Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

1. (12 pts) Give the definition for each of the following:
   a. A set \( K \) is convex if . . .
   b. A region \( G \) is simply connected if . . .
   c. A function \( f \) has an isolated singularity at \( z = a \) if . . .

2. (20 pts) State and prove Morera’s Theorem.

3. (28 pts) Identify each of the isolated singularities for each of the following functions. Determine whether the singularities are removable singularities, poles or essential singularities. If the singularity is removable, determine what value the function should be assigned at the singularity to analytically extend the function at the singularity. If the singularity is a pole, determine the order of the pole and, if the pole occurs at \( z = 0 \), then determine the singular part of the function at 0. If the singularity is an essential singularity, determine the residue of the function at the singularity.
   
   a. (10) \( f(z) = \frac{z^3 + 1}{z^2(z + 1)} \)   
   b. (6) \( f(z) = z^3 e^z \)
   c. (6) \( f(z) = \frac{\cos z}{z^2 + 1} + 4z \)   
   d. (6) \( f(z) = \frac{\sin(3z)}{z^2} - \frac{3}{z} \)

4. (20 pts) Consider the region \( G = \text{ann}(0;0,2) \). Consider the function \( f(z) = \frac{1}{z} \) which is analytic on \( G \). Consider the set \( S = \{ z : |z| = 1 \} \) which is a compact subset of \( G \). Show that it is not possible to find a sequence of polynomials \( \{ p_n \} \) such that \( \{ p_n \} \rightarrow f \) on \( S \).

   (Consider using a proof by contradiction, i.e., suppose that there existed a sequence of polynomials \( \{ p_n \} \) such that \( \{ p_n \} \rightarrow f \) on \( S \). Proceed from that supposition to a contradiction.)
5. (21 pts) Classify each of the following regions at to whether they are convex or not convex, starlike (with respect to some point \( a \in G \)) or not starlike (with respect to any point \( a \in G \)), simply connected or not simply connected.

Notation: \([a,b]\) denotes the (closed) straight line segment between \( a \) and \( b \).

Use the table on the next page to report your answers

a. \( G_a = B(0,1)\left[\frac{1}{2}, \frac{1}{2}\right] \)

b. \( G_b = B(0,1)\left[-1, -\frac{1}{2}\right] \)

c. \( G_c = B(0,1)\left[-1, \frac{1}{2}\right] \)

d. \( G_d = B(0,1)\left[i, \frac{1}{2}\right] \)

e. \( G_e = B(0,1) \setminus B\left(-\frac{1}{2}, \frac{1}{2}\right) \)

f. \( G_f = B(0,1)\setminus B\left(\frac{1+i}{4}, \frac{1}{4}\right) \)

g. \( G_g = B(0,1) \setminus B\left(\frac{1+i}{2}, \frac{1}{2}\right) \)

h. \( G_h = B(0,1) \cap B(2,2) \)

i. \( G_i = B(0,1) \setminus \{B(-1+i,1) \cup B(-1-i,1)\} \)

j. \( G_j = B(0,1) \setminus \{B(-1+i, \frac{3}{4}) \cup B(-1-i, \frac{3}{4})\} \)

k. \( G_k = B(0,1) \setminus H_k \) where \( H_k = \{z : \Re z \leq -\frac{1}{2}\} \)

l. \( G_l = B(0,1) \setminus H_l \) where \( H_l = \{z : \Re z \geq -\frac{1}{2}\} \)

m. \( G_m = B(0,1) \setminus H_m \) where \( H_m = \{z = x+iy : \frac{x^2}{\left(\frac{1}{4}\right)} + \frac{y^2}{4} \geq 1\} \)

n. \( G_n = B(0,1) \setminus H_n \) where \( H_n = \{z = x+iy : \frac{x^2}{\left(\frac{1}{4}\right)} - \frac{y^2}{4} \geq 1\} \)
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