Complex Analysis

Preliminary Examination

May 2004

Do all 9 problems. Notation: $D = \{z \in \mathbb{C} : |z| < 1\}$.

1. Find a one-to-one conformal map $f$ of $D \setminus (-1, 0]$ onto $D$ such that $f(1/2) = 0$.

2. Give an explicit formula for a one-to-one conformal map defined on $D$ whose range is dense in $\mathbb{C}$.

3. State and prove Rouche’s Theorem.

4. Let $g$ be a rational function with $|g(z)| = 1$ whenever $|z| = 1$. Prove that
   $$g(z) = \frac{b_1(z)}{b_2(z)},$$
   where $b_1$ and $b_2$ are finite products of the form
   $$e^{i\theta} \prod_{j=1}^{n} \frac{\alpha_j - z}{1 - \alpha_j z^2},$$
   where $|\alpha_j| < 1$, $j = 1, 2, \ldots, n$, and $\theta \in \mathbb{R}$.

5. Suppose $f$ is entire and $\lim_{z \to \infty} \frac{f(z)}{z} = 0$. Prove that $f$ must be a constant function.

6. Let $f$ be a one-to-one, analytic map of $D$ into $D$. Prove that
   $$|f'(z)| \leq \frac{1}{1 - |z|^2},$$
   for all $z \in D$.

7. Let $h(z) = z + \frac{z^2}{2}$. Find the area of $h(D)$.

8. Let $\Omega \subset \mathbb{C}$ be a simply connected region and $u : \Omega \to \mathbb{R}$ a harmonic function. Prove that there exists $v : \Omega \to \mathbb{R}$ such that $u + iv$ is analytic on $\Omega$.

9. Let
   $$p_n(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}.$$  
   Prove that for every $R > 0$ there exists $N > 0$ such that for all $n \geq N$ the zeros of $p_n$ belong to the set $\{z : |z| > R\}$. 