Do all eight problems.

1. State and prove Schwarz’s Lemma.

2. Find a conformal, one-to-one map $f$ from $\mathbb{D} = \{z : |z| < 1\}$ onto
   
   $G = \{w : \text{Im } w < \pi/2\} \setminus \{w : w \leq -1\}$

   such that $f(0) = 1$.

3. Evaluate the integral
   
   $$\int_0^{2\pi} \frac{d\theta}{1 + \sin^2 \theta}$$

4. Prove the reflection principle: If $H = \{z \in \mathbb{C} : \text{Im } z > 0\}$, and if $f$ is a continuous function on $\overline{H}$, analytic on $H$ and if $f$ is real on the real axis, then $f$ can be analytically extended from $H$ to all of $\mathbb{C}$.

5. A function $f$ is said to satisfy the Lipschitz condition on $\mathbb{C}$ if there exists a positive constant $M$ such that
   
   $$|f(z_1) - f(z_2)| \leq M \cdot |z_1 - z_2|$$

   for all $z_1, z_2 \in \mathbb{C}$.

   Find all entire functions that satisfy the Lipschitz condition on $\mathbb{C}$.

6. Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$ has the property that the series $\sum_{n=0}^{\infty} f^{(n)}(c)$ converges. Show that $f$ is an entire function.

7. Classify [type (and order where applicable)] all of the isolated singularities of the following functions (including any isolated singularities at the point at $\infty$):
   
   a) $f(z) = \frac{\sin^2 z}{z^4}$

   b) $f(z) = \sin \left( \frac{1}{z} \right) + \frac{1}{z^2(z-1)}$

   c) $f(z) = \csc z - \frac{1}{z}$

8. Let $w_1$ and $w_2$ be distinct points in $\mathbb{C}$ and let $L$ be the perpendicular bisector of the line segment connecting them. Describe the image of $L$ under the map

   $$F(z) = \frac{z - w_1}{z - w_2}.$$