Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

1. (10) Let $z = -2\sqrt{3}i - 2$ and $w = 4 - 4i$. Write in rectangular form, $a + bi$, and polar form, $r \text{cis} \theta$, each of the following:

   a. $\frac{z^2}{w^3}$

   b. $\left(\frac{z + w}{z + w}\right)^3$

2. (10) Prove Proposition 1.13 d.

3. (22) Give examples of sets in $\mathbb{C}$ with the usual topology:

   a. i) A set $A$ such that $\text{int}(\overline{A \setminus \text{int} A}) = \emptyset$

      ii) A set $A$ such that $\text{int}(\overline{A \setminus \text{int} A}) \neq \emptyset$

   b. i) A set $A$ such that $A$ has only finitely many components

      ii) A set $A$ such that $A$ has countably infinitely many components

      iii) A set $A$ such that $A$ has uncountably many components

   A countable collection $\{A_n\}_{n=1}^{\infty}$ is said to be distinct if $j \neq k \Rightarrow A_j \neq A_k$ for all $j, k \in \mathbb{N}$

   c. i) A countable collection of distinct closed sets $\{G_n\}_{n=1}^{\infty}$ such that

      a) $\bigcup_{n=1}^{\infty} G_n$ is open ($\neq \mathbb{C}$)

      b) $\bigcup_{n=1}^{\infty} G_n$ is closed ($\neq \mathbb{C}$)

      c. $\bigcup_{n=1}^{\infty} G_n$ is neither open nor closed
ii) A countable collection of distinct open sets \( \{ F_n \}_{n=1}^{\infty} \) such that

a) \( \bigcap_{n=1}^{\infty} F_n \) is closed (\( \neq \emptyset \))
b) \( \bigcap_{n=1}^{\infty} F_n \) is open (\( \neq \emptyset \))
c) \( \bigcap_{n=1}^{\infty} F_n \) is neither open nor closed

4. (10) Prove the following proposition: Let \( (X, d) \) be a metric space. Let \( f, g : X \to \mathbb{C} \) be uniformly continuous on \( X \). Then, \( f + g \) is uniformly continuous on \( X \).

5. (9) Give examples of sequences \( \{ x_n \}, \{ y_n \} \) in \( \mathbb{C} \) such that

a. \( \lim x_n y_n = L, \lim x_n = M, \lim y_n \) does not exist
b. \( \lim x_n y_n = L, \lim x_n \) does not exist, \( \lim y_n \) does not exist
c. \( \lim \frac{x_n}{y_n} = L, \lim x_n = M, \lim y_n = N, L \neq \frac{M}{N} \)

6. (6) Give examples of distinct sequences \( \{ x_n \} \) in \( \mathbb{C} \) such that

a. \# \( \{ x_n \} \) = 5 (i.e., the sequence has exactly five limit points)
b. the sequence \( \{ x_n \} \) is given by a formula \( x_n = f(n) \), where \( f : \mathbb{N} \to \mathbb{C} \), such that \( x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 8, x_5 = 16 \), but \( x_6 \neq 32 \)
   (i.e., give a closed formula for the function \( f \) defining the sequence \( x_n \)).

7. (8) For \( a, b, c \in \mathbb{C}, a \neq 0 \), show that \( ab = ac \Rightarrow b = c \).

8. (6) Provide a counterexample to each of the following assertions:

a. In a metric space \( (X, d) \), if a set \( A \) is closed and bounded, then \( A \) is compact.
b. Let \( (X, d), (\Omega, \rho) \) be metric spaces. Let \( f : X \to \Omega \). If \( f \) is uniformly continuous on \( X \), then \( f \) is Lipschitz on \( X \).
9. (20) Classification Problem. Correctly identify whether the following subsets of \( \mathbb{C} \) are: (a) open; (b) closed; (c) connected; (d) polygonally path connected; (e) compact; (f) complete; (g) bounded; (h) region. You do not need to provide a rationale for your classification. Fill out the classification information on the attach table.

A. \( B(0,5) \setminus \bar{E} \) where \( E = \{ z = x + iy : \frac{x^2}{25} + \frac{y^2}{4} < 1 \} \)

B. \( \overline{B(0,5)} \setminus \bar{E} \) where \( E = \{ z = x + iy : \frac{x^2}{25} + \frac{y^2}{4} < 1 \} \)

C. \( B(0,1) \setminus B(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \)

D. \( \{ B(0,4) \setminus B(1,1) \} \cap \{ z : \text{Re } z \geq 0 \} \)

E. \( \bigcup_{n=1}^{\infty} l_n \), where each \( l_n = [0, \frac{\text{cis}(\frac{\pi}{2n})}{n}] \)

Classification Table for Problem 9

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