

and it follows by Lemma (6.1.15) and Remark (5.1.8) that

$$\operatorname{supp}_R M = \emptyset \iff \mathrm{H}(M) = 0. \tag{6.4.2.3}$$

(6.4.3) **Lemma.** *For every R-complex M there is an inclusion of sets $\operatorname{supp}_R M \subseteq \operatorname{Supp}_R M$; equality holds if $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$.*

Proof. The inclusion is immediate by (6.4.2.1). If $M \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$, then $M_\mathfrak{p} \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R_\mathfrak{p})$ and $\mathrm{H}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} M_\mathfrak{p}) \neq 0$ by NAK; cf. Observation (5.2.2). □

(6.4.4) **Lemma.** *Let M and N be R-complexes. There is an equality of sets*

$$\operatorname{supp}_R M \otimes^{\mathbf{L}}_R N = \operatorname{supp}_R M \cap \operatorname{supp}_R N.$$

Proof. For every $\mathfrak{p} \in \operatorname{Spec} R$ there are isomorphisms

$$\begin{aligned} k(\mathfrak{p}) \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R N) &\simeq k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (M_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} N_\mathfrak{p}) \\ &\simeq (M_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p})) \otimes^{\mathbf{L}}_{k(\mathfrak{p})} (k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} N_\mathfrak{p}) \\ &\simeq \mathrm{H}(M_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} \mathrm{H}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} N_\mathfrak{p}), \end{aligned}$$

by (6.4.2.1), associativity (4.4.2), commutativity (4.4.1), and Proposition (2.1.19). It follows that for every $\mathfrak{p} \in \operatorname{Spec} R$ and every $m \in \mathbb{Z}$

$$\mathrm{H}_m(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R N)) \cong \coprod_{i\in\mathbb{Z}} \mathrm{H}_i(M_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} \mathrm{H}_{m-i}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} N_\mathfrak{p}).$$

Now the claim follows by (6.4.2.2). □

Finiteness of width and depth

(6.4.5) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a local ring and K be the Koszul complex on a set of generators for $\mathfrak{m}$. For an R-complex M, the following conditions are equivalent:*

- (*i*) $\mathrm{H}(k \otimes^{\mathbf{L}}_R M) = 0$;
- (*ii*) $\mathrm{H}(K \otimes_R M) = 0$;
- (*iii*) $\mathrm{H}(\operatorname{Hom}_R(K, M)) = 0$;
- (*iv*) $\mathrm{H}(\mathbf{R}\operatorname{Hom}_R(k, M)) = 0$.

Proof. (*i*) $\iff$ (*ii*): Let $x_1, \dots, x_e$ be a set of generators for $\mathfrak{m}$ and set $K = \mathrm{K}^R(x_1, \dots, x_e)$. Note that $\operatorname{supp}_R K = \operatorname{Supp}_R K = \{\,\mathfrak{m}\,\}$. By Lemma (6.4.4),

$$\mathrm{H}(k \otimes^{\mathbf{L}}_R M) = 0 \iff \mathfrak{m} \notin \operatorname{supp}_R M \iff \mathrm{H}(K \otimes^{\mathbf{L}}_R M) = 0.$$

(*ii*) $\iff$ (*iii*): Up to a shift, the complexes $K \otimes_R M$ and $\operatorname{Hom}_R(K, M)$ are isomorphic. Indeed, for an elementary Koszul complex

$$\mathrm{K}^R(x) = 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

it is clear that $\operatorname{Hom}_R(\mathrm{K}^R(x), R) \cong \Sigma^{-1}\mathrm{K}^R(x)$. By induction,

$$\begin{aligned} \operatorname{Hom}_R(\mathrm{K}^R(x_1, \dots, x_e), R) &\cong \operatorname{Hom}_R(\mathrm{K}^R(x_1, \dots, x_{e-1}) \otimes_R \mathrm{K}^R(x_e), R) \\ &\cong \operatorname{Hom}_R(\mathrm{K}^R(x_1, \dots, x_{e-1}), \operatorname{Hom}_R(\mathrm{K}^R(x_e), R)) \\ &\cong \operatorname{Hom}_R(\mathrm{K}^R(x_1, \dots, x_{e-1}), R) \otimes_R \operatorname{Hom}_R(\mathrm{K}^R(x_e), R) \\ &\cong \Sigma^{1-e}\mathrm{K}^R(x_1, \dots, x_{e-1}) \otimes_R \Sigma^{-1}\mathrm{K}^R(x_e) \\ &\cong \Sigma^{-e}\mathrm{K}^R(x_1, \dots, x_e), \end{aligned}$$

where the third isomorphism is tensor evaluation (2.5.5). It follows, again by (2.5.5), that

$$K \otimes_R M \cong \Sigma^e \mathrm{Hom}_R(K,R) \otimes_R M \cong \Sigma^e \mathrm{Hom}_R(K,M).$$

(*iii*) $\Longleftrightarrow$ (*iv*): If $\mathbf{R}\mathrm{Hom}_R(k,M)$ is acyclic, then $\mathrm{Hom}_R(K,M)$ is the same by (E 6.4.2) as $\mathrm{H}(K)$ is a k-vector space. If $\mathrm{Hom}_R(K,M)$ is acyclic, then so is

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(k, \mathrm{Hom}_R(K,M)) &\simeq \mathbf{R}\mathrm{Hom}_R(k \otimes_R K, M) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\bigoplus_{i=0}^{e} \Sigma^i k^{\binom{e}{i}}, M),\end{aligned}$$

whence $\mathbf{R}\mathrm{Hom}_R(k,M)$ is acyclic. □

(6.4.6) **Theorem.** *Let $(R,\mathfrak{m},k)$ be a local ring. For an R-complex M, the following conditions are equivalent:*

(*i*) $\mathfrak{m} \in \mathrm{supp}_R\, M$;

(*ii*) $\mathrm{depth}_R\, M < \infty$;

(*iii*) $\mathrm{width}_R\, M < \infty$.

Proof. By the definitions (6.4.1) and (5.2.1)

$$\mathfrak{m} \in \mathrm{supp}_R\, M \iff \mathrm{H}(k \otimes^{\mathbf{L}}_R M) \neq 0 \iff \mathrm{width}_R\, M < \infty,$$

and by the previous lemma and Definition (5.2.4)

$$\mathrm{H}(k \otimes^{\mathbf{L}}_R M) \neq 0 \iff \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k,M)) \neq 0 \iff \mathrm{depth}_R\, M < \infty.$$ □

Exercises

(E 6.4.1) Give an example of an R-complex M and prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ such that $\mathfrak{p} \in \mathrm{supp}_R\, M$ but $\mathfrak{q} \notin \mathrm{supp}_R\, M$.

(E 6.4.2) Let M and N be R-complexes. Prove that if $M \in \mathsf{D}_\sqsubset(R)$, then $\sup \mathbf{R}\mathrm{Hom}_R(M,N) \leqslant \sup\{ \sup \mathbf{R}\mathrm{Hom}_R(\mathrm{H}_v(M),N) - v \mid v \in \mathbb{Z} \}$.

6.5. Intersection results

New Intersection Theorem (without proof)

(6.5.1) **New Intersection Theorem.** *Let $(R,\mathfrak{m},k)$ be a local ring and $L = 0 \to L_u \to \cdots \to L_0 \to 0$ a complex of finitely generated free R-modules. If $\mathrm{H}(L) \neq 0$ and all the homology modules $\mathrm{H}_v(L)$ have finite length, then $\dim R \leqslant u$.*

(6.5.2) **Remark.** Let $(R,\mathfrak{m},k)$ be a Cohen–Macaulay local ring and $\boldsymbol{x}$ a maximal R-sequence. The module $R/(\boldsymbol{x})$ has finite length and finite projective dimension.

The next corollary shows that that existence of a module of finite length and finite projective dimension implies Cohen–Macaulayness of the ring.

(6.5.3) **Corollary.** *Let $(R,\mathfrak{m},k)$ be a local ring and $M \in \mathsf{D}^{\mathrm{f}}_\square(R)$ an R-complex. If M has finite projective dimension, and $\mathrm{H}(M) \neq 0$ has finite length, then*

$$\dim R \leqslant \mathrm{pd}_R\, M - \inf M.$$

In particular, if there exists an R-module $M \neq 0$ of finite length and finite projective dimension, then R is Cohen–Macaulay.

Proof. Set $u = \operatorname{pd}_R M$ and $w = \inf M$. By Theorem (3.1.10) the complex M has a semifree resolution L with L_v finitely generated for all v, and $L_v = 0$ when $v < w$ or $v > u$. Since $\mathrm{H}(L) \cong \mathrm{H}(M)$ has finite length, it follows by (6.5.1) that $\dim R \leqslant u - w$.

If M is a module, then $w = 0$ and $u = \operatorname{depth}_R R - \operatorname{depth}_R M$ by the Auslander–Buchsbaum Formula (5.2.14). This implies $\dim R = \operatorname{depth} R$ as desired. $\square$

The boundedness condition on N in the next theorem is not necessary. To lift it takes a different version of tensor evaluation (4.4.5).

(6.5.4) **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Let M and N be R-complexes in $\mathsf{D}^{\mathrm{f}}_{\square}(R)$. If $\mathrm{H}(M) \neq 0$ and $\operatorname{pd}_R M < \infty$, then*

$$\operatorname{cmd}_R N \leqslant \operatorname{cmd}_R M \otimes^{\mathbf{L}}_R N.$$

Proof. First note that

$$\begin{aligned}\operatorname{cmd}_R M \otimes^{\mathbf{L}}_R N - \operatorname{cmd}_R N &= \dim_R M \otimes^{\mathbf{L}}_R N - \operatorname{depth}_R M + \operatorname{depth} R - \dim_R N \\ &= \dim_R M \otimes^{\mathbf{L}}_R N + \operatorname{pd}_R M - \dim_R N\end{aligned}$$

by Theorem (5.2.12) and the Auslander–Buchsbaum Formula (5.2.14). Thus, it is sufficient to prove the inequality

$$\dim_R N \leqslant \operatorname{pd}_R M + \dim_R M \otimes^{\mathbf{L}}_R N.$$

This is done in six steps.

1° Suppose R is a catenary domain and $N = R$. Choose $\mathfrak{p}$ minimal in $\operatorname{Supp}_R M$, then $\mathrm{H}(M_{\mathfrak{p}})$ has finite length, and $\operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$; see (6.1.8.1). By Corollary (6.5.3), there is an inequality $\dim R_{\mathfrak{p}} \leqslant \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \inf M_{\mathfrak{p}}$, and because R is a catenary domain

$$\begin{aligned}\dim R = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} &\leqslant \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \inf M_{\mathfrak{p}} + \dim R/\mathfrak{p} \\ &\leqslant \operatorname{pd}_R M + \dim_R M,\end{aligned}$$

where the inequalities use (6.1.8.1) and the definition of dimension (6.3.1).

2° Suppose R is catenary and $N = R$. Choose $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R = \dim R/\mathfrak{p}$. Set $R' = R/\mathfrak{p}$ and $M' = R' \otimes^{\mathbf{L}}_R M$. By definition of projective dimension, $\operatorname{pd}_{R'} M' \leqslant \operatorname{pd}_R M$. (Actually, equality holds by Theorem (5.2.13).) By definition of dimension, Proposition (6.1.5), and Lemma (4.3.8) there are inequalities $\dim_{R'} M' = \dim_R M' \leqslant \dim_R M$. Because R' is a catenary domain, 1° yields

$$\dim R = \dim R' \leqslant \operatorname{pd}_{R'} M' + \dim_{R'} M' \leqslant \operatorname{pd}_R M + \dim_R M.$$

3° Suppose $N = R$. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R, then $\dim R = \dim \widehat{R}$ and $\dim_R \widehat{R} \otimes_R M = \dim_R M$ by Lemma (6.3.5) and faithful flatness of $\widehat{R}$. Moreover, $\widehat{R} \otimes_R M$ is in $\mathsf{D}^{\mathrm{f}}_{\square}(\widehat{R})$ and $\operatorname{pd}_{\widehat{R}} \widehat{R} \otimes_R M = \operatorname{pd}_R M$ by Theorem (5.2.13). By Proposition (7.1.12) the ring $\widehat{R}$ is catenary, so the desired equality follows from 2°.

4° Suppose $N = S$ is a cyclic module. By definition of projective dimension, $\operatorname{pd}_S M \otimes^{\mathbf{L}}_R S \leqslant \operatorname{pd}_R M$ and by definition of dimension $\dim_S M \otimes^{\mathbf{L}}_R S \leqslant \dim_R M \otimes^{\mathbf{L}}_R S$; see also the argument for 2°. Thus, 3° yields

$$\dim_R S = \dim S \leqslant \operatorname{pd}_S M \otimes^{\mathbf{L}}_R S + \dim_S M \otimes^{\mathbf{L}}_R S \leqslant \operatorname{pd}_R M + \dim_R M \otimes^{\mathbf{L}}_R S.$$

5° Suppose N is a finitely generated R-module. Set $S = R/\operatorname{Ann}_R N$, then

$$\dim_R N = \dim_R S \leqslant \operatorname{pd}_R M + \dim_R M \otimes^{\mathbf{L}}_R S = \operatorname{pd}_R M + \dim_R M \otimes^{\mathbf{L}}_R N$$

by 4° and Proposition (6.3.9)(a).

6° The general case now follows by Lemma (6.3.5), 5°, and Proposition (6.3.9)(b):

$$\begin{aligned}\dim_R N &= \sup\{\, \dim_R \mathrm{H}_n(N) - n \mid n \in \mathbb{Z}\,\} \\ &\leqslant \sup\{\, \operatorname{pd}_R M + \dim_R M \otimes^{\mathbf{L}}_R \mathrm{H}_n(N) - n \mid n \in \mathbb{Z}\,\} \\ &= \operatorname{pd}_R M + \dim_R M \otimes^{\mathbf{L}}_R N. \qquad \square\end{aligned}$$

The special case $N = R$ shows that existence of a Cohen–Macaulay complex $M \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ of finite projective dimension implies Cohen–Macaulayness of the ring.

(6.5.5) **Corollary.** *Let $(R, \mathfrak{m}, k)$ be a local ring and $M \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ an R-complex. If $\mathrm{H}(M) \neq 0$ and $\operatorname{pd}_R M < \infty$, then*

$$\operatorname{cmd}_R M \geqslant \operatorname{cmd} R. \qquad \square$$

(6.5.6) **Corollary.** *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $M \neq 0$ and N be finitely generated R-modules. If $\operatorname{pd}_R M < \infty$, then*

$$\dim_R N \leqslant \operatorname{pd}_R M + \dim_R M \otimes_R N.$$

Proof. Immediate from Theorem (6.5.4) and (E 6.3.1). □

The next two results are special cases of (6.5.6).

(6.5.7) **Corollary.** *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M and N be finitely generated R-modules. If $\operatorname{pd}_R M < \infty$ and $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \{\, \mathfrak{m} \,\}$, then*

$$\dim_R N \leqslant \operatorname{pd}_R M. \qquad \square$$

(6.5.8) **Corollary.** *Let $(R, \mathfrak{m}, k)$ be a local ring. If $M \neq 0$ is a finitely generated R-module with $\operatorname{pd}_R M < \infty$, then*

$$\dim R - \dim_R M \leqslant \operatorname{pd}_R M. \qquad \square$$

CHAPTER 7

Duality

7.1. Dualizing complexes

(7.1.1) Let $(R, \mathfrak{m}, k)$ be a local ring and $\mathrm{E}_R(k)$ the injective hull of the residue field. The functor $\mathrm{Hom}_R(-, \mathrm{E}_R(k))$ is exact and faithful; it is called the *Matlis duality* functor. It follows that

$$\sup \mathrm{Hom}_R(M, \mathrm{E}_R(k)) = -\inf M \qquad \text{and} \qquad \inf \mathrm{Hom}_R(M, \mathrm{E}_R(k)) = -\sup M$$

for every R-complex M. Moreover

$$\mathrm{Hom}_R(k, \mathrm{E}_R(k)) \cong k \qquad \text{and} \qquad \mathrm{Hom}_R(\mathrm{E}_R(k), \mathrm{E}_R(k)) \cong \widehat{R}.$$

For $M \in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$, homomorphism evaluation (2.5.6) yields an isomorphism

$$\mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathrm{E}_R(k)), \mathrm{E}_R(k)) \cong M \otimes_R \widehat{R}.$$

If R is artinian, then there are isomorphisms $\mathrm{Hom}_R(\mathrm{E}_R(k), \mathrm{E}_R(k)) \cong R$ and $M \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathrm{E}_R(k)), \mathrm{E}_R(k))$ for every $M \in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$. Moreover, $\mathrm{E}_R(k)$ is finitely generated.

(7.1.2) **Definition.** For an R-complex M, the assignment $r \mapsto r^M$ defines a natural morphism of R-complexes

$$\chi^R_M \colon R \longrightarrow \mathrm{Hom}_R(M, M),$$

called the *homothety morphism*. The same name is used for the map $R \to \mathbf{R}\mathrm{Hom}_R(M, M)$.

(7.1.3) **Definition.** A complex $D \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ is *dualizing* for R if it has finite injective dimension and $\chi^R_D \colon R \to \mathbf{R}\mathrm{Hom}_R(D, D)$ is an isomorphism in $\mathsf{D}(R)$.

(7.1.4) **Remark.** A ring R of finite Krull dimension is Gorenstein if and only if R is dualizing for R.

(7.1.5) **Lemma.** *Let $R \to S$ be a ring homomorphism. If S is finitely generated as module over R, and R has a dualizing complex D, then $\mathbf{R}\mathrm{Hom}_R(S, D)$ is a dualizing complex for S.*

Proof. Let I be a bounded semiinjective resolution of D over R. The complex $\mathrm{Hom}_R(S, I)$ is a bounded complex of injective S-modules, in particular semiinjective, and the homology modules of $\mathbf{R}\mathrm{Hom}_R(S, D) \simeq \mathrm{Hom}_R(S, I)$ are finitely generated over R and hence over S. The commutative diagram of S-complexes

$$\begin{tikzcd}
S \arrow[r, "\chi^S_{\mathrm{Hom}(S,I)}"] \arrow[d, "\cong"] & \mathrm{Hom}_S(\mathrm{Hom}_R(S,I),\mathrm{Hom}_R(S,I)) \\
S \otimes_R R \arrow[d, "\simeq", "S\otimes_R\chi^R_I"'] & \mathrm{Hom}_R(\mathrm{Hom}_R(S,I)\otimes_S S, I) \arrow[u, "\cong"', "\rho_{\mathrm{Hom}(S,I)SI}"] \\
S \otimes_R \mathrm{Hom}_R(I,I) \arrow[r, "\theta_{SII}", "\cong"'] & \mathrm{Hom}_R(\mathrm{Hom}_R(S,I),I) \arrow[u, "\cong"']
\end{tikzcd}$$

shows that the homothety morphism $\chi^S_{\mathrm{Hom}_R(S,I)}$ is an isomorphism in $\mathsf{D}(S)$. $\square$

(7.1.6) **Remark.** An immediate consequence of this lemma is, that if S is a homomorphic image of a Gorenstein ring of finite Krull dimension, then S has a dualizing complex. By [3] the converse is also true.

It follows by Cohen's Structure Theorem that every complete local ring has a dualizing complex.

(7.1.7) **Observation.** If D is a dualizing complex for R, then so is $\Sigma^n D$ for every $n \in \mathbb{Z}$. This is immediate from Lemmas (2.3.10) and (2.3.16).

(7.1.8) **Proposition.** *If D is a dualizing complex for R, then*

$$\mathrm{supp}_R\, D = \mathrm{Supp}_R\, D = \mathrm{Spec}\, R.$$

and for every $\mathfrak{p} \in \mathrm{Spec}\, R$, the complex $D_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$.

Proof. Let I be a bounded semiinjective resolution of D over R, then $I_\mathfrak{p}$ is a bounded semiinjective resolution of $D_\mathfrak{p}$ over $R_\mathfrak{p}$, and the homology complex $\mathrm{H}(D_\mathfrak{p}) \cong \mathrm{H}(D)_\mathfrak{p}$ is degreewise finitely generated over $R_\mathfrak{p}$. There is a commutative diagram in $\mathsf{D}(R)$

$$\begin{tikzcd}
R_\mathfrak{p} \arrow[r, "\chi^{R_\mathfrak{p}}_{D_\mathfrak{p}}"] \arrow[d, "\simeq"] & \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(D_\mathfrak{p}, D_\mathfrak{p}) \\
R_\mathfrak{p} \otimes^{\mathbf{L}}_R R \arrow[r, "R_\mathfrak{p}\otimes^{\mathbf{L}}_R\chi^R_D", "\simeq"'] & R_\mathfrak{p} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,D) \arrow[u, "\simeq"']
\end{tikzcd}$$

where the right vertical arrow is by Lemma (6.1.7). It shows that $D_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$; in particular, $\mathrm{H}(D_\mathfrak{p}) \neq 0$ and the claim about supports follows. $\square$

(7.1.9) **Definition.** Let $(R, \mathfrak{m}, k)$ be a local ring. A dualizing complex D for R is *normalized* if $\sup D = \dim R$.

(7.1.10) **Remark.** If $(R, \mathfrak{m}, k)$ is an artinian local ring, then the injective hull $\mathrm{E}_R(k)$ is a normalized dualizing complex for R.

(7.1.11) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be a local ring. If C is a dualizing complex for R, then the Bass series $\mathrm{I}^C_R(t)$ is a monomial, and for every $\mathfrak{p} \in \mathrm{Spec}\, R$ there is an equality*

$$\mathrm{depth}_R\, C = \mathrm{depth}_{R_\mathfrak{p}}\, C_\mathfrak{p} + \dim R/\mathfrak{p}.$$

Moreover, $D = \Sigma^{\dim R - \sup C} C$ *is a normalized dualizing complex for* R*, and the following hold:*

(a) *The Bass series* $\mathrm{I}_R^D(t)$ *is* 1*; in particular,* $\mathrm{id}_R D = 0 = \operatorname{depth}_R D$.

(b) $\sup D = \dim R$ *and* $\inf D = \operatorname{depth} R$*; in particular,* $\operatorname{amp} D = \operatorname{cmd}_R R$.

Proof. By Lemma (6.2.12) there are equalities of formal Laurent series

$$1 = \mathrm{P}_R^R(t) = \mathrm{P}^R_{\mathbf{R}\mathrm{Hom}_R(C,C)}(t) = \mathrm{I}_R^C(t)\,\mathrm{I}_R^C(t^{-1}).$$

In particular, $0 = \operatorname{ord} \mathrm{I}_R^C(t)\,\mathrm{I}_R^C(t^{-1}) = \operatorname{ord} \mathrm{I}_R^C(t) - \deg \mathrm{I}_R^C(t)$, so $\mathrm{I}_R^C(t) = ct^n$ for some $c \in \mathbb{N}$ and $n \in \mathbb{Z}$. Since, $1 = (ct^n)(ct^{-n}) = c^2$ the coefficient c is 1, and $\mathrm{I}_R^C(t)$ is a monomial. It follows by Lemmas (6.1.11) and (6.1.12) that there are (in)equalities

$$\mathrm{id}_R C = \operatorname{depth}_R C \leqslant \operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p} + \dim R/\mathfrak{p} \leqslant \mathrm{id}_{R_\mathfrak{p}} C_\mathfrak{p} + \dim R/\mathfrak{p} \leqslant \mathrm{id}_R C$$

for every $\mathfrak{p} \in \operatorname{Spec} R$. This yields the desired equality

$$\operatorname{depth}_R C = \operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p} + \dim R/\mathfrak{p}. \tag{1}$$

(a): The complex $D = \Sigma^{\dim R - \sup C} C$ is dualizing by Observation (7.1.7) and clearly normalized. The Bass series of D is a monomial, so $\mathrm{id}_R D = \operatorname{depth}_R D$, and it suffices to prove $\operatorname{depth}_R D = 0$. To prove the inequality $\operatorname{depth}_R D \geqslant 0$, choose $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R/\mathfrak{p} = \dim R$. Now (1) yields

$$\operatorname{depth}_R D = \operatorname{depth}_{R_\mathfrak{p}} D_\mathfrak{p} + \dim R \geqslant -\sup D_\mathfrak{p} + \dim R = \sup D - \sup D_\mathfrak{p} \geqslant 0,$$

where the last equality uses that D is normalized. For the opposite inequality, set $d = \dim R = \sup D$ and choose $\mathfrak{q} \in \operatorname{Ass}_R \mathrm{H}_d(D)$. Then (1) and Observation (5.2.5) yield

$$\operatorname{depth}_R D = \operatorname{depth}_{R_\mathfrak{q}} D_\mathfrak{q} + \dim R/\mathfrak{q} = -d + \dim R/\mathfrak{q} \leqslant 0.$$

(b): The first equality holds by definition. The second one follows from (a) and the Bass Formula (5.2.14). □

Rings with dualizing complexes

(7.1.12) **Proposition.** *If* R *has a dualizing complex, then* R *is catenary and of finite Krull dimension.*

Proof. For each $\mathfrak{p} \in \operatorname{Spec} R$ the complex $\Sigma^{\dim R_\mathfrak{p} - \sup D_\mathfrak{p}} D_\mathfrak{p}$ is a normalized dualizing complex for $R_\mathfrak{p}$; cf. Propositions (7.1.8) and (7.1.11). By Lemma (6.1.19), Remark (5.1.5), and Proposition (7.1.11) there are (in)equalities

$$\begin{aligned}
\mathrm{id}_R D &= \sup\{\, \mathrm{id}_{R_\mathfrak{p}} D_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\
&= \sup\{\, \mathrm{id}_{R_\mathfrak{p}} (\Sigma^{\dim R_\mathfrak{p} - \sup D_\mathfrak{p}} D_\mathfrak{p}) + \dim R_\mathfrak{p} - \sup D_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\
&= \sup\{\, \dim R_\mathfrak{p} - \sup D_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} \\
&\geqslant \sup\{\, \dim R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \,\} - \sup D \\
&= \dim R - \sup D.
\end{aligned}$$

By assumption, $\mathrm{id}_R D$ and $\sup D$ are finite, and hence so is $\dim R$.

To prove that R is catenary, we may assume that it is local. It suffices to show that there is a function $f \colon \operatorname{Spec} R \to \mathbb{N}_0$ such that $\dim R_\mathfrak{q}/\mathfrak{p}_\mathfrak{q} = f(\mathfrak{q}) - f(\mathfrak{p})$ for all prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in R. For such ideals, Proposition (7.1.11) yields equalities

$$\operatorname{depth}_{R_\mathfrak{q}} D_\mathfrak{q} + \dim R/\mathfrak{q} = \operatorname{depth}_R D = \operatorname{depth}_{R_\mathfrak{p}} D_\mathfrak{p} + \dim R/\mathfrak{p}$$

and

$$\operatorname{depth}_{R_{\mathfrak{q}}} D_{\mathfrak{q}} = \operatorname{depth}_{(R_{\mathfrak{q}})_{\mathfrak{p}_{\mathfrak{q}}}} (D_{\mathfrak{q}})_{\mathfrak{p}_{\mathfrak{q}}} + \dim R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}} = \operatorname{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} + \dim R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}},$$

which combine to yield

$$\dim R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}} = \dim R/\mathfrak{p} - \dim R/\mathfrak{q}. \qquad \square$$

The next result holds in general, but the proof is easier when a dualizing complex is available.

(7.1.13) **Theorem.** *Assume R has a dualizing complex. Every R-complex of finite flat dimension has finite projective dimension.*

Proof. Let D be a dualizing complex for R. Assume $\operatorname{fd}_R M < \infty$, then

$$n = \max\{ \operatorname{id}_R D + \sup D \otimes^{\mathbf{L}}_R M \,,\, \sup M \}$$

is finite by Theorem (5.1.9). Let $P \xrightarrow{\simeq} M$ be a semiprojective resolution. Since $n \geqslant \sup M = \sup P$, there is a quasi-isomorphism $P \xrightarrow{\simeq} P_{\subset n}$, and it suffices to prove that the R-module $\mathrm{C}_n(P)$ is projective. This is tantamount to showing that $\operatorname{Ext}^1_R(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P))$ vanishes. Because $n \geqslant \sup M$, the complex $\mathbf{\Sigma}^{-n} P_{\geqslant n}$ is a projective resolution of $\mathrm{C}_n(P)$. Therefore,

$$\begin{aligned} \operatorname{Ext}^1_R(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P)) &\cong \mathrm{H}_{-1}(\operatorname{Hom}_R(\mathbf{\Sigma}^{-n} P_{\geqslant n}, \mathrm{C}_{n+1}(P))) \\ &\cong \mathrm{H}_{-(n+1)}(\operatorname{Hom}_R(P_{\geqslant n}, \mathrm{C}_{n+1}(P))) \\ &\cong \mathrm{H}_{-(n+1)}(\mathbf{R}\!\operatorname{Hom}_R(M, \mathrm{C}_{n+1}(P))), \end{aligned}$$

and it suffices to prove the inequality $\inf \mathbf{R}\!\operatorname{Hom}_R(M, \mathrm{C}_{n+1}(P)) \geqslant -n$. The module $\mathrm{C}_{n+1}(P)$ has finite flat dimension, because a truncation $P_{\subset m}$ for some $m > n$ is a semiflat resolution of M; see Theorem (5.1.9). In particular, tensor evaluation (4.4.5) yields an isomorphism

$$\text{(1)} \qquad \mathrm{C}_{n+1}(P) \simeq \mathbf{R}\!\operatorname{Hom}_R(D, D) \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P) \simeq \mathbf{R}\!\operatorname{Hom}_R(D, D \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P)).$$

Moreover, for every cyclic R-module T, tensor evaluation (4.4.5) and Lemma (4.3.8) yield

$$\begin{aligned} -\inf \mathbf{R}\!\operatorname{Hom}_R(T, D \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P)) &= -\inf \mathbf{R}\!\operatorname{Hom}_R(T, D) \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P) \\ &\leqslant -\inf \mathbf{R}\!\operatorname{Hom}_R(T, D); \end{aligned}$$

in particular, $\operatorname{id}_R D \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P) \leqslant \operatorname{id}_R D$. Finally, (1) and adjointness (4.4.3) in combination with this inequality yield

$$\begin{aligned} -\inf \mathbf{R}\!\operatorname{Hom}_R(M, \mathrm{C}_{n+1}(P)) &= -\inf \mathbf{R}\!\operatorname{Hom}_R(M, \mathbf{R}\!\operatorname{Hom}_R(D, D \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P))) \\ &= -\inf \mathbf{R}\!\operatorname{Hom}_R(D \otimes^{\mathbf{L}}_R M, D \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P)) \\ &\leqslant \operatorname{id}_R D \otimes^{\mathbf{L}}_R \mathrm{C}_{n+1}(P) + \sup D \otimes^{\mathbf{L}}_R M \\ &\leqslant \operatorname{id}_R D + \sup D \otimes^{\mathbf{L}}_R M \\ &\leqslant n. \end{aligned} \qquad \square$$

7.2. Duality with respect to a dualizing complex

(7.2.1) **Construction.** Let M and N be R-complexes. The assignment

$$m \longmapsto [\psi \mapsto (-1)^{|m||\psi|}\psi(m)],$$

for $m \in M$ and $\psi \in \operatorname{Hom}_R(M,N)$, defines a morphism in $\mathsf{C}(R)$

$$\delta_N^M \colon M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M,N),N),$$

which is natural in M and N. It is called the *biduality* morphism for M with respect to N. The same name and notation is used for the induced morphism

$$\delta_N^M \colon M \longrightarrow \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,N),N)$$

in the derived category $\mathsf{D}(R)$.

The next result is Grothendieck's duality theorem.

(7.2.2) **Theorem.** *Assume R has a dualizing complex D. For every R-complex $M \in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$ the biduality morphism*

$$\delta_D^M \colon M \longrightarrow \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,D),D)$$

is an isomorphism in $\mathsf{D}(R)$.

Proof. Immediate from the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\delta_D^M"] \arrow[d, "\simeq"] & \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,D),D) \\
M \otimes^{\mathbf{L}}_R R \arrow[r, "M\otimes^{\mathbf{L}}_R \chi^R_D", "\simeq"'] & M \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(D,D), \arrow[u, "\simeq"', "\theta_{MDD}"]
\end{tikzcd}$$

where the right vertical isomorphism is homomorphism evaluation (4.4.6). $\square$

Formal invariants

(7.2.3) **Definition.** Let $(R,\mathfrak{m},k)$ be a local ring with a normalized dualizing complex D. For an R-complex M, set $M^\dagger = \mathbf{R}\operatorname{Hom}_R(M,D)$.

(7.2.4) **Proposition.** *Let $(R,\mathfrak{m},k)$ be a local ring with a dualizing complex. For an R-complex $M \in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$ there is an equality of formal Laurent series*

$$\mathrm{I}_R^{M^\dagger}(t) = \mathrm{P}_M^R(t).$$

In particular,

$$\operatorname{id}_R M^\dagger = \operatorname{pd}_R M \qquad \textit{and} \qquad \operatorname{depth}_R M^\dagger = \inf M.$$

(7.2.5) **Proof.** Immediate from Lemma (6.2.10) and Observation (6.2.9). $\square$

Homological dimensions

(7.2.6) **Corollary.** *Let $(R,\mathfrak{m},k)$ be a local ring with a dualizing complex. For an R-complex $M \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$,*

$$\operatorname{pd}_R M < \infty \iff \operatorname{id}_R M^\dagger < \infty \quad \textit{and}$$
$$\operatorname{id}_R M < \infty \iff \operatorname{pd}_R M^\dagger < \infty.$$

Proof. Immediate from Theorem (7.2.2) and the previous proposition. $\square$

Krull dimension and depth

(7.2.7) **Proposition.** *Let $(R, \mathfrak{m}, k)$ be a local ring with a dualizing complex. The next equalities hold for R-complexes $M \in \mathsf{D}^{\mathsf{f}}_{\Box}(R)$:*

(a) $\dim_R M = \sup M^{\dagger}$;
(b) $\operatorname{depth}_R M = \inf M^{\dagger}$; *in particular,*
(c) $\operatorname{cmd}_R M = \operatorname{amp} M^{\dagger}$.

Proof. (a): Let D be a normalized dualizing complex. The first equality in the chain below is from Lemma (5.2.8), the last two follow by Proposition (7.1.11) and the definition of dimension.

$$\begin{aligned} \sup M^{\dagger} &= -\inf\{\operatorname{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} + \inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{-\operatorname{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} - \inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\dim R/\mathfrak{p} - \operatorname{depth}_R D - \inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \dim_R M \end{aligned}$$

Part (b) follows from Theorem (7.2.2) and Proposition (7.2.4); part (c) follows from (a) and (b). $\Box$

7.3. Applications of dualizing complexes

(7.3.1) Let $(R, \mathfrak{m}, k)$ be a local ring and $\mathrm{E}_R(k)$ the injective hull of the residue field. There are equalities of supports

$$\operatorname{supp}_R \mathrm{E}_R(k) = \operatorname{Supp}_R \mathrm{E}_R(k) = \{\mathfrak{m}\}.$$

There is always an equality

$$\operatorname{depth}_R \operatorname{Hom}_R(M, \mathrm{E}_R(k)) = \operatorname{width}_R M,$$

and if $M \in \mathsf{D}_{\sqsubset}(R)$, then

$$\operatorname{width}_R \operatorname{Hom}_R(M, \mathrm{E}_R(k)) = \operatorname{depth}_R M.$$

In particular, $\operatorname{width}_R \mathrm{E}_R(k) = \operatorname{depth} R$.

If $M \in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$, then

$$\operatorname{pd}_R M = \operatorname{id}_R \operatorname{Hom}_R(M, \mathrm{E}_R(k)).$$

If $M \in \mathsf{D}^{\mathsf{f}}_{\sqsubset}(R)$, then

$$\operatorname{id}_R M = \operatorname{fd}_R \operatorname{Hom}_R(M, \mathrm{E}_R(k)).$$

Local duality

(7.3.2) **Observation.** Let $(R, \mathfrak{m}, k)$ be a local ring with a dualizing complex D. It is immediate from Lemma (6.1.7)

$$\operatorname{supp}_R \operatorname{Hom}_R(D, \mathrm{E}_R(k)) = \operatorname{Supp}_R \operatorname{Hom}_R(D, \mathrm{E}_R(k)) = \{\mathfrak{m}\}.$$

(7.3.3) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a local ring and $M \in \mathsf{D}_{\sqsubset}(R)$ an R-complex. If D is a normalized dualizing complex for R, then*

$$\operatorname{depth}_R M = -\sup \operatorname{Hom}_R(D, \mathrm{E}_R(k)) \otimes^{\mathbf{L}}_R M.$$

Proof. By the previous observation and Theorem (6.4.6), $\operatorname{depth}_R M$ is finite exactly when $\mathrm{H}(\operatorname{Hom}_R(D, \mathrm{E}_R(k)) \otimes^{\mathbf{L}}_R M) \neq 0$. We may assume this is the case and set $s = \sup \operatorname{Hom}_R(D, \mathrm{E}_R(k)) \otimes^{\mathbf{L}}_R M > -\infty$. It now follows by Observation (5.2.5), Theorem (5.2.12), and (7.3.1) that

$$\begin{aligned}
-\sup \operatorname{Hom}_R(D, \mathrm{E}_R(k)) \otimes^{\mathbf{L}}_R M \\
&= \operatorname{depth}_R \operatorname{Hom}_R(D, \mathrm{E}_R(k)) \otimes^{\mathbf{L}}_R M \\
&= \operatorname{depth}_R \operatorname{Hom}_R(D, \mathrm{E}_R(k)) + \operatorname{depth}_R M - \operatorname{depth} R \\
&= \operatorname{depth}_R M. \qquad \square
\end{aligned}$$

(7.3.4) It can also be proved that $\dim_R M \geqslant -\inf \operatorname{Hom}_R(D, \mathrm{E}_R(k)) \otimes^{\mathbf{L}}_R M$, when $M \in \mathsf{D}_{\sqsupset}(R)$. (This will follow from the Local Duality Theorem, which will be added to a later version.) It follows that

$$\dim_R M \geqslant \operatorname{depth}_R M$$

for $M \in \mathsf{D}_{\sqsubset}(R)$ with $\mathfrak{m} \in \operatorname{supp}_R M$.

Bass' question

(7.3.5) **Remark.** Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring and $\boldsymbol{x}$ a maximal R-sequence. The module $\operatorname{Hom}_R(R/(\boldsymbol{x}), \mathrm{E}_R(k))$ has finite length and finite injective dimension; cf. Remark (6.5.2) and (7.3.1). In [1] H. Bass raised the question whether existence of a finitely generated module of finite injective dimension would imply Cohen–Macaulayness of the ring.

(7.3.6) **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring and $M \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ an R-complex. If $\mathrm{H}(M) \neq 0$ and $\operatorname{id}_R M < \infty$, then*

$$\operatorname{cmd}_R R \leqslant \operatorname{amp} M.$$

Proof. Since $\dim R = \dim \widehat{R}$ and $\operatorname{depth} R = \operatorname{depth} \widehat{R}$, we have $\operatorname{cmd} R = \operatorname{cmd} \widehat{R}$. Moreover, $\operatorname{amp} M = \operatorname{amp} \widehat{R} \otimes_R M$ by faithful flatness of $\widehat{R}$, and $\operatorname{id}_{\widehat{R}} \widehat{R} \otimes_R M = \operatorname{id}_R M$ by the Bass Formula (5.2.14). Thus, we may assume R is complete and let D be a normalized dualizing complex for R. By Theorem (7.2.2) and Proposition (7.2.7) we have $\operatorname{amp} M = \operatorname{cmd}_R M^{\dagger}$. The complex $M^{\dagger}$ has finite flat dimension, and by Lemma (4.2.9) and Theorem (5.1.6) it belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$. Thus, $M^{\dagger}$ has finite projective dimension by Theorem (5.2.13), and then $\operatorname{cmd} R \leqslant \operatorname{cmd}_R M^{\dagger}$ by Corollary (6.5.5). $\square$

The affirmative answer to Bass' question is a special case of this theorem.

(7.3.7) **Corollary.** *Let $(R, \mathfrak{m}, k)$ be a local ring. If there exists a finitely generated R-module $M \neq 0$ of finite injective dimension, then R is Cohen–Macaulay.* $\square$

Vasconcelos' conjecture

(7.3.8) **Observation.** Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring and set $d = \dim R$. Suppose $\mu^d_R(R) = \operatorname{rank}_k \operatorname{Ext}^d_R(k, R) = 1$, then also $\mu^d_{\widehat{R}}(\widehat{R}) = 1$ by adjointness (4.4.3) and tensor evaluation (4.4.5). Let D be a normalized dualizing complex for $\widehat{R}$, by Proposition (7.2.4) we have $\beta^{\widehat{R}}_d(D) = 1$. By Proposition (7.1.11) the complex $\Sigma^{-d} D$ is isomorphic in $\mathsf{D}(\widehat{R})$ to a module C with $\beta^{\widehat{R}}_0(C) = 1$ and $\operatorname{Hom}_{\widehat{R}}(C, C) \cong \widehat{R}$. It follows that $C \cong \widehat{R}$, so $\widehat{R}$, and thereby R, is Gorenstein.

In [6] Vasconcelos conjectured the next result.

(7.3.9) **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring and set $d = \dim R$. If $\mu^d_R(R) = 1$, then R is Gorenstein.*

Proof. As in Observation (7.3.8), we may assume R is complete. Then R has a normalized dualizing complex D, and it follows by Proposition (7.2.4) that $\beta^R_d(D) = 1$. Take a semifree resolution $L \xrightarrow{\simeq} D$ with L_v finitely generated for all v.

$$\cdots \longrightarrow L_{d+1} \xrightarrow{\partial^L_{d+1}} L_d \xrightarrow{\partial^L_d} L_{d-1} \longrightarrow \cdots \tag{1}$$

(1) First note that after splitting off contractible summands, we may assume that $L_d = R$ and the differentials ∂^L_{d+1} and ∂^L_d are given by matrices with entries in $\mathfrak{m}$.

If R is a domain, then either ∂^L_{d+1} or ∂^L_d is the 0-map. Thus, L decomposes as a direct sum $L = L_{\geqslant n} \oplus L_{\leqslant n-1}$ where n is either $d+1$ or d. Because $R \simeq \operatorname{Hom}_R(L, L)$ is indecomposable, one of the summands must vanish. Since $L_d \neq 0$ there are only two possibilities. If $\partial^L_{d+1} = 0$, then $L_{\geqslant d+1} = 0$. This implies that $\operatorname{pd}_R D$ is finite, and then R is Gorenstein by Corollary (7.2.6). If $\partial^L_d = 0$, then $L_{\leqslant d-1} = 0$. This implies that $d \leqslant \inf D = \operatorname{depth} R$, see Proposition (7.1.11), and then R is Cohen–Macaulay. In this case Observation (7.3.8) shows that R is Gorenstein.

A proof of the general case will be added in a later version. □

APPENDIX A

Half-exact Functors

(A.1) **Lemma.** *Let* F *be a half-exact functor on the category of R-modules. If T is a finitely generated R-module such that* $\mathrm{F}(T) \neq 0$, *then there is a prime ideal* $\mathfrak{p} \in \operatorname{Supp}_R T$ *such that* $\mathrm{F}(R/\mathfrak{p}) \neq 0$.

Proof. Choose a filtration $0 = T_0 \subset T_1 \subset \cdots \subset T_{n-1} \subset T_n = T$ such that $T_i/T_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} R$. The short exact sequences

$$0 \longrightarrow T_{i-1} \longrightarrow T_i \longrightarrow R/\mathfrak{p}_i \longrightarrow 0$$

induce exact sequences

$$\mathrm{F}(T_{i-1}) \longrightarrow \mathrm{F}(T_i) \longrightarrow \mathrm{F}(R/\mathfrak{p}_i)$$

or

$$\mathrm{F}(R/\mathfrak{p}_i) \longrightarrow \mathrm{F}(T_i) \longrightarrow \mathrm{F}(T_{i-1})$$

depending on the variance of F. In either case it follows that $\mathrm{F}(R/\mathfrak{p}_i) \neq 0$ for at least one $i \in \{ 1, \dots, n \}$. □

(A.2) **Lemma.** *Let $(R, \mathfrak{m}, k)$ be local. Let* F *be a covariant linear half-exact functor on the category of finitely generated R-modules. If* F *is not the* 0*-functor, then* $\mathrm{F}(k) \neq 0$.

Proof. Choose a finitely generated R-module T such that $\mathrm{F}(T) \neq 0$. By the previous lemma, the set $\{ \mathfrak{p} \in \operatorname{Supp}_R T \mid \mathrm{F}(R/\mathfrak{p}) \neq 0 \}$ is non-empty and hence has a maximal element $\mathfrak{q}$. Set $C = R/\mathfrak{q}$ and choose any element $x \in R \setminus \mathfrak{q}$. It follows from the previous lemma that F vanishes on $C/xC = R/(\mathfrak{q} + (x))$. Application of F to the short exact sequence

$$0 \longrightarrow C \xrightarrow{x} C \longrightarrow C/xC \longrightarrow 0$$

therefore yields an exact sequence

$$\mathrm{F}(C) \xrightarrow{x} \mathrm{F}(C) \longrightarrow 0.$$

NAK now implies that $x \in r \setminus \mathfrak{m}$ and hence $\mathfrak{q} = \mathfrak{m}$. □

(A.3) **Lemma.** *Let* F *be a half-exact functor on the category of R-modules. Assume that to each finitely generated module M and each M-regular element x there is an exact sequence*

$$\mathrm{F}(M/xM) \longrightarrow \mathrm{F}(M) \xrightarrow{x} \mathrm{F}(M).$$

If T is a finitely generated R-module such that $\mathrm{F}(T) \neq 0$, *then there is a* $\mathfrak{q} \in \operatorname{Supp}_R T$ *such that* $\mathrm{F}(R/\mathfrak{q})_{\mathfrak{q}} \neq 0$.

Note that this lemma applies if F is half-exact, contravariant, and linear.

Proof. By Lemma (A.1) the set $\{\, \mathfrak{p} \in \operatorname{Supp}_R T \mid \mathrm{F}(R/\mathfrak{p}) \neq 0 \,\}$ is non-empty and hence has a maximal element $\mathfrak{q}$. Set $C = R/\mathfrak{q}$, for every element $x \in R \setminus \mathfrak{q}$ it follows from (A.1) that F vanishes on $C/xC = R/(\mathfrak{q} + (x))$. By assumption, there are now exact sequences

$$0 \longrightarrow \mathrm{F}(C) \xrightarrow{x} \mathrm{F}(C),$$

so every $x \in R \setminus \mathfrak{q}$ is $\mathrm{F}(C)$-regular. In particular, the canonical map $\mathrm{F}(C) \to \mathrm{F}(C)_{\mathfrak{q}}$ is injective, and the assertion follows. $\square$

Bibliography

1. Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
2. David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, New York, 1995, Graduate Texts in Mathematics, vol. 150. MR **97a:**13001
3. Takesi Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 123–149 (electronic). MR **2002i:**13001
4. T. Y. Lam, *A first course in noncommutative rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR **MR1838439 (2002c:**16001)
5. Saunders MacLane, *Categories for the working mathematician*, Springer-Verlag, New York, 1971, Graduate Texts in Mathematics, vol. 5. MR 50 #7275
6. Wolmer V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. [Notes on Mathematics, No. 53]. MR MR0498530 (58 #16637)
7. Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR **1269324 (95f:**18001)

Glossary

For terms that are used but not defined in the text, please see one of the following standard references: *Commutative algebra with a view toward algebraic geometry* [2] by Eisenbud; *A first course in noncommutative rings* [4] by Lam; and *Categories for the working mathematician* [5] by MacLane.

Once completed, this glossary will give brief definitions and precise references to the above mentioned monographs.

NAK: Nakayama's lemma; see [4, lem. (4.22)]

noetherian ring: associative ring that satisfies the ascending chain condition; see [2, sec. 1.4]

semisimple ring: ring over which all modules are projective; see [4, thm. (2.6) and (2.8)]

List of Symbols

Bold face page numbers refer to definitions. Eventually, references to [2; 4; 5] will be added for definitions of the symbols that are not defined in this text.

$\sim$	homotopy relation in $\mathsf{C}(R)$, **8**
$\simeq$	quasiisomorphism in $\mathsf{C}(R)$, **7**
	isomorphism in $\mathsf{D}(R)$, **36**
$\cong$	isomorphism in $\mathsf{C}(R)$, **5**
$\coprod$	coproduct,
$\prod$	product,
$\bigoplus$	finite (co-)product,
$\lvert m \rvert$	degree of an element m, **5**
$- \otimes_R -$	tensor product functor, **16**
$- \otimes_R^{\mathbf{L}} -$	left derived tensor product functor, **41**
$M \otimes_R N$	tensor product of the R-complexes M and N, **15**
$M^\natural$	the graded module underlying the complex M, **5**
$M_{\leqslant n}$	hard truncation above of M at n, **9**
$M_{\geqslant n}$	hard truncation below of M at n, **9**
$M_{\subset n}$	soft truncation above of M at n, **9**
$M_{\supset n}$	soft truncation below of M at n, **10**
1^M	identity morphism on the complex M, **5**
∂^M	differential of the complex M, **5**
$\Sigma -$	shift functor, **9**
$\Sigma^n M$	n-fold shift of the complex M, **9**
$\beta_m^R(M)$	nth Betti number of the R-complex M, **56**
δ_N^M	biduality morphism for the complex M with respect to N, **72**
θ_{KMN}	homomorphism evaluation homomorphism for modules K, M, and N, **2**
	homomorphism evaluation morphism in $\mathsf{C}(R)$ for complexes K, M, and N, **22**
	homomorphism evaluation morphism in $\mathsf{D}(R)$ for complexes K, M, and N, **44**
$\mu_R^n(M)$	nth Bass number of the R-complex M, **56**
ρ_{KMN}	adjointness homomorphism for modules K, M, and N, **2**
	adjointness morphism in $\mathsf{C}(R)$ for complexes K, M, and N, **19**

	adjointness morphism in $\mathsf{D}(R)$ for complexes K, M, and N, **43**
σ_{KMN}	associativity homomorphism for modules K, M, and N, **1**
	associativity morphism in $\mathsf{C}(R)$ for complexes K, M, and N, **19**
	associativity morphism in $\mathsf{D}(R)$ for complexes K, M, and N, **43**
ς_{KMN}	swap homomorphism for modules K, M, and N, **2**
	swap morphism in $\mathsf{C}(R)$ for complexes K, M, and N, **20**
	swap morphism in $\mathsf{D}(R)$ for complexes K, M, and N, **43**
τ_{KMN}	commutativity homomorphism for modules K, M, and N, **1**
	commutativity morphism in $\mathsf{C}(R)$ for complexes K, M, and N, **18**
	commutativity morphism in $\mathsf{D}(R)$ for complexes K, M, and N, **43**
χ_M^R	homothety morphism of the R-complex M, **69**
ω_{KMN}	tensor evaluation homomorphism for modules K, M, and N, **2**
	tensor evaluation morphism in $\mathsf{C}(R)$ for complexes K, M, and N, **21**
	tensor evaluation morphism in $\mathsf{D}(R)$ for complexes K, M, and N, **44**
$\mathbb{N}$	natural numbers,
$\mathbb{N}_0$	$(=\mathbb{N}\cup\{0\})$, non-negative integers,
$\mathbb{Q}$	rational numbers,
$\mathbb{Z}$	integers,
$\mathbb{Z}(\!(t)\!)$	formal Laurent series with coefficients in $\mathbb{Z}$,
$\mathsf{C}(R)$	category of R-complexes and their morphisms, **5**
$\mathsf{D}(R)$	derived category of the category of R-modules, **35**
$\mathsf{D}^{\mathsf{f}}(R)$	full subcategory of $\mathsf{D}(R)$ whose objects have finitely generated homology modules, **37**
$\mathsf{D}_{\sqsubset}(R)$	full subcategory of $\mathsf{D}(R)$ whose objects have bounded above homology complexes, **37**
$\mathsf{D}_{\sqsupset}(R)$	full subcategory of $\mathsf{D}(R)$ whose objects have bounded below homology complexes, **37**
$\mathsf{D}_{\Box}(R)$	full subcategory of $\mathsf{D}(R)$ whose objects have bounded homology complexes, **37**
$\mathsf{K}(R)$	homotopy category of R-complexes, **35**
$\operatorname{Ann}_R M$	annihilator of the R-module M,
$\operatorname{Ass}_R M$	the set of associated prime ideals of the R-module M,
$\mathrm{B}(M)$	boundary subcomplex of M, **6**
$\mathrm{B}_v(M)$	$(=\operatorname{Im}\partial_{v+1}^M)$, boundaries of M in degree v, **6**
$\mathrm{C}(M)$	cokernel subcomplex of M, **6**
$\mathrm{C}_v(M)$	$(=\operatorname{Coker}\partial_{v+1}^M)$, **6**
$\operatorname{Coker}\alpha$	cokernel of the map α,
$\operatorname{Cone}\alpha$	mapping cone of the morphism α, **10**
$\mathrm{E}_R(k)$	injective envelope of the residue field of the local ring $(R,\mathfrak{m},k)$,
$\operatorname{Ext}_R^m(M,N)$	mth Ext group of the R-modules M and N,
$\mathrm{H}(-)$	Homology functor, **8**
$\mathrm{H}(M)$	homology complex of the complex M,
$\mathrm{H}_v(M)$	Homology of the complex M in degree v, **6**
$\operatorname{Hom}_R(-,-)$	homomorphism functor, **12**
$\operatorname{Hom}_R(M,N)$	complex of homomorphisms from the R-complex M to N, **11**

Index