**Definition.** Let $G$ be a region and let $\alpha = a + bi = (a, b) \in G$. A function $g : G \to \mathbb{R}$ is differentiable at $\alpha$ if there exist constants $A$ and $B$ and functions $\varepsilon_1, \varepsilon_2 : G \to \mathbb{R}$ which are continuous at $\alpha$ with $\varepsilon_1(a, b) = \varepsilon_2(a, b) = 0$ such that

$$g(x, y) = g(a, b) + (x - a)[A + \varepsilon_1(x, y)] + (y - b)[B + \varepsilon_2(x, y)]$$

If $g$ is differentiable at $\alpha$, we will define $g_x(a, b) = A$ and $g_y(a, b) = B$.

**Proposition.** Let $G$ be a region and let $\alpha = a + bi = (a, b) \in G$. Suppose that $u, v : G \to \mathbb{R}$ such that $u, v$ are (continuously) differentiable at $\alpha$ and that $u, v$ satisfy the Cauchy-Riemann equations at $\alpha$. Then, $f = u + iv$ is (continuously) differentiable at $\alpha$.

**Proof.** Since $u, v$ are differentiable at $\alpha$, there exist functions $\varepsilon_r : G \to \mathbb{R}$, $r = 1, 2, 3, 4$, which are continuous at $\alpha$ with $\varepsilon_r(a, b) = 0$, $r = 1, 2, 3, 4$, such that

$$u(x, y) = u(a, b) + (x - a)[u_x(a, b) + \varepsilon_1(x, y)] + (y - b)[u_y(a, b) + \varepsilon_2(x, y)] \quad (1)$$

$$v(x, y) = v(a, b) + (x - a)[v_x(a, b) + \varepsilon_3(x, y)] + (y - b)[v_y(a, b) + \varepsilon_4(x, y)] \quad (2)$$

If in (1) we replace $u_y(a, b) = -v_x(a, b) = i \cdot v_x(a, b)$ and in (2) we replace $v_y(a, b) = u_x(a, b)$ and then multiply (2) by $i$ and add the result to (1) we obtain (after a rearrangement)

$$f(z) - f(\alpha) = (z - \alpha)[u_x(a, b) + i v_x(a, b) + w(z)]$$

where

$$w(z) = \frac{x - a}{z - \alpha} \left[ \varepsilon_1(x, y) + i\varepsilon_3(x, y) \right] + \frac{y - b}{z - \alpha} \left[ \varepsilon_2(x, y) + i\varepsilon_4(x, y) \right].$$

Since $|w(z)| \leq |\varepsilon_1(x, y)| + |\varepsilon_2(x, y)| + |\varepsilon_3(x, y)| + |\varepsilon_4(x, y)|$, then

$$\lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = \lim_{z \to \alpha} (u_x(a, b) + i v_x(a, b) + w(z)) = u_x(a, b) + i v_x(a, b)$$
since \( \lim_{z \to \infty} w(z) = 0 \).